Full Length Research Paper

Mixed equilibrium variational inequalities

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In this paper, we introduce and consider a new class of equilibrium problems and variational inequalities, which is called the mixed equilibrium-variational inequality. We use the auxiliary principle technique to suggest and analyze some explicit and proximal-point iterative methods for solving the mixed equilibrium-variational inequalities. Convergence of these iterative methods is proved under very mild and suitable assumptions. Several special cases are also considered. Results proved in this paper can be viewed as refinement and improvement of the previously known and new results. The ideas and techniques of this paper may stimulate further research in this dynamic area of pure and applied sciences.

Key words: Mixed variational inequality, equilibrium problems, convergence, auxiliary principle technique.

INTRODUCTION

Variational inequalities, the origin of which can be traced back to Stampacchia (1964), are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of variational inequalities have been considered in different directions using novel and innovative techniques related to the variational inequalities, we have the equilibrium problems, which were introduced and studied (Blum et al., 1994). These problems provide us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. For the formulation, applications and numerical methods of equilibrium problems (Blum et al., 1994; Noor, 1975, 2003, 2004, 2004a, 2004b, 2004c; Noor et al., 1993, 1994, 2004, 2004a, 2008; Yao et al., 2009, 2011).

It is known that the mixed variational inequalities and the mixed equilibrium problems are two different classes of variational inequalities and the equilibrium problems. It is natural to consider the unification of these different problems. Motivated by this fact, we consider and study a

new class of mixed equilibrium problems and variational inequalities, which is known as the mixed equilibrium-variational inequality. This class includes the mixed variational inequalities and mixed variational inequalities as special cases. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving the equilibrium problems and variational inequalities. However, it is known that projection, Wiener-Hopf equations and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving mixed equilibrium-variational inequalities due to the nature of the problem. This fact motivated us to use the auxiliary principle technique of Glowinski et al. (1981) as developed by these studies (Noor, 2004, 2004a, 2004b; Noor et al., 1993, 2004, 2004a, 2008). This is the main motivation of this paper. This approach is used to suggest some explicit and proximal-point iterative methods for these problems. We also consider the convergence criteria of the proposed methods under suitable mild conditions, which are the main results (Theorem 3.1, Theorem 3.2 and Theorem 3.3) of this paper. Several special cases of our main results are also considered. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results. For the recent results in

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this direction (Noor et al., 2011, 2011a, 2011b, 2011c). The ideas and techniques of this paper stimulate further research in this area of pure and applied sciences.

PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $F : H \times H \to R$ be a bifunction and $T : H \to H$ be an operator F. Let $\phi(.)$ be proper, semicontinuous and differentiable convex function. We consider the problem of finding $u \in H$ such that:

$$F(u,v) + \langle Tu, v-u \rangle + \phi(v) - \phi(u) \geq 0, \ \forall v \in H , \quad (1)$$

which is called the mixed equilibrium-variational inequality. A variant form of this problem is considered and studied (Yao et al., 2008, 2011).

   We now discuss some important special cases of the problem (Equation 1).

Special cases

1. If $F(u,v) = 0$, then problem (Equation 1) is equivalent to finding $u \in H$ such that:

$$\langle Tu, v-u \rangle + \phi(v) - \phi(u) \geq 0, \ \forall v \in H, \quad (2)$$

which is known as a mixed variational inequality. A wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems of Equations 1 and 2. For the applications, formulation, numerical results and other aspects of the mixed variational inequalities.

2. If $\langle Tu, v-u \rangle = 0$ then problem (Equation 1) turns into the problem of finding $u \in H$ such that:

$$F(u,v) + \phi(v) - \phi(u) \geq 0, \ \forall v \in H , \quad (3)$$

which is known as the mixed equilibrium problems considered (Noor, 2004, 2004a, 2004b).

3. If $\phi(.)$ is an indicator function on the closed convex set K in H, that is,

$$\phi(u) = \begin{cases} 0, & u \in K \\ \infty, & u \notin K, \end{cases}$$

then problem (Equation 1) is equivalent to finding $u \in K$ satisfying:

$$F(u,v) + \langle Tu, v-u \rangle \geq 0, \quad \forall v \in K, \quad (4)$$

which is known as equilibrium-variational inequality and was studied (Takahashi and Takahashi, 2008).

