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Full Length Research Paper

Weighted pseudo almost periodic solutions for differential equations involving reflection of the argument

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By means of the fixed point methods and the properties of the weighted pseudo almost periodic functions, the existence and uniqueness of weighted pseudo almost periodic solutions are obtained for differential equations involving reflection of the argument.

Key words: Almost periodic functions, weighted pseudo almost periodic functions, fixed point methods, differential equations, reflection of argument.

INTRODUCTION

The almost periodic functions are closely connected with harmonic analysis, differential equations, dynamical systems, etc. (Diagana, 2008; Fink, 1974). Almost periodic functions are the generalization of continuous periodic and quasi-periodic functions. And they are generalized by many mathematicians in various ways (Sharkovskii, 1978). What we are interested in here is the pseudo almost periodic functions. This almost periodic type functions was introduced by Zhang in 1994. In Zhang, 1994, 1995), he also investigated their applications to differential equations. In 2006, Diagana (Diagana, 2006) generalized the notion of pseudo almost periodic function, and introduced some new classes of functions called weighted pseudo almost periodic functions and their applications to differential equations. Some new references focus on the weighted pseudo almost periodic solutions of differential equations; (Diagana, 2006, 2008, 2009a; Zhang, 2003).

Motivated by the aforementioned works, we are going to investigate the weighted pseudo almost periodic solutions of differential equations involving reflection of argument. These kind of differential equations have applications in the study of stability of differential-difference equations, see Sarkovskii (1978), and such equations show very interesting properties by themselves. So many authors have worked on them.

Aftabizadeh et al. (1985) initiated to study boundary value problems involving reflection of the argument. Gupta (1987a, 1987b) investigated two point boundary value problems for this kind of equations. Afidabizadeh et al. (1988) studied the existence of unique bounded solution of

$$\dot{x}(t) = f(t, x(t), x(-t)). \tag{1}$$

They proved that $x(t)$ is almost periodic by assuming the existence of bounded solution $x(t)$ of Equation 1. The existence and uniqueness of periodic, almost periodic and pseudo almost periodic solutions of the equations

$$\dot{x}(t) + ax(t) + bx(-t) = g(t), \quad b \neq 0, \quad t \in \mathrm{R}, \tag{2}$$

and

$$\dot{x}(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), \quad b \neq 0, \quad t \in \mathrm{R}, \tag{3}$$

were investigated in Piao (2004 a; 2004 b). Here $f(t, x, y)$ is almost periodic or pseudo almost periodic on t uniformly with respect to x and y in any compact subset of R^2.

Now we are concerned with weighted pseudo almost periodic solutions of Equation 2 and 3, where $f(t, x, y)$

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weighted pseudo almost periodic on t uniformly with respect to x and y in any compact subset of R^2. And we obtain the existence and uniqueness of weighted pseudo almost periodic solution.

Remark 1

Let $y(t)=x(-t)$, then Equation 2 is changed into system

$\dot{x}=-ax-by+g(t)$, $\dot{y}=bx+ay-g(-t)$, (4)

which is formally a Hamilton system with Hamiltonian

$$H(x,y,t)=\frac{1}{2}bx^2+\frac{1}{2}by^2+axy-g(-t)x-g(t)y.$$

So one may say that some first order scalar differential equations can also generate Hamilton systems.

PRELIMINARIES

Throughout the paper, let $(BC(\mathrm{R},\mathrm{R}),\|\cdot\|)$ denote the collection of all R-valued bounded continuous functions equipped with the sup norm $\|\phi\|_\infty:=\sup_{t\in\mathrm{R}}|\phi(t)|$ for each $\phi\in BC(\mathrm{R},\mathrm{R})$. Let U denote the collection of functions (weights) $\rho:\mathrm{R}\to(0,\infty)$, which are locally integrable over R such that $\rho>0$ almost everywhere. Let U_B denote the set of all $\rho\in U$ such that ρ is bounded with $\inf_{t\in\mathrm{R}}\rho(t)>0$. From now on, if $\rho\in U$ and for $T>0$, we then set:

$$\mu(T,\rho):=\int_{-T}^{T}\rho(t)\mathrm{d}t.$$

As in the particular case when $\rho(t)=1$ for each $t\in\mathrm{R}$, we are exclusively interested in those weights, ρ, for which $\lim_{T\to\infty}\mu(T,\rho)=\infty$. Thus the space of weights U_∞ defined by

$$U_\infty:=\{\rho\in U:\lim_{T\to\infty}\mu(T,\rho)=\infty\}$$

plays a key role. Of course, one can always consider those weights $\rho\in U$ for which $\lim_{T\to\infty}\mu(T,\rho)\neq\infty$.

