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Full Length Research Paper

The characterizations of constant slope surfaces and Bertrand curves

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In this paper, we give some characterizations of constant slope surfaces and Bertrand curves in Euclidean 3-space. We find parametrization of constant slope surfaces for spherical images of tangent indicatrix, principal normal indicatrix, binormal indicatrix and the Darboux indicatrix of a space curve. Furthermore, we investigate Bertrand curves corresponding to constant parameter curves of constant slope surfaces.

Key words: Bertrand curve, helix, spherical images, constant slope surface.

INTRODUCTION

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, $\alpha-$helices, DNA double and collagen triple helix; the double helix shape is commonly associated with DNA, since the double helix is structure of DNA (Camci et al., 2009). This fact was published for the first time by Watson and Crick (1953). They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases, guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds.

Helix is one of the most fascinating curve in science and nature. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices (Toledo-Suarez, 2009). The helix may be called a circular helix or W-curve (Ilarslan and Boyacioglu, 2007; Monterde, 2009). Circular helix is the simplest three-dimensional spirals. One of the most interesting spiral examples are k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related golden mean or golden section appears very often in theoretical physics and physics of the high energy particles (El Naschie, 2001, 2005). Three-dimensional k-Fibonacci spirals were studied from a geometric point of view in (Falson and Plaza, 2008).

We are thinking of a curve as the path traced out by a particle moving in Euclidean 3-space. So, position vector of the curve is very important to determine behaviour of the curve, because a position vector is a vector which describes the position of a point P in space in relation to an arbitrary reference origin O. It is equaivalent to an imaginary displacement from O to P.

A curve of constant slope or general helix in Euclidean 3-space is defined by the property that the tangent makes a constant angle with a fixed straight line. A classical result about helix stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (Struik, 1988) says that: A necessary and sufficient condition that a curve be a general helix is that the ratio $\frac{\kappa}{\tau}$ is constant along the curve, where κ and $\tau \neq 0$ denote the curvature and the torsion, respectively. If both of $\kappa(s)$ and $\tau(s)$ are non-zero constant, it is called a circular helix. Thereafter Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topic of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with

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another special curve (called Bertrand mate). The curve is a Bertrand curve if and only if there exist non-zero real numbers A, B such that $A\kappa(s)+B\tau(s)=1$ for any $s\in I$ (Carmo, 1976; Izumiya and Takeuchi, 2003). So a circular helix is a Bertrand curve. Note that Bertrand mates are particular examples of offset curves used in computer-aided design (Nutbourne and Martin, 1988).
Izumiya and Takeuchi (2002) have shown that Bertrand curves can be constructed from the spherical curves. Thereafter they Izumiya and Takeuchi, (2004) intro-duced the concept of slant helix by saying that the normal lines of the curve make a constant angle with a fixed direction and given a characterization of slant helix in Euclidean 3-space by the fact that the function

$$\psi(s)=\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s) \tag{1}$$

is constant. After them, Kula and Yayli (2005) investigated spherical images of tangent indicatrix and binormal indicatrix of a slant helix. Moreover, they showed that the spherical images are spherical helices. Boyadzhiev (2007) explored three-dimensional versions of these two properties: Surfaces that are equiangular and those that are self-similar. He investigated the relationships among these surfaces and gave some examples. Munteanu (2010) studied constant slope surfaces in Euclidean 3-space and found the parametric equations which characterized these surfaces and showed that a constant slope surface could be constructed by using an arbitrary curve on the sphere S^2.

In this study, we give some characterizations of constant slope surfaces and Bertrand curves in Euclidean 3-space. We find parametrization of constant slope surfaces for spherical images of tangent indicatrix, principal normal indicatrix, binormal indicatrix and the Darboux indicatrix of a space curve. Furthermore we investigate Bertrand curves corresponding to constant parameter curves of constant slope surfaces.