4. If $\phi(.)$ is an indicator function, then (Equation 2) is equivalent to finding $u \in K$ such that:

$$\langle Tu, v-u \rangle \geq 0, \quad \forall v \in K, \quad (5)$$

which is classical variational inequality introduced and studied (Stampacchia, 1964).

5. Finally, if $\phi(.)$ is an indicator function in (Equation 3), then we obtained the problem of finding $u \in K$ such that:

$$F(u,v) \geq 0, \quad \forall v \in K, \quad (6)$$

which is known as classical equilibrium problem introduced and studied (Blum et al., 2004; Noor et al., 2004).

   For suitable and appropriate choice of the operator and spaces, one can obtain several new and known problems as special cases of the mixed equilibrium-variational inequalities problems (Equation 1). For the applications, formulations, numerical methods and other aspects of the equilibrium problems and variational inequalities.

Definition 1

An operator $T : H \to H$ is said to be:

1.   Monotone,   if   and   only   if,
$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in H.$$
2. Partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that
$$\langle Tu - Tv, z - v \rangle \geq -\alpha \|z - u\|^2, \quad \forall u, v, z \in H .$$

Note that, for $z = u$, partially relaxed strong monotonicity reduces to monotonicity of the operator T.

Definition 2

A bifunction $F(.,.) : H \times H \to R$ is said to be:

1.   Monotone,   if   and   only   if,
$$F(u,v) + F(v,u) \leq 0, \quad \forall u, v \in H.$$
2. Partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that
$$F(u,v) + F(v,z) \leq \mu \|z - u\|^2, \quad \forall u, v, z \in H .$$

It is clear that for $z = u$, partially relaxed strongly monotone bifunction is simply a monotone.

RESULTS

Here, we suggest and analyze an iterative method for solving the mixed equilibrium-variational inequality problem (Equation 1) by using the auxiliary principle technique. This technique is mainly due to Glowinski et

al. (1981) as developed (Noor, 2004, 2004a, 2004b; Noor et al., 2011, 2011a, 2011b, 2011c).

For a given $u \in H$, consider the problem of finding a $w \in H$, such that:

$$\rho F(u,v) + \langle \rho Tu, v-w \rangle + \langle w-u, v-w \rangle \geq \rho\phi(w) - \rho\phi(v), \; \forall v \in H,$$
(7)

where $\rho > 0$ is a constant.

Note that if $w = u$, then $w \in H$ is a solution of (Equation 1). This observation enables us to suggest and analyze the following iterative method for solving mixed equilibrium-variational inequalities problem (Equation 1).

Algorithm 1

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\rho F(u_n,v) + \langle \rho Tu_n, v-u_{n+1} \rangle + \langle u_{n+1}-u_n, v-u_{n+1} \rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \; \forall v \in H,$$
(8)

We now discuss some special cases.

1. If $F(u,v) = 0$, then Algorithm (Equation 1) reduces to the following scheme for mixed variational inequalities (Equation 2).

Algorithm 2

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\langle \rho Tu_n, v-u_{n+1} \rangle + \langle u_{n+1}-u_n, v-u_{n+1} \rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \; \forall v \in H.$$

1. If $\langle Tu, v-u \rangle = 0,$ then Algorithm 1 reduces to algorithm 3.

Algorithm 3

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$F(u_n,v) + \langle u_{n+1}-u_n, v-u_{n+1} \rangle \geq +\rho\phi(u_{n+1}) - \rho\phi(v), \; \forall v \in H,$$

which is used for finding the solution of mixed equilibrium problems (Equation 3).

1. If $\phi(.)$ is an indicator function on the closed convex set K, then Algorithm 3 reduces to the following method for solving the problem (Equation 4), which appears to be new one.

Algorithm 4

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\rho F(u_n,v) + \langle \rho Tu_n, v-u_{n+1} \rangle + \langle u_{n+1}-u_n, v-u_{n+1} \rangle \geq 0, \; \forall v \in K.$$

For suitable and appropriate choice of $F(.,.), T, \varphi(.)$ and spaces, one can define iterative algorithms to find the solutions to different classes of equilibrium problems and variational inequalities.

We now study the convergence analysis of Algorithm 1 using the technique (Noor et al., 2011) and this is the main motivation of our next result.

