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Full Length Research Paper

A study of fuzzy Abel-Grassmann's groupoids

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Fuzzy sets were first introduced by Zadeh in 1965 which have a wide range of applications in various fields of Engineering Sciences, Computer Science and Management Sciences. It is worth mentioning here that for applications of fuzzy sets mostly associative algebraic structures are used such as in 2003, Mordeson, Malik and Kuroki discovered the applications of fuzzy semigroups in fuzzy coding, fuzzy finite-state machines and fuzzy languages. Our aim in this paper is to apply fuzzy sets to non-associative algebraic structures and to develop some new related properties. Specifically we have characterized intra-regular Abel-Grassmann's groupoids using generalized fuzzy bi-ideals.

Key words: Abel-Grassman's groupoid (AG-groupoid), intra-regular, fuzzy sets, fuzzy AG-groupoids, $\left(\in,\in V_q\right)$ -fuzzy ideals.

INTRODUCTION

Fuzzy set theory and its applications in several branches of Science are growing day by day. Since pacific models of real world problems in various fields such as computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many others, may not be constructed because we are unfortunately uncertain in many occasions. For handling such difficulties, we need some natural tools such as probability theory and theory of fuzzy sets (Zadeh, 1965) which have already been developed. Moreover fuzzy sets are closely connected with some other soft computing models such as rough sets, random sets and soft sets (Feng et al., 2010, 2011). Associative algebraic structures are mostly used for applications of fuzzy sets. Mordeson et al. (2003) have discovered the vast field of fuzzy semigroups, where theoretical exploration of fuzzy semi-groups and their applications are used in fuzzy coding, fuzzy finite-state machines and fuzzy languages. The use of fuzzification in automata and formal language has widely been explored. Moreover the complete l-semi-groups have wide range of applications in the theories of automata, formal languages and programming. It is worth mentioning that some recent investigations of l-semigroups are closely connected with algebraic logic and non-classical logics. In this paper we introduced a new class of a non-associative algebraic structure (Abel-Grassman's groupoid, AG-groupoid) namely intra-regular AG-groupoid. An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly, it works like a commutative semigroup for instance $a^2b^2=b^2a^2$, for all a,b holds in a commutative semi-group while this equation also holds for an AG-groupoid with left identity. Moreover, if an AG-groupoid contains left identity e , then $ab=(ba)e$ for any subset $\{a,b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for intra-regular AG-groupoids using the new generalized concept of fuzzy sets.

FUZZY AG-GROUPOIDS

A groupoid is called an AG-groupoid if it satisfies the left invertive law, that is, $(ab)c=(cb)a$. Every AG-groupoid satisfies the medial law $(ab)(cd)=(ac)(bd)$. It is basically a non-associative algebraic structure in between a groupoid and a commutative semigroup. It is important to mention here that if an AG-groupoid contains identity or even right identity, then it becomes a commutative monoid. An AG-groupoid is not necessarily contains a left

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identity and if it contains a left identity then it is unique (Mushtaq and Yousuf, 1978). An AG-groupoid S with left identity satisfies the paramedial law, that is, $(ab)(cd)=(db)(ca)$ and $S=S^2$. Moreover S satisfies the following law

$$a(bc)=b(ac),\ \text{for all}\ a,b,c,d\in S. \qquad (1)$$

Let S be an AG-groupoid. By an AG-subgroupoid of S, we means a non-empty subset A of S such that $A^2\subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S if $SA\subseteq A$ $(AS\subseteq A)$ and it is called a two-sided ideal if it is both left and a right ideal of S. A non-empty subset A of an AG-groupoid S is called quasi-ideal of S if $SQ\cap QS\subseteq Q$. A non-empty subset A of an AG-groupoid S is called a generalized bi-ideal of S if $(AS)A\subseteq A$ and an AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A\subseteq A$. A non-empty subset A of an AG-groupoid S is called an interior ideal of S if $(SA)S\subseteq A$.

If S is an AG-groupoid with left identity e then $S=S^2$ and $Sa=\{sa\ :\ s\in S\}$ is both left ideal and bi ideal of S.

The subsequently given definitions are available in Mordeson et al. (2003).