PRELIMINARIES

We now recall basic concepts on classical differential geometry of space curves and the definitions of the spherical images and constant slope surfaces in Euclidean 3-space. After that, we give the parametrizations about Bertrand curves and constant slope surfaces. Let $\tilde{f}: I\to R^3$ be a curve with $\tilde{f}'(t)\neq 0$ where $\tilde{f}'(t)=d\tilde{f}/dt(t)$. We also denote the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|$. The arc-length of a curve $\tilde{f}$, measured from $\tilde{f}(t_0)$, $t_0\in I$ is;

$$s(t)=\int_{t_0}^{t}\left\|\tilde{f}'(v)\right\|dv. \tag{2}$$

We say that a curve $\tilde{f}$ is parameterized by the arc-length if it satisfies $\left\|\tilde{f}'(s)\right\|=1$. Let us denote $\boldsymbol{T}(s)=\tilde{f}'(s)$ and we call $\boldsymbol{T}(s)$ a unit tangent vector of $\tilde{f}$ at s. We define the curvature of $\tilde{f}$ by $\kappa(s)=\left\|\tilde{f}''(s)\right\|$. If $\kappa(s)\neq 0$, then the unit principal normal vector $\boldsymbol{N}(s)$ of curve $\tilde{f}$ at s is given by $\tilde{f}''(s)=\kappa(s)\boldsymbol{N}(s)$. The unit vector $\boldsymbol{B}(s)=\boldsymbol{T}(s)\times\boldsymbol{N}(s)$ is called the unit binormal vector of $\tilde{f}$ at s. We recall the well known Frenet-Serret formulae:

$$\begin{aligned}\boldsymbol{T}'(s)&=\kappa(s)\boldsymbol{N}(s)\\ \boldsymbol{N}'(s)&=-\kappa(s)\boldsymbol{T}(s)+\tau(s)\boldsymbol{B}(s)\\ \boldsymbol{B}'(s)&=-\tau(s)\boldsymbol{N}(s).\end{aligned} \tag{3}$$

For any unit speed curve $\tilde{f}: I\to R^3$, we call $\boldsymbol{W}(s)=\tau(s)\boldsymbol{T}(s)+\kappa(s)\boldsymbol{B}(s)$ the Darboux vector field of $\tilde{f}$. Let us define the curve $\boldsymbol{C}$ on S^2 with the help of vector field $\boldsymbol{C}(s)=\dfrac{\boldsymbol{W}(s)}{\|\boldsymbol{W}(s)\|}$. This curve is called the spherical Darboux image or the Darboux indicatrix of $\tilde{f}$. The unit tangent vectors along the curve $\tilde{f}$ generate a curve $(\boldsymbol{T})$ on S^2. The curve $(\boldsymbol{T})$ is called the spherical indicatrix of $\boldsymbol{T}$ or more commonly tangent indicatrix of the curve $\tilde{f}$. If $\tilde{f}=\tilde{f}(s)$ is a natural representation of $\tilde{f}$, then $(\boldsymbol{T})=\boldsymbol{T}(s)$ will be a representation of $(\boldsymbol{T})$. Similarly, one considers the principal normal indicatrix $(\boldsymbol{N})=\boldsymbol{N}(s)$ and binormal indicatrix $(\boldsymbol{B})=\boldsymbol{B}(s)$ (Struik, 1988).

For a general parameter t of a space curve $\tilde{f}$, we can calculate the curvature and the torsion as follows:

$$\kappa(t)=\frac{\left\|\dot{\tilde{f}}(t)\times\ddot{\tilde{f}}(t)\right\|}{\left\|\dot{\tilde{f}}(t)\right\|^3},\quad \tau(t)=\frac{\det(\dot{\tilde{f}}(t),\ddot{\tilde{f}}(t),\tilde{f}^{(3)}(t))}{\left\|\dot{\tilde{f}}(t)\times\ddot{\tilde{f}}(t)\right\|^2}. \tag{4}$$

Constant slope surfaces are those for which the position vector of a point of the surface makes constant angle with the normal at the surface in that point in the Euclidean 3-space (Munteanu, 2010).

Let $f : I \to S^2$ be unit speed spherical curve. We denote v as the arc-lenght parameter of f. Let us denote $t(v) = \dot{f}(v)$ and we call $t(v)$ a unit tangent vector of f at v, where $\dot{f} = \frac{df}{dv}$. We now set a vector $s(v) = f(v) \times t(v)$, where f denotes either the position vector or the point of the curve. By definition of the curve f, we have an orthonormal frame $\{f(v), t(v), s(v)\}$ along f. This frame is called the Sabban frame of f (Koenderink, 1990). Then we have the following spherical Frenet-Serret formulae of f:

$$\begin{aligned} \dot{f}(v) &= t(v) \\ \dot{t}(v) &= -f(v) + \kappa_g(v) s(v) \\ \dot{s}(v) &= -\kappa_g(v) t(v), \end{aligned} \tag{5}$$

where $\kappa_g(v)$ is the geodesic curvature of the curve f in S^2 which is given by $\kappa_g(v) = \det(f(v), t(v), \dot{t}(v))$. We now define a space curve

$$\tilde{\gamma}(v) = a\int_{v_0}^{v} f(t)dt + a\cot\theta \int_{v_0}^{v} f(t) \times f'(t)dt + c, \tag{6}$$

where a, θ are constant numbers and c is a constant vector.