Theorem 1

Let $u \in H$ be a solution of (Equation 1) and $u_{n+1} \in H$ be an approximate solution obtained from Algorithm 1. If the bifunction $F(.,.)$ and the operator $T(.)$ are partially relaxed strongly monotone operators with constants $\mu > 0$ and $\sigma > 0$, respectively, then:

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1-2\rho(\mu+\sigma))\|u_{n+1} - u_n\|^2$$
(9)

Proof

Let $u \in H$ be a solution of (Equation 1). Then, replacing v by u_{n+1} in (Equation 1), we have:

$$\rho F(u,u_{n+1}) + \rho\langle Tu, u_{n+1}-u \rangle + \rho\phi(u_{n+1}) - \rho\phi(u) \geq 0, \quad \rho > 0.$$
(10)

Let $u_{n+1} \in H$ be the approximate solution obtained from Algorithm 1. Taking $v = u$ in (Equation 8), we have:

$$\rho F(u_n,u) + \langle \rho Tu_n, u-u_{n+1} \rangle + \rho\phi(u) - \rho\phi(u_{n+1}) + \langle u_{n+1}-u_n, u-u_{n+1} \rangle \geq 0.$$
(11)

Adding Equations 10 and 11, we have:

$\rho[F(u,u_{n+1})+F(u_n,u)]+\rho\langle Tu-Tu_n,u_{n+1}-u\rangle+\langle u_{n+1}-u_n,u-u_{n+1}\rangle\geq 0,$

which implies that

$$\langle u_{n+1}-u_n,u-u_{n+1}\rangle\geq-\rho[F(u_n,u)+F(u,u_{n+1})]+\rho\langle Tu_n-Tu,u_{n+1}-u\rangle$$
$$\geq-\rho(\mu+\sigma)\|u_{n+1}-u_n\|^2$$

(12)

where we have used partially relaxed strong monotonicity of the bifunction $F(.,.)$ and operator T.

Using   the   relation
$2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2,\quad \forall u,v\in H$, and from (Equation 12), one can have

$\|u-u_{n+1}\|^2\leq\|u-u_n\|^2-(1-2\rho(\mu+\sigma))\|u_{n+1}-u_n\|^2,$

which is the required result (Equation 9).

Theorem 2

Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u\in H$ be a solution of problem (Equation 1). If

$0<\rho<\dfrac{1}{2(\mu+\sigma)}$, then $\lim_{n\to\infty}u_n=u$.

Proof

Let $u\in H$ be a solution of Definition 1. For $0<\rho<\dfrac{1}{2(\mu+\sigma)}$, we see that the sequence $\{\|u-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (Equation 9), we have:

$$\sum_{n=0}^{\infty}(1-2(\mu+\sigma))\|u_{n+1}-u_n\|^2\leq\|u-u_0\|^2,$$

which implies that:

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \qquad (12)$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of this sequence converges to $\hat{u}\in H$. Replacing u_n by u_{n_j} in (Equation 8) and taking the limit as $n_j\to\infty$ and using (Equation 12), we have:

$F(\hat{u},v)+\langle T\hat{u},v-\hat{u}\rangle+\phi(v)-\phi(\hat{u})\geq 0,\ \forall v\in H$, which shows $\hat{u}$ solves the mixed equilibrium-variational inequality (Equation 1) and $\|u_{n+1}-\hat{u}\|^2\leq\|u_n-\hat{u}\|^2$. Thus, it follows from the aforementioned inequality that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n\to\infty}u_n=\hat{u}$, the required result.

We again use the auxiliary principle technique to suggest and analyze several proximal point algorithms for solving the mixed equilibrium-variational inequalities (Equation 1) and this is another motivation of this paper.

1. For a given $u\in H$, consider the problem of finding $w\in H$ such that:

$\rho F(w,v)+\langle\rho Tw,v-w\rangle+\langle w-u,v-w\rangle\geq\rho\phi(w)-\rho\phi(v),\ \ \forall v\in H.$

(14)

Note that if $w=u$, then $w\in K$ is a solution of (Equation 1). This observation enables us to suggest and analyze the following iterative method for solving mixed equilibrium-variational inequalities (Equation 1).