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy)\geq f(x)\wedge f(y)$ for all x, $y\in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy)\geq f(y)$ $(f(xy)\geq f(x))$ for all x, $y\in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy two-sided ideal of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of an AG-groupoid S is called a fuzzy quasi-ideal of S if $f\circ S\cap S\circ f\subseteq f$. A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xa)y)\geq f(x)\wedge f(y)$, for all x, a and $y\in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy bi-ideal of S if $f((xa)y)\geq f(x)\wedge f(y)$, for all x, a and $y\in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy interior ideal of S if $f((xa)y)\geq f(a)$, for all x, a and $y\in S$.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f\circ g$ is defined by,

$$(f\circ g)(a)=\begin{cases}\bigvee_{a=bc}\{f(b)\wedge g(c)\}, & \text{if there exist } b,c\in S, \text{ such that } a=bc.\\ 0, & \text{otherwise.}\end{cases}$$

The symbols $f\cap g$ and $f\cup g$ will means the following fuzzy subsets of S

$(f\cap g)(x)=\min\{f(x),\ g(x)\}=f(x)\wedge g(x),$ for all x in S

and

$(f\cup g)(x)=\max\{f(x),\ g(x)\}=f(x)\vee g(x),$ for all x in S.

Let f be a fuzzy subset of an AG-groupoid S and $t\in(0,1]$. Then $x_t\in f$ means $f(x)\geq t$; x_tqf means $f(x)+t>1$; $x_t\alpha\vee\beta f$ means $x_t\alpha f$ or $x_t\beta f$; where α,β denotes any one of $\in$, q, $\in\vee q$, $\in\wedge q$. $x_t\alpha\wedge\beta f$ means $x_t\alpha f$ and $x_t\beta f$; $x_t\overline{\alpha}f$ means $x_t\alpha f$ does not holds.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f\circ_{0.5}g$ is defined by,

$$(f\circ_{0.5}g)(a)=\begin{cases}\bigvee_{a=bc}\{f(b)\wedge g(c)\wedge 0.5\}, & \text{if there exist } b,c\in S, \text{ such that } a=bc.\\ 0, & \text{otherwise.}\end{cases}$$

The following definitions for AG-groupoids are same as for semigroups in Shabir et al. (2010).

Definition 1

A fuzzy subset δ of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy AG-subgroupoid of S if for all $x,y\in S$ and $t,r\in(0,1]$, it satisfies, $x_t\in\delta$, $y_r\in\delta$ implies that $(xy)_{\min\{t,r\}}\in\vee q\delta$.

Definition 2

A fuzzy subset δ of S is called an $(\in,\in\vee q)$ -fuzzy left (right) ideal of S if for all $x,y\in S$ and $t,r\in(0,1]$, it satisfies, $x_t\in\delta$ implies $(yx)_t\in\vee q\delta$ $\left(x_t\in\delta\right.$ implies $\left.(xy)_t\in\vee q\delta\right)$.

Definition 3

A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy interior ideal of S if for all $x,y,z\in S$ and $t,r\in(0,1]$ the following condition holds. $y_t\in f$ implies $((xy)z)_t\in\vee qf$.

Definition 4

A fuzzy subset f of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy quasi-ideal of S if it satisfies, $f(x)\geq\min(f\circ C_S(x),C_S\circ f(x),0.5)$, where C_S is the fuzzy subset of S mapping every element of S on 1 .

Definition 5

A fuzzy subset f of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy generalized bi-ideal of S if $x_t\in f$ and $z_r\in S$ implies $((xy)z)_{\min\{t,r\}}\in\vee qf$, for all $x,y,z\in S$ and $t,r\in(0,1]$.

Definition 6

A fuzzy subset f of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy bi-ideal of S if for all $x,y,z\in S$ and $t,r\in(0,1]$, the following conditions hold

(i) If $x_t\in f$ and $y_r\in S$ implies $(xy)_{\min\{t,r\}}\in\vee qf$,

(ii) If $x_t\in f$ and $z_r\in S$ implies $((xy)z)_{\min\{t,r\}}\in\vee qf$

The proofs of the following four theorems are same as in Shabir et al. (2010).

Theorem 1

Let δ be a fuzzy subset of S . Then δ is an $(\in,\in\vee q)$ -fuzzy AG-subgroupoid of S if and only if $\delta(xy)\geq\min\{\delta(x),\delta(y),0.5\}$.

Theorem 2

A fuzzy subset δ of an AG-groupoid S is called an $(\in,\in\vee q)$ -fuzzy left (right) ideal of S if and only if $\delta(xy)\geq\min\{\delta(y),0.5\}\left(\delta(xy)\geq\min\{\delta(x),0.5\}\right)$.

Theorem 3

A fuzzy subset f of an AG-groupoid S is an $(\in,\in\vee q)$ -fuzzy interior ideal of S if and only if it satisfies the following conditions.

(i) $f(xy)\geq\min\{f(x),f(y),0.5\}$ for all $x,y\in S$.