Theorem 1

Under the above notation, $\tilde{\gamma}$ is a Bertrand curve. Moreover all Bertrand curves can be constructed by the aforementioned method (Izumiya and Takeuchi, 2002). This theorem gives parametrization of Bertrand curves and we can give parametrization of constant slope surfaces as follows.

Theorem 2

Let $\tilde{f} : S \to R^3$ be an isometric immersion of a surface S in the Euclidean 3-space. Then S is of constant slope if and only if either it is an open part of the Euclidean 2-sphere centered in the origin, or it can be parametrized by

$$\tilde{f}(u,v) = u\sin\theta(\cos\xi\, f(v) + \sin\xi\, f(v) \times f'(v)), \tag{7}$$

where θ is a constant (angle) different from 0, $\xi = \xi(u) = \cot\theta \log u$ and f is a unit speed curve on the Euclidean sphere S^2 (Munteanu, 2010).

Then we have the following corollary of Theorem 2.

Corollary 1

The position vector of $\tilde{f}(u,v)$ lies on the $Sp\{f(v), s(v)\}$ plane, where $s(v) = f(v) \times f'(v)$.

THE CHARACTERIZATIONS OF CONSTANT SLOPE SURFACES AND BERTRAND CURVES

Here, we give some characterizations of constant slope surfaces and Bertrand curves in Euclidean 3-space. We find parametrization of constant slope surfaces for spherical images of tangent indicatrix, principal normal indicatrix, binormal indicatrix and the Darboux indicatrix of a space curve. Furthermore we investigate Bertrand curves corresponding to constant parameter curves of constant slope surfaces and give an example of constant slope surfaces and Bertrand curves.

The parametrization of Bertrand curves had been given previously, follow is the lemma.

Lemma 1

Let $f : I \to S^2$ be unit speed spherical curve. Then

$$\tilde{\gamma}(v) = a\int_0^v f(t)dt + a\tan\xi \int_0^v f(t) \times f'(t)dt \tag{8}$$

is a Bertrand curve, where a, ξ are constant numbers. Moreover all Bertrand curves can be constructed by this method.

Proof

By using the method in Izumiya and Takeuchi (2002), we compute the curvature and the torsion of $\tilde{\gamma}(v)$. Taking the derivative of Equation 8 with respect to v, we have

$$\begin{aligned} \dot{\tilde{\gamma}}(v) &= a(f(v) + \tan\xi s(v)) \\ \ddot{\tilde{\gamma}}(v) &= a(1 - \tan\xi\kappa_g(v))t(v) \\ \tilde{\gamma}^{(3)}(v) &= a(-1 + \tan\xi\kappa_g(v))f(v) - a\tan\xi\dot{\kappa}_g(v)t(v) + a(\kappa_g(v) - \tan\xi\kappa_g^2(v))s(v) \end{aligned} \tag{9}$$

Therefore, by Equation 4, we can calculate as follows:

$$\kappa(v)=\varepsilon\frac{\cos^2\xi(1-\kappa_g(v)\tan\xi)}{a} \text{ and } \tau(v)=\frac{\cos^2\xi(\kappa_g(v)+\tan\xi)}{a}, \quad (10)$$

where $\varepsilon=\pm 1$.

It follows from these formulae that

$$a(\varepsilon\kappa(v)+\tan\xi\tau(v))=1, \quad (11)$$

so that $\tilde{\gamma}(v)$ is a Bertrand curve.
Conversely, let $\tilde{\gamma}(s)$ be a Bertrand curve. There exist real numbers $A, \; B$ different from 0 such that $A\kappa(s)+B\tau(s)=1$. We take that $a=A, \; \tan\xi=\frac{B}{a}$. We assume that $a>0$ and choose $\varepsilon\pm 1, \; \frac{\varepsilon\cos\xi}{a}>0$. If we consider the Frenet frame $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ for the curve $\tilde{\gamma}(s)$, we define a spherical curve

$$\boldsymbol{f}(s)=\varepsilon(\cos\xi\boldsymbol{T}(s)-\sin\xi\boldsymbol{B}(s)). \quad (12)$$