Obviously, $U_B\subset U_\infty\subset U$, with strict inclusions.

Definition 1

Let $\rho_1,\rho_2\in U$. One says ρ_1 is equivalent to ρ_2 or $\rho_1\sim\rho_2$ if whenever $\rho_1/\rho_2\in U_B$ (Diagana, 2006, 2008).

Definition 2

(Corduneanu, 1989; Fink, 1974) A function $f:\mathrm{R}\to\mathrm{R}$ is almost periodic, if the $\varepsilon-$translation of f $T(f,\varepsilon)=\{\tau\in\mathrm{R}:|f(t+\tau)-f(t)|<\varepsilon,\forall t\in\mathrm{R}\}$ is relatively dense in R. We denote the set of all such functions by $AP(\mathrm{R})$.

Definition 3

A function $F:\mathrm{R}\times\mathrm{R}^2\to\mathrm{R}$ is almost periodic for t uniformly on R^2, if for any compact $\mathrm{W}\subset\mathrm{R}^2$, the $\varepsilon-$translation of F (Corduneanu, 1989; Fink, 1974)

$$T(F,\varepsilon,W)=\{\tau\in\mathrm{R}:|F(t+\tau,x,y)-F(t,x,y)|<\varepsilon,\forall(t,x,y)\in\mathrm{R}\times W\}$$

is relatively dense in R. We denote the set of all such functions by $AP(\mathrm{R}\times\mathrm{R}^2)$.

Let $\rho\in U_\infty$. We denote by $PAP_0(\mathrm{R},\rho)$ the set that

$$\{f\in BC(\mathrm{R}):\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|f(\sigma)|\rho(\sigma)\mathrm{d}\sigma=0\},$$

and by $PAP_0(\mathrm{R}\times\mathrm{R}^2,\rho)$ the set that

$$\{F\in BC(\mathrm{R}\times\mathrm{R}^2):\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|F(\sigma,x,y)|\rho(\sigma)\mathrm{d}\sigma=0,\forall(x,y)\in W\}.$$

Clearly, when $\rho(t)=1$ for each $t\in\mathrm{R}$, one retrieves the so-called ergodic space of Zhang, that is, $PAP_0(\mathrm{R})$, defined by

$$\{f\in BC(\mathrm{R}):\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma)|\mathrm{d}\sigma=0\}.$$

Clearly, the spaces $PAP_0(\mathrm{R},\rho)$ are richer than

$PAP_0(\mathrm{R})$ and give rise to an enlarged space of pseudo-almost periodic functions.

Definition 4

Let $\rho \in U_\infty$. A function $f \in BC(\mathrm{R})$ is called weighted pseudo almost periodic (or $\rho-$pseudo almost periodic) if it can be decomposed as $f = g + \phi$, where $g \in AP(\mathrm{R})$ and $\phi \in PAP_0(\mathrm{R},\rho)$. The collection of such functions will be denoted by $PAP(\mathrm{R},\rho)$.
The decomposition of a $\rho-$pseudo almost periodic function is also unique (Diagana, 2006, 2008).

Lemma 1

Let $\rho \in U_\infty$.

(1) If $g(t) \in PAP(\mathrm{R},\rho)$, $|g(\cdot)|$ is even, then $g(-t) \in PAP(\mathrm{R},\rho)$.
(2) Let $g(t) \in PAP(\mathrm{R},\rho)$. If $\rho(t) \sim \rho(-t)$, then $g(-t) \in PAP(\mathrm{R},\rho)$.
Particularly if $\rho(t) = \rho(-t)$, then $g(-t) \in PAP(\mathrm{R},\rho)$.
Proof (1) If $|g(\cdot)|$ is even, the proof is direct.

(2) If $|g(\cdot)|$ is not even, write $g = h + \varphi$, where $h \in AP(\mathrm{R})$ and $\varphi \in PAP_0(\mathrm{R},\rho)$. $h(-t) \in AP(\mathrm{R})$ (Diagana, 2006). If $\rho(t) \sim \rho(-t)$, we have

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} |\varphi(-\sigma)| \rho(\sigma) \mathrm{d}\sigma$$

$$=\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} |\varphi(\sigma)| \rho(\sigma) \frac{\rho(-\sigma)}{\rho(\sigma)} \mathrm{d}\sigma = 0,$$

hence $\varphi(-t) \in PAP_0(\mathrm{R},\rho)$. So $g(-t) \in PAP(\mathrm{R},\rho)$.