(ii) $f((xy)z)\geq\min\{f(y),0.5\}$ for all $x,y,z\in S$.

Theorem 4

Let f be a fuzzy subset of S . Then f is an $(\in,\in\vee q)$ -fuzzy bi-ideal of S if and only if

(i) $f(xy)\geq\min\{f(x),f(y),0.5\}$ for all $x,y\in S$,

(ii)) $f((xy)z)\geq\min\{f(x),f(z),0.5\}$ for all $x,y,z\in S$.

Here we begin with examples of AG-groupoids.

Example 1

Let us consider an AG-groupoid $S=\{1,2,3\}$ in the following multiplication table.

.	1	2	3
1	3	2	3
2	2	2	2
3	2	2	2

Clearly S is non-commutative and non-associative, because $1.3\neq3.1$ and $(1.1).3\neq1.(1.3)$. Note that S has no left identity. Define a fuzzy subset $F\ :\ S\rightarrow[0,1]$ as follows:

$$F(x)=\begin{cases}0.8 \text{ for } x=1\\0.7 \text{ for } x=2\\0.6 \text{ for } x=3\end{cases}$$

Then clearly F is an ($\in,\in\vee q)$ -fuzzy ideal of S .

Lemma 1

Intersection of two ideals of an AG-groupoid with left identity is either empty or an ideal.

Proof

It is easy.

Generalized Fuzzy bi-ideals of an intra-regular AG-groupoid

An element a of an AG-groupoid S is called intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called intra-regular, if every element of S is intra-regular.

Example 2

Let $S = \{a, b, c, d, e\}$, and the binary operation " $\cdot$ " be defined on S as follows:

$\cdot$	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	5	6	3	4
4	1	1	4	5	6	3
5	1	1	3	4	5	6
6	1	1	6	3	4	5

Then clearly $(S, \cdot)$ is an AG-groupoid. Also $1 = (2 \cdot 1^2) \cdot 3$, $2 = (2 \cdot 2^2) \cdot 1$, $3 = (4 \cdot 3^2) \cdot 4$, $4 = (3 \cdot 4^2) \cdot 6$, $5 = (4 \cdot 5^2) \cdot 6$ and $6 = (3 \cdot 6^2) \cdot 4$. Therefore $(S, \cdot)$ is an intra-regular AG-groupoid. Define a fuzzy subset $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.8 \text{ for } x = 2 \\ 0.8 \text{ for } x = 3 \\ 0.6 \text{ for } x = 4 \\ 0.5 \text{ for } x = 5 \\ 0.5 \text{ for } x = 6 \end{cases}$$

Then clearly f is an $(\in, \in \vee q)$ -fuzzy bi-ideal of S .

Theorem 5

In an intra-regular AG-groupoid S with left identity the following are equivalent.

(i) A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy right ideal
(ii) A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy left ideal
(iii) A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy bi-ideal
(iv) A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy interior ideal
(v) A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy quasi-ideal.

Proof

It is easy.

Definition 7

Let f and g be fuzzy subsets of an AG-groupoid S . We define the fuzzy subsets $f_{0.5}$, $f \wedge_{0.5} g$ and $f \circ_{0.5} g$ of S as follows,

(1) $f_{0.5}(a) = f(a) \wedge 0.5$.
(ii) $(f \wedge_{0.5} g)(a) = (f \wedge g)(a) \wedge 0.5$.
(iii) $(f \circ_{0.5} g)(a) = (f \circ g)(a) \wedge 0.5$, for all $a \in S$.

Definition 8

Let A be any subset of an AG-groupoid S , then the characteristic function $(C_A)_{0.5}$ is defined as,

$$(C_A)_{0.5}(a) = \begin{cases} 0.5 \text{ if } a \in A \\ 0 \quad \text{otherwise.} \end{cases}$$

Lemma 2

The following properties hold in an AG-groupoid S.

(i) A is an AG-subgroupoid of S if and only if $(C_A)_{0.5}$ is an $(\in, \in \vee q)$ -fuzzy AG-subgroupoid of S .
(ii) A is a left (right, two-sided) ideal of S if and only if $(C_A)_{0.5}$ is an $(\in, \in \vee q)$ -fuzzy left (right, two-sided) ideal of S .

(iii) A is left (bi) ideal of an AG-groupoid S if and only if $(C_A)_{0.5}$ is $(\in,\in\vee q)$ -fuzzy left (bi)-ideal.