Thus we have

$$\boldsymbol{f}'(s)=\varepsilon\cos\xi(\kappa(s)+\tan\xi\tau(s))\boldsymbol{N}(s)=\frac{\varepsilon}{a}\cos\xi\boldsymbol{N}(s). \quad (13)$$

Let v be the arc-length parameter of $\boldsymbol{f}$, then we have $\frac{dv}{ds}=\frac{\varepsilon}{a}\cos\xi$. Moreover we have

$$a\boldsymbol{f}(s)\frac{dv}{ds}=a\varepsilon(\cos\xi\boldsymbol{T}(s)-\sin\xi\boldsymbol{B}(s))\frac{\varepsilon}{a}\cos\xi=\cos\xi(\cos\xi\boldsymbol{T}(s)-\sin\xi\boldsymbol{B}(s)) \quad (14)$$

And

$$\begin{aligned} a\tan\xi\boldsymbol{f}(v)\times\frac{d\boldsymbol{f}}{dv}\frac{dv}{ds} &= a\tan\xi\varepsilon(\cos\xi\boldsymbol{T}(s)-\sin\xi\boldsymbol{B}(s))\times\frac{\varepsilon}{a}\cos\xi\boldsymbol{N}(s) \\ &= \sin\xi(\cos\xi\boldsymbol{B}(s)+\sin\xi\boldsymbol{T}(s)). \end{aligned} \quad (15)$$

Since $s=\boldsymbol{f}\times\frac{d\boldsymbol{f}}{dv}$, we have

$$\begin{aligned} a\int_0^v \boldsymbol{f}(t)dt+a\tan\xi\int_0^v \boldsymbol{f}(t)\times\boldsymbol{f}'(t)dt &= \int_{s_0}^s \cos\xi(\cos\xi\boldsymbol{T}(t)-\sin\xi\boldsymbol{B}(t))dt \\ &\quad +\int_{s_0}^s \sin\xi(\cos\xi\boldsymbol{B}(t)+\sin\xi\boldsymbol{T}(t))dt \\ &= \int_{s_0}^s \boldsymbol{T}(t)dt=\tilde{\gamma}(s). \end{aligned} \quad (16)$$

This completes the proof.

Then we have the following theorem.

Theorem 3

Let $\tilde{\gamma}(v)$ be a Bertrand curve. Then, $\tilde{\gamma}'(v)$ lies on the constant slope surface.

Proof

By Lemma 1, taking the derivative of Equation 8 with respect to v , we obtain

$$\tilde{\gamma}'(v)=a\boldsymbol{f}(v)+a\tan\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v). \quad (17)$$

We can take that $a=u\sin\theta\cos\xi$ and then $a\tan\xi=u\sin\theta\sin\xi$, where $u, \; \theta$ are constant. Thus by Theorem 2, $\tilde{\gamma}'(v)$ is $u=$constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}(u,v)$ and $\tilde{\gamma}'(v)$ lies on it. This completes the proof.
We now show the relation between Bertrand curves and constant slope surfaces. We have the following theorem.

Theorem 4

Let $\tilde{\boldsymbol{f}} : S\rightarrow R^3$ be an isometric immersion of a surface S in the Euclidean 3-space and $\tilde{\boldsymbol{f}}(v)$ be $u=$ constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}(u,v)$. Then $\int_0^v \tilde{\boldsymbol{f}}(v)dv$ is a Bertrand curve.

Proof

By Equation 7, we get $\tilde{\boldsymbol{f}}(v)=u\sin\theta\cos\xi\boldsymbol{f}(v)+u\sin\theta\sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v)$ for $u=$constant. If we integrate $\tilde{\boldsymbol{f}}(v)$ then the equation is

$$\int_0^v \tilde{\boldsymbol{f}}(v)dv=u\sin\theta\cos\xi\int_0^v \boldsymbol{f}(v)dv+u\sin\theta\sin\xi\int_0^v \boldsymbol{f}(v)\times\boldsymbol{f}'(v)dv. \quad (18)$$

Since coefficients of $\boldsymbol{f}(v)$ and $\boldsymbol{f}(v)\times\boldsymbol{f}'(v)$ are constant, we can take that $a=u\sin\theta\cos\xi$ and then $a\tan\xi=u\sin\theta\sin\xi$. Therefore we obtain $\int_0^v \tilde{\boldsymbol{f}}(v)dv=a\int_0^v \boldsymbol{f}(v)dv+a\tan\xi\int_0^v \boldsymbol{f}(v)\times\boldsymbol{f}'(v)dv.$

By Lemma 1, $\int_0^v \tilde{\boldsymbol{f}}(v)dv$ is a Bertrand curve. This completes the proof.