Lemma 2

Let $\rho \in U_\infty$ and $W_1, W_2 \subset \mathrm{R}$ be closed, set $W = W_1 \times W_2 \subset \mathrm{R}^2$ (Diagana, 2008). Let $f \in PAP(\mathrm{R} \times \mathrm{W},\rho)$ satisfy

$$|f(t,Z') - f(t,Z'')| \le L|Z' - Z''| \quad (t \in \mathrm{R}, Z', Z'' \in W),$$

where $L > 0$. If $F \in PAP(\mathrm{R},\rho)^2$ and $F(t) \in W$ for all $t \in \mathrm{R}$, then $f \circ (\iota \times F) \in PAP(\mathrm{R},\rho)$. Here $F = (f_1, f_2)$, and $\iota \times F(t) = (t, f_1(t), f_2(t))$.

MAIN RESULTS

Our main results can be stated as follows.

Theorem 1

$\rho \in U_\infty$. For any $g(t) \in PAP(\mathrm{R},\rho)$, $\lambda^2 = a^2 - b^2$, $\lambda > 0$. Setting $P(\lambda,\rho) := \sup_{T>0}\{\int_{-T}^{T} e^{-\lambda(t+T)} \rho(t)\mathrm{d}t\}$, we suppose that $P(\lambda,\rho) < \infty$, and $\rho(t) \sim \rho(-t)$. Then Equation 2 has a unique $\rho-$weighted pseudo almost periodic solution $x(t)$.

Proof existence

From Lemma 2 and 3 of Afidabizadeh et al. (1985), we can derive

$$x(t) = -\frac{1}{2\lambda}[e^{\lambda t}\int_t^\infty e^{-\lambda s}((\lambda - a)g(s) + bg(-s))]\mathrm{d}s$$
$$+\frac{1}{2\lambda}[e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}((\lambda + a)g(s) - bg(-s))]\mathrm{d}s$$

is a particular solution of Equation 2 for any $g(t) \in PAP(R,\rho)$. Now we show $x(t) \in PAP(R,\rho)$. Suppose $g(t) = h(t) + \varphi(t)$, $h(t) \in AP(R)$, $\varphi(t) \in PAP_0(R,\rho)$. Let

$$H(t) = -\frac{1}{2\lambda}[e^{\lambda t}\int_t^\infty e^{-\lambda s}((\lambda - a)h(s) + bh(-s))]\mathrm{d}s$$
$$+\frac{1}{2\lambda}[e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}((\lambda + a)h(s) - bh(-s))]\mathrm{d}s$$

and

$$\Phi(t) = -\frac{1}{2\lambda}[e^{\lambda t}\int_t^\infty e^{-\lambda s}((\lambda - a)\varphi(s) + b\varphi(-s))]\mathrm{d}s$$

$$+\frac{1}{2\lambda}[e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}((\lambda+a)\varphi(s)-b\varphi(-s))]\mathrm{d}s$$

then $x(t)=H(t)+\Phi(t)$. Similar to the proof of Theorem 2.1 in Piao (2004a), we have $H(t)\in AP(\mathrm{R})$.

Next, we show that $\Phi(t)\in PAP_0(\mathrm{R},\rho)$.

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}|\Phi(t)|\,\rho(t)\mathrm{d}t$$
$$\le\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{t}^{\infty}e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$+\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{-\infty}^{t}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$=\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{t}^{T}e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$+\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{T}^{\infty}e^{\lambda(t-s)}(|\lambda-a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$+\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{-\infty}^{-T}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$+\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{-T}^{T}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$

For convenience, we denote the last four terms by I_1, I_2, I_3 and I_4 respectively.

$$I_3(\rho)=\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{-\infty}^{-T}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$=\frac{1}{2\lambda\mu(T,\rho)}\int_{-\infty}^{-T}e^{\lambda s}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s\int_{-T}^{T}e^{-\lambda t}\rho(t)\mathrm{d}t$$

$$\le\frac{P(\lambda,\rho)}{2\lambda^2\mu(T,\rho)}(|\lambda+a|+|b|)\|\varphi\|$$

$\to 0$, as $T\to\infty$,

$$I_4(\rho)=\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\rho(t)\mathrm{d}t\int_{-T}^{t}e^{\lambda(s-t)}(|\lambda+a||\varphi(s)|+|b||\varphi(-s)|)\mathrm{d}s$$
$$=\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}(|\lambda+a||\varphi(t)|+|b||\varphi(-t)|)\rho(t)\mathrm{d}t\int_{-T}^{t}e^{-\lambda(t-s)}\mathrm{d}s$$