(iv) For any non-empty subsets A and B of S, $C_A \circ_{0.5} C_B = (C_{AB})_{0.5}$ and $C_A \wedge_{0.5} C_B = (C_{A\cap B})_{0.5}$.

Proof

It is same as in Shabir et al. (2010).

Lemma 3

Every left ideal in an AG-groupoid with left identity is a bi-ideal.

Proof

Let B any left ideal of an AG-groupoid S with left identity e . Then using paramedial and medial laws, we get $(BS)B=(BS)(eB)\subseteq(BS)(SS)=(SS)(SB)=S(SB)\subseteq SB\subseteq B.$

Hence B is a bi-ideal.

Lemma 4

Every $(\in,\in\vee q)$ -fuzzy left ideal of an intra-regular AG-groupoid with left identity is an $(\in,\in\vee q)$ -fuzzy bi-ideal of S.

Proof

Let f be an $(\in,\in\vee q)$ -fuzzy left ideal of an intra-regular AG-groupoid S with left identity. Now for a,x,b in S , we have $(ax)b=(bx)a$. Then $f((ax)b)\geq f(b)\wedge 0.5$ and $f((bx)a)\geq f(a)\wedge 0.5$. Combining both, we got $f((ax)b)\geq f(b)\wedge 0.5\wedge f(a)\wedge 0.5 = f(a)\wedge f(b)\wedge 0.5$

Theorem 6

For an AG-groupoid S with left identity, the following are equivalent.

(i) S is intra-regular

(ii) $I\cap J = IJ$, for all bi-ideals I and J .

(iii) $f\wedge_{0.5} g = f\circ_{0.5} g$, for all $(\in,\in\vee q)$ -fuzzy bi-ideals f and g .

(iv) $f\wedge_{0.5} g = f\circ_{0.5} g$, for all $(\in,\in\vee q)$ -fuzzy generalized bi-ideals f and g .

Proof

$(i)\Rightarrow(iv)$ Let f and g be $(\in,\in\vee q)$ -fuzzy generalized bi-ideals of an intra-regular AG-groupoid S with left identity. Then by theorem 5, f and g become $(\in,\in\vee q)$ -fuzzy ideals of S . For each a in S there exist x,y in S such that $a=(xa^2)y$. Now using (1) and left invertive law, we get $a=(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a.$ Then,

$$\begin{aligned}(f\circ_{0.5} g)(a) &= \bigvee_{a=pq}\{f(p)\wedge g(q)\wedge 0.5\}\\ &= \bigvee_{a=(y(xa))a}\{f(y(xa))\wedge g(a)\wedge 0.5\}\\ &\geq f(a)\wedge g(a)\wedge 0.5 = f\wedge_{0.5} g(a)\end{aligned}$$

Therefore $f\circ_{0.5} g\geq f\wedge_{0.5} g$. Moreover it is obvious to see that $f\circ_{0.5} g\leq f\wedge_{0.5} g$. Hence $f\circ_{0.5} g(a) = f\wedge_{0.5} g(a).$

$(iv)\Rightarrow(iii)$ is obvious.

$(iii)\Rightarrow(ii)$

Let I and J be the bi-ideals of an AG-groupoid S with left identity. Now by hypothesis and lemma 2, we get

$$\begin{aligned}(C_{IJ})_{0.5}(a) &= (C_I\circ_{0.5} C_J)(a) = (C_I\wedge_{0.5} C_J)(a)\\ &= (C_{I\cap J})_{0.5}(a)\geq 0.5.\end{aligned}$$

Thus $IJ = I\cap J$.

$(ii)\Rightarrow(i)$.

Since Sa is a bi-ideal of an AG-groupoid S with left identity containing a . Therefore using medial law, paramedial law and (1) we get

$$\begin{aligned}a\in Sa\cap Sa &= (Sa)(Sa) = (SS)(aa) = (aa)(SS) = a^2(SS)\\ &= S(a^2S) = (SS)(a^2S) = (Sa^2)(SS) = (Sa^2)S.\end{aligned}$$

Theorem 7

For an AG-groupoid S with left identity, the following are equivalent,

(i) S is intra-regular

(ii) $B \cap L = BL \quad (B \cap L \subseteq BL)$, for every bi-ideal B and left ideal L .

(iii) $f \wedge_{0.5} g = f \circ_{0.5} g \quad \left(f \wedge_{0.5} g \leq f \circ_{0.5} g\right)$, where f is any $(\in, \in \vee q)$ fuzzy bi-ideal, g is any $(\in, \in \vee q)$ fuzzy left ideal.

(iv) $f \wedge_{0.5} g = f \circ_{0.5} g \quad \left