We have the following parametrizations of constant slope surfaces.

Proposition 1

Let $\alpha: I \to R^3$ be a space curve parametrized by the arc-length s and $(\boldsymbol{T}): I \to S^2$ be spherical image of tangent indicatrix of space curve α. Then constant slope surface can be parametrized by

$$\tilde{\boldsymbol{f}}_{\boldsymbol{T}}(u,v) = u\sin\theta\cos\xi\boldsymbol{T}(v) + u\sin\theta\sin\xi\boldsymbol{B}(v), \text{ where } v = \int_0^s \|\boldsymbol{T}'(s)\| ds. \quad (19)$$

Proof

From Theorem 2, we have $\tilde{\boldsymbol{f}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{f}(v) + \sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v))$.

Since $(\boldsymbol{T}): I \to S^2$ is a spherical curve, we can take that $(\boldsymbol{T}) = \boldsymbol{f}(v)$. Thus

$$\tilde{\boldsymbol{f}}_{\boldsymbol{T}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{T}(v) + \sin\xi\boldsymbol{T}(v)\times\boldsymbol{T}'(v)) \quad (20)$$

and from the Frenet frame and the Frenet-Serret formulae, we get $\tilde{\boldsymbol{f}}_{\boldsymbol{T}}(u,v) = u\sin\theta\cos\xi\boldsymbol{T}(v) + u\sin\theta\sin\xi\boldsymbol{B}(v)$.

This completes the proof.

We have the following corollaries of Proposition 1.

Corollary 2

Let $\tilde{\boldsymbol{f}}(v)$ be the position vector of $u =$constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}_{\boldsymbol{T}}(u,v)$. Then we have the following relations:

$$<\tilde{\boldsymbol{f}}(v),\boldsymbol{T}(v)>=<\tilde{\boldsymbol{f}}(v),\boldsymbol{B}(v)>=\text{constant and } <\tilde{\boldsymbol{f}}(v),\boldsymbol{N}(v)>=0. \quad (21)$$

Corollary 3

Let $\tilde{\boldsymbol{f}}(v)$ be $u =$constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}_{\boldsymbol{T}}(u,v)$. Then

$$\int_0^v \tilde{\boldsymbol{f}}(v)dv = u\sin\theta\cos\xi\int_0^v \boldsymbol{T}(v)dv + u\sin\theta\sin\xi\int_0^v \boldsymbol{B}(v)dv \quad (22)$$

is a Bertrand curve.

In the following, we present analogue results in terms of spherical images of principal normal, binormal and the Darboux indicatrices of space curve α.

Proposition 2

Let $\alpha: I \to R^3$ be a space curve parametrized by the arc-length s and $(\boldsymbol{N}): I \to S^2$ be spherical image of principal normal indicatrix of space curve α. Then constant slope surface can be parametrized by

$$\tilde{\boldsymbol{f}}_{\boldsymbol{N}}(u,v) = u\sin\theta\cos\xi\boldsymbol{N}(v) + u\sin\theta\sin\xi\boldsymbol{C}(v), \text{ where } v = \int_0^s \|\boldsymbol{N}'(s)\| ds. \quad (23)$$

Proof

From Theorem 2, we have $\tilde{\boldsymbol{f}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{f}(v) + \sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v))$.

Since $(\boldsymbol{N}): I \to S^2$ is a spherical curve, we can take that $(\boldsymbol{N}) = \boldsymbol{f}(v)$. Thus

$$\tilde{\boldsymbol{f}}_{\boldsymbol{N}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{N}(v) + \sin\xi\boldsymbol{N}(v)\times\boldsymbol{N}'(v)) \quad (24)$$

and from $\boldsymbol{N}(v)\times\boldsymbol{N}'(v) = \boldsymbol{C}(v)$, we get $\tilde{\boldsymbol{f}}_{\boldsymbol{N}}(u,v) = u\sin\theta\cos\xi\boldsymbol{N}(v) + u\sin\theta\sin\xi\boldsymbol{C}(v)$.

This completes the proof.

We have the following corollaries of Proposition 2.