Algorithm 5

For a given $u_0\in H$, compute $u_{n+1}\in H$ from the iterative scheme:

$\rho F(u_{n+1},v)+\langle\rho Tu_{n+1},v-u_{n+1}\rangle+\langle u_{n+1}-u_n,v-u_{n+1}\rangle\geq\rho\phi(u_{n+1})-\rho\phi(v),\ \ \forall v\in H.$

(15)

1. For a given $u\in H$, consider the problem of finding a $w\in H$ such that

$\rho F(u,v)+\langle\rho Tw,v-w\rangle+\langle w-u,v-w\rangle\geq\rho\phi(w)-\rho\phi(v),\ \ \forall v\in H.$

Note that if $w=u$, then $w\in K$ is a solution of (Equation 1). This observation enables us to suggest and analyze the following proximal iterative method for solving mixed equilibrium-variational inequalities (Equation 1).

Algorithm 6

For a given $u_0\in H$, compute $u_{n+1}\in H$ from the iterative scheme:

$\rho F(u_n,v)+\langle\rho Tu_{n+1},v-u_{n+1}\rangle+\langle u_{n+1}-u_n,v-u_{n+1}\rangle\geq\rho\phi(u_{n+1})-\rho\phi(v),\ \ \forall v\in H.$

1. For a given $u_0\in H$, consider the problem of finding a

$w \in H$ such that:

$$\rho F(w,v) + \langle \rho Tu, v-w \rangle + \langle w-u, v-w \rangle \geq \rho\phi(w) - \rho\phi(v), \forall v \in H.$$

Note that if $w = u$, then $w \in K$ is a solution of (Equation 1). This observation enables us to suggest and analyze the following iterative method for solving mixed equilibrium-variational inequalities (Equation 1).

Algorithm 7

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\rho F(u_{n+1}, v) + \langle \rho Tu_n, v-u_{n+1} \rangle + \langle u_{n+1} - u_n, v-u_{n+1} \rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \forall v \in H.$$

Some special cases of these algorithms are as under:

If $F(u,v) = 0$, then Algorithm 5 reduces to the following scheme for mixed variational inequalities given as (Equation 2).

Algorithm 8

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\langle \rho Tu_{n+1}, v-u_{n+1} \rangle + \langle u_{n+1} - u_n, v-u_{n+1} \rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \forall v \in H.$$

If $\langle Tu, v-u \rangle = 0$, then Algorithm 1 reduces to Algorithm 9.

Algorithm 9

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$F(u_{n+1}, v) + \langle u_{n+1} - u_n, v-u_{n+1} \rangle \geq +\rho\phi(u_{n+1}) - \rho\phi(v), \forall v \in H,$$

which is used for finding the solution of mixed equilibrium problems given as (Equation 3).

If $\phi(.)$ is an indicator function on the closed convex set K in H, then Algorithm 5 can be used to find the approximate solution of problem (Equation 4) and which appears to be new one.

Algorithm 10

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\rho F(u_{n+1}, v) + \langle \rho Tu_{n+1}, v-u_{n+1} \rangle + \langle u_{n+1} - u_n, v-u_{n+1} \rangle \geq 0, \forall v \in K.$$

For suitable and appropriate choice of $F(.,.), T, \varphi(.)$ and spaces, one can define iterative algorithms as special cases of Algorithms 6 and 7 to find the solutions to different classes of equilibrium problems and variational inequalities.

We now study the convergence analysis of Algorithm 5 using the technique of Theorems 1 and 2. However, for the sake of completeness and to convey the main ideas, we include the main steps of the proof. Convergence analysis of other algorithms can be proved using the same technique.

Theorem 3

Let $u \in H$ be a solution of (Equation 1) and $u_{n+1} \in H$ be an approximate solution obtained from Algorithm 5. If bifunction $F(.,.)$ and operator T are monotone, then:

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \|u_{n+1}-u_n\|^2. \tag{16}$$

Proof

Let $u \in H$ be a solution of (Equation 1). Then, replacing v by u_{n+1} in (Equation 1), we have:

$$\rho F(u, u_{n+1}) + \rho \langle Tu, u_{n+1} - u \rangle + \rho\phi(u_{n+1}) - \rho\phi(u) \geq 0. \tag{17}$$