$$=\frac{1}{2\lambda\mu(T,\rho)}\int_{-T}^{T}\frac{1}{\lambda}(1-e^{-\lambda(t+T)})(|\lambda+a||\varphi(t)|+|b||\varphi(-t)|)\rho(t)\mathrm{d}t$$

$$\le\frac{1}{2\lambda^2\mu(T,\rho)}\int_{-T}^{T}(|\lambda+a||\varphi(t)|+|b||\varphi(-t)|)\rho(t)\mathrm{d}t$$

$$=\frac{|\lambda+a|}{2\lambda^2\mu(T,\rho)}\int_{-T}^{T}|\varphi(t)|\,\rho(t)\mathrm{d}t+\frac{|b|}{2\lambda^2\mu(T,\rho)}\int_{-T}^{T}|\varphi(-t)|\,\rho(t)\mathrm{d}t$$

$\to 0$, as $T\to\infty$.

Similarly, we can show that $I_1(\rho),I_2(\rho)\to 0$ as $T\to\infty$. So, $x(t)$ is weighted pseudo almost periodic solution of Equation 2.

Uniqueness

If there is another $\rho-$weighted pseudo almost periodic solution $x_1(t)$ for Equation 2, then the difference $x(t)-x_1(t)$ should be a solution of the homogeneous equation

$$\dot{x}(t)+ax(t)+bx(-t)=0,\ b\neq 0,\ t\in\mathrm{R}. \tag{5}$$

According to the Lemma 2 of Fink (1974), we can derive

$$x(t)-x_1(t)=C\left(\frac{\lambda-a}{b}e^{\lambda t}+e^{-\lambda t}\right),\ t\in\mathrm{R}, \tag{6}$$

for some constant C. If $C\neq 0$, then $x(t)-x_1(t)$ will be unbounded. This is a contradiction to the boundedness of weighted pseudo almost periodic function. So $x(t)-x_1(t)\equiv 0$, that is, $x(t)\equiv x_1(t)$.

Theorem 2

Let $\rho\in U_\infty$. Suppose $f(t,x,y)\in PAP(\mathrm{R}\times\mathrm{R}^2,\rho)$ and satisfies Lipschitz condition

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le L(|x_1-x_2|+|y_1-y_2|)$$

for any $(x_1,y_1),(x_2,y_2)\in\mathrm{R}^2$, where $L<\frac{2\lambda^2}{|\lambda-a|+|\lambda+a|+2|b|}$, $\lambda^2=a^2-b^2$, $\lambda>0$.

Setting $P(\lambda,\rho):=\sup_{T>0}\{\int_{-T}^{T}e^{-\lambda(t+T)}\rho(t)\mathrm{d}t\}$, we suppose that $P(\lambda,\rho)<\infty$, and $\rho(t)\sim\rho(-t)$. Then Equation 3 has a unique $\rho-$weighted pseudo almost periodic solution $x(t)$.

Proof

From Remark 2.5 in Diagana (2008), we know the space $PAP(\mathrm{R},\rho)$ is a Banach with the supremum norm. And from Lemma 2, we see $f(t,\varphi(t),\varphi(-t)) \in PAP(\mathrm{R},\rho)$. According to Theorem 1, we see the equation

$$\dot{x}(t)+ax(t)+bx(-t) = f(t,\varphi(t),\varphi(-t)),\ b \neq 0,\ t \in \mathrm{R}, \quad (7)$$

has a unique $\rho-$weighted pseudo almost periodic solution, denote it by $(T\varphi)(t)$. Then we define a mapping $T: PAP(\mathrm{R},\rho) \to PAP(\mathrm{R},\rho)$. Now we show T is contracted.

For $\varphi(t), \psi(t) \in PAP(\mathrm{R},\rho)$, the equation

$$\dot{x}(t)+ax(t)+bx(-t) = f(t,\varphi(t),\varphi(-t)) - f(t,\psi(t),\psi(-t)),\ b \neq 0,\ t \in \mathrm{R}, \quad (8)$$

has a unique $\rho-$weighted pseudo almost periodic solution $(T\varphi - T\psi)(t)$, and

$$\|T\varphi - T\psi\| \leq \frac{|\lambda - a| + |\lambda + a| + 2|b|}{2\lambda^2} L \|\varphi - \psi\|$$

Since $\frac{|\lambda - a| + |\lambda + a| + 2|b|}{2\lambda^2} L < 1$, T is a contraction mapping, and so T has a unique fixed point in $PAP(\mathrm{R},\rho)$. That is to say the Equation 3 has a unique $\rho-$weighted pseudo almost periodic solution $x(t)$. The proof is completed.

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