Corollary 4

Let $\tilde{\boldsymbol{f}}(v)$ be the position vector of $u =$constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}_{\boldsymbol{N}}(u,v)$. Then we have the following:

$$<\tilde{\boldsymbol{f}}(v),\boldsymbol{N}(v)>=\text{constant}. \quad (25)$$

Corollary 5

Let $\tilde{\boldsymbol{f}}(v)$ be $u =$constant parameter curve of constant slope surface $\tilde{\boldsymbol{f}}_{\boldsymbol{N}}(u,v)$. Then

$$\int_0^v \tilde{\boldsymbol{f}}(v)dv = u\sin\theta\cos\xi\int_0^v \boldsymbol{N}(v)dv + u\sin\theta\sin\xi\int_0^v \boldsymbol{C}(v)dv. \quad (26)$$

is a Bertrand curve.

Proposition 3

Let $\alpha : I \to R^3$ be a space curve parametrized by the arc-length s and $(\boldsymbol{B}) : I \to S^2$ be spherical image of binormal indicatrix of space curve α. Then constant slope surface can be parametrized by

$$\tilde{f}_{\boldsymbol{B}}(u,v) = u\sin\theta\cos\xi\boldsymbol{B}(v) + u\sin\theta\sin\xi\boldsymbol{T}(v), \text{ where } v = \int_0^s \|\boldsymbol{B}'(s)\| ds. \quad (27)$$

Proof

From Theorem 2, we have $\tilde{f}(u,v) = u\sin\theta(\cos\xi f(v) + \sin\xi f(v)\times f'(v))$. Since $(\boldsymbol{B}) : I \to S^2$ is a spherical curve, we can take that $(\boldsymbol{B}) = \boldsymbol{f}(v)$. Thus

$$\tilde{f}_{\boldsymbol{B}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{B}(v) + \sin\xi\boldsymbol{B}(v)\times\boldsymbol{B}'(v)) \quad (28)$$

and from the Frenet frame and the Frenet-Serret formulae, we get $\tilde{f}_{\boldsymbol{B}}(u,v) = u\sin\theta\cos\xi\boldsymbol{B}(v) + u\sin\theta\sin\xi\boldsymbol{T}(v)$.
This completes the proof.

We have the following corollaries of Proposition 3.

Corollary 6

Let $\tilde{f}(v)$ be the position vector of $u =$constant parameter curve of constant slope surface $\tilde{f}_{\boldsymbol{B}}(u,v)$. Then we have the following:

$$< \tilde{f}(v), \boldsymbol{T}(v) > = < \tilde{f}(v), \boldsymbol{B}(v) > = \text{constant and } < \tilde{f}(v), \boldsymbol{N}(v) > = 0. \quad (29)$$

Corollary 7

Let $\tilde{f}(v)$ be $u =$constant parameter curve of constant slope surface $\tilde{f}_{\boldsymbol{B}}(u,v)$. Then

$$\int_0^v \tilde{f}(v)dv = u\sin\theta\cos\xi\int_0^v \boldsymbol{B}(v)dv + u\sin\theta\sin\xi\int_0^v \boldsymbol{T}(v)dv. \quad (30)$$

is a Bertrand curve.

Proposition 4

Let $\alpha : I \to R^3$ be a space curve parametrized by the arc-length s and $(\boldsymbol{C}) : I \to S^2$ be the Darboux indicatrix of space curve α. Then constant slope surface can be parametrized by

$$\tilde{f}_C(u,v) = u\sin\theta\cos\xi\boldsymbol{C}(v) + \varepsilon u\sin\theta\sin\xi\boldsymbol{N}(v) \text{ where } \varepsilon = \pm 1,\ v = \int_0^s \|\boldsymbol{C}'(s)\| ds. \quad (31)$$

Proof

From Theorem 2, we have

$$\tilde{f}(u,v) = u\sin\theta(\cos\xi\boldsymbol{f}(v) + \sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v)).$$

Since $(\boldsymbol{C}) : I \to S^2$ is a spherical curve, we can take that $(\boldsymbol{C}) = \boldsymbol{f}(v)$. Thus we get

$$\tilde{f}_{\boldsymbol{C}}(u,v) = u\sin\theta(\cos\xi\boldsymbol{C}(v) + \sin\xi\boldsymbol{C}(v)\times\boldsymbol{C}'(v)) \quad (32)$$

and since $\boldsymbol{C} = \dfrac{\tau\boldsymbol{T} + \kappa\boldsymbol{B}}{\sqrt{\tau^2+\kappa^2}}$ and $\boldsymbol{C}' = \varepsilon\dfrac{-\kappa\boldsymbol{T} + \tau\boldsymbol{B}}{\sqrt{\tau^2+\kappa^2}}$, we obtain $\boldsymbol{C}\times\boldsymbol{C}' = \varepsilon\boldsymbol{N}$. Therefore we have $\tilde{f}_{\boldsymbol{C}}(u,v) = u\sin\theta\cos\xi\boldsymbol{C}(v) + \varepsilon u\sin\theta\sin\xi\boldsymbol{N}(v)$.
This completes the proof.
We have the following corollary of Proposition 4.