Let $u_{n+1} \in H$ be the approximate solution obtained from Algorithm 1. Taking $v = u$ in (Equation 15), we have:

$$\rho F(u_{n+1}, u) + \langle \rho Tu_{n+1}, u-u_{n+1} \rangle + \rho\phi(u) - \rho\phi(u_{n+1}) + \langle u_{n+1} - u_n, u-u_{n+1} \rangle \geq 0. \tag{18}$$

Adding Equations 17 and 18, we have:

$$\rho[F(u, u_{n+1}) + F(u_n, u)] + \rho \langle Tu - Tu_{n+1}, u_{n+1} - u \rangle + \langle u_{n+1} - u_n, u-u_{n+1} \rangle \geq 0,$$

which implies that

$$\langle u_{n+1} - u_n, u-u_{n+1} \rangle \geq -\rho[F(u_n, u) + F(u, u_{n+1})] + \rho \langle Tu_{n+1} - Tu, u_{n+1} - u \rangle \geq 0, \tag{19}$$

where we have used the monotonicity of the operator T and the bifunction $F(.,.)$.

Using the relation $2\langle u,v \rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2$, $\forall u, v \in H$ and from (Equation 19), we have:

$$\left\| u - u_{n+1} \right\|^2 \leq \left\| u - u_n \right\|^2 - \left\| u_{n+1} - u_n \right\|^2,$$

which is the required result (Equation 16).

Theorem 4

Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 5 and $u \in H$ is a solution of problem (Equation 1), then $\lim_{n \to \infty} u_n = u$.

Proof

Let $u \in H$ be a solution of Definition 1. Then, we see that the sequence $\left\{ \left\| u - u_n \right\| \right\}$ is nonincreasing and consequently $\left\{ u_n \right\}$ is bounded. Also from (Equation 16), we have:

$$\sum_{n=0}^{\infty} \left\| u_{n+1} - u_n \right\|^2 \leq \left\| u - u_0 \right\|^2,$$

which implies that

$$\lim_{n \to \infty} \left\| u_{n+1} - u_n \right\| = 0. \tag{20}$$

Let $\hat{u}$ be the cluster point of $\left\{ u_n \right\}$ and the sub sequence $\left\{ u_{n_j} \right\}$ of this sequence converges to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (Equation 15) and taking the limit $n_j \to \infty$ and using (Equation 20), we have:

$$F(\hat{u}, v) + \left\langle T\hat{u}, v - \hat{u} \right\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \ \forall v \in H,$$

which shows $\hat{u}$ solves the mixed equilibrium-variational inequality (Equation 1) and $\left\| u_{n+1} - \hat{u} \right\|^2 \leq \left\| u_n - \hat{u} \right\|^2$. Thus, it follows from the aforementioned inequality that the sequence $\left\{ u_n \right\}$ has exactly one cluster point and $\lim_{n \to \infty} u_n = \hat{u}$, the required result.

CONCLUSION

In this paper, we have used the auxiliary principle technique to suggest and analyze some explicit and proximal point algorithms for solving the mixed equilibrium-variational inequality problem. We have also discussed the convergence criteria of the proposed new iterative methods under some suitable weaker conditions. In this sense, our results can be viewed as refinement and improvement of the previously known results. Note the auxiliary principle technique does not involve the projection and the resolvent operators. We have also shown that this technique can be used to suggest several iterative methods for solving various classes of equilibrium and variational inequalities problems. Results proved in this paper may inspire further research in this area.

Future directions

We would like to mention that the problem considered in this paper can be studied from different point of views, such as sensitivity analysis, dynamical and stability analysis. It is an interesting problem from both application point of view and numerical analysis to verify the implementation and efficiency of the proposed iterative methods for solving the mixed equilibrium-variational inequalities. Recently much attention has been given to study the existence of a solution of the equilibrium problems and variational inequalities in the topological vector spaces using the Knaster-Kuratowski Mazurkiewicz (KKM) mapping theorem. This is an open problem to study the existence of a solution of the mixed equilibrium-variational inequalities in the Banach and topological spaces. This is another direction of future research in the dynamic and fast growing field of mathematical and engineering sciences. It is known that the equilibrium-variational inequalities can be characterized by systems of variational equal ions using the penalty methods. These variational equations can be solved using the variational of parameters method and homotopy methods (Noor et al., 2010, 2011d, 2011e, 2011d, 2011f, 2011g). We hope to explore these aspects in details along with some applications.

ACKNOWLEDGEMENTS

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