Corollary 8

Let $\tilde{f}(v)$ be $u =$constant parameter curve of constant slope surface $\tilde{f}_C(u,v)$. Then

$$\int_0^v \tilde{f}(v)dv = u\sin\theta\cos\xi\int_0^v \boldsymbol{C}(v)dv + \varepsilon u\sin\theta\sin\xi\int_0^v \boldsymbol{N}(v)dv \quad (33)$$

is a Bertrand curve.

We now give an example of constant slope surfaces and Bertrand curves and draw their pictures by using Mathematica.

Example

We consider a spherical curve $\boldsymbol{f}(v) = (\cos v, \sin v, 0)$. Then we have $\boldsymbol{f}(v)\times\boldsymbol{f}'(v) = (0,0,1)$. The constant slope surface is

$$\tilde{f}(u,v) = u\sin\theta(\cos\xi\boldsymbol{f}(v) + \sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v)) \text{ (Munteanu, 2010)},$$

thus we get

$$\tilde{f}(v) = u\sin\theta(\cos\xi\boldsymbol{f}(v) + \sin\xi\boldsymbol{f}(v)\times\boldsymbol{f}'(v)) \text{ for}$$

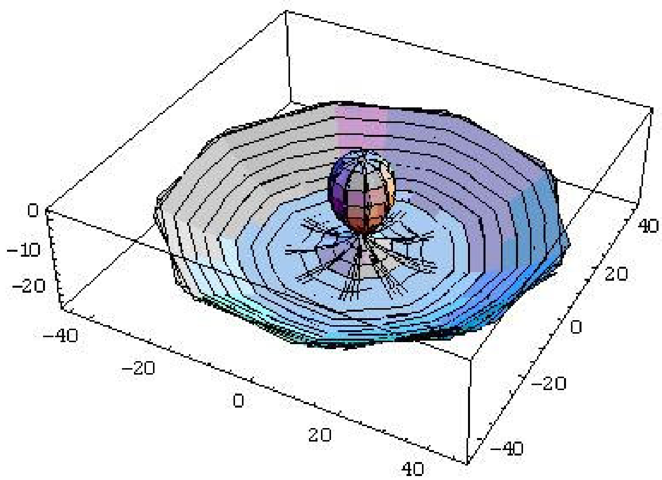

Figure 1. $\tilde{f}(u,v)$ constant slope surface (Munteanu, 2010).

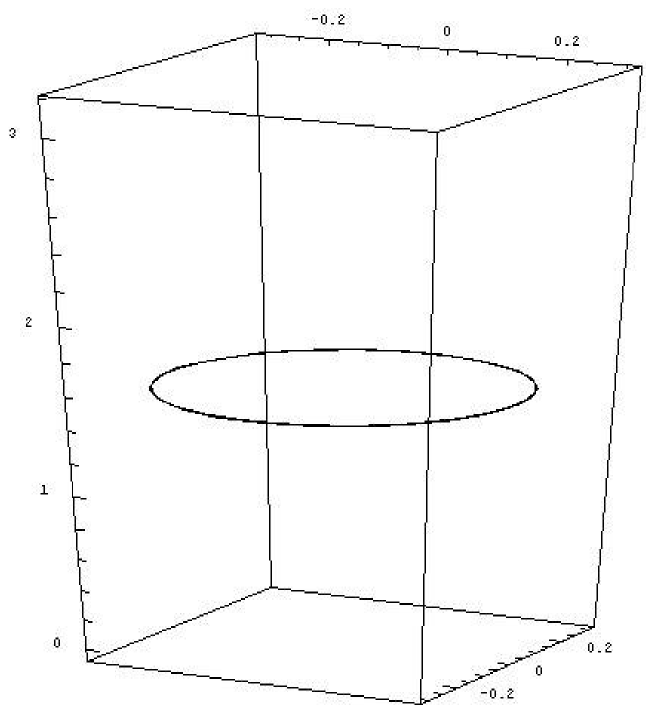

Figure 2. $\tilde{f}(v)$ $u =$ constant parameter curve of $\tilde{f}(u,v)$.

$u =$ constant. By using Theorem 4, we have the following Bertrand curve $\left(\theta = \frac{\pi}{5}, u = e, \xi = \xi(u) = \cot\frac{\pi}{5}\right)$:

$$\int_0^v \tilde{f}(v)dv = e\sin(\frac{\pi}{5})\cos(\cot\frac{\pi}{5})\int_0^v (\cos v, \sin v, 0)dv + e\sin(\frac{\pi}{5})\sin(\cot\frac{\pi}{5})\int_0^v (0,0,1)dv.$$

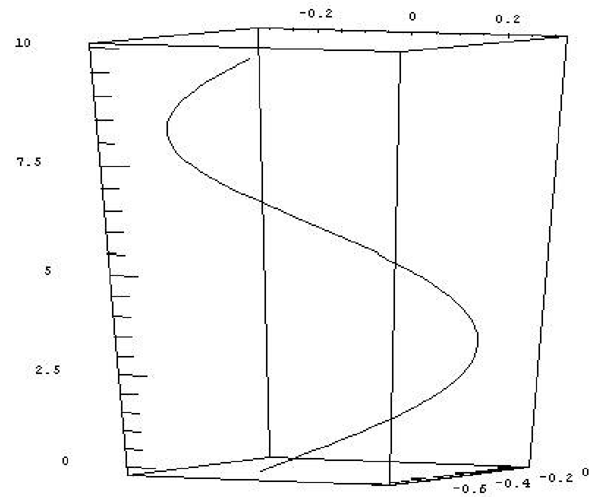

Figure 3. $\int_0^v \tilde{f}(v)dv$ Bertrand curve corresponding to $\tilde{f}(v)$.

We can draw pictures of $\tilde{f}(u,v)$, $\tilde{f}(v)$ and $\int_0^v \tilde{f}(v)dv$, respectively (Figures 1 to 3).

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REFERENCES

Boyadzhiev KN (2007). Equiangular Surfaces, Self-Similar Surfaces, and the Geometry of Seashells. Coll. Math. J., 38(4): 265-271.

Camci C, Ilarslan K, Kula L, Hacisalihoglu HH (2009). Harmonic curvatures and generalized helices in E^n. Chaos Soliton Fract., 40: 2590-2596.

Carmo MP (1976). Differential Geometry of Curves and Surfaces. Prentice-Hall, New Jersey.

El Naschie MS (2001). Notes on superstings and the infinite sums of Fibonacci and Lucas numbers. Chaos Soliton Fract. 12: 1937-1940.

El Naschie MS (2005). Experimental and theoretical arguments for the number and mass of the Higgs particle. Chaos Soliton Fract., 23: 1901-1908.

Falson S, Plaza A (2008). On the 3-dimensional k-Fibonacci spirals. Chaos Soliton Fract., 38: 993-1003.

Ilarslan K, Boyacioglu O (2007). Position vectors of a spacelike W-curve in Minkowski space E_1^3. . Bull. Korean Math. Soc., 44: 429-438.

Izumiya S, Takeuchi N (2002). Generic properties of helices and Bertrand curves. J. Geom., **7**4: 97-109.

Izumiya S, Takeuchi N (2003). Special Curves and Ruled Surfaces. Contrib. Algebra Geom., 44: 203-212.

Izumiya S, Takeuchi N (2004). New special curves and developable surfaces. Turk. J. Math., 28: 153-163.

Koenderink J (1990). Solid shape. MIT Press, Cambridge, MA.

Kula L, Yayli Y (2005). On slant helix and its spherical indicatrix. Appl. Math. Comput., 169: 600-607.

Monterde J (2009). Salkowski curves revisted: A family of curves with constant curvature and non-constant torsion. Comput. Aided Geom. D., 26: 271-278.

Munteanu MI (2010). From Golden Spirals to Constant Slope Surfaces. J. Math. Phys., 51: 1-9.

Nutbourne AW, Martin RR (1988). Differential Geometry Applied to Design of Curves and Surfaces. Ellis Horwood, Chichester, UK.

Struik DJ (1988). Lectures on Classical Differential Geometry. New-York, Dover, pp 23-35.

Toledo-Suarez CD (2009). On the arithmetic of fractal dimension using hyperhelices. Chaos Soliton Fract., 39: 342-349.

Watson JD, Crick FH (1953). Molecular structures of nucleic acids. Nature, 171: 737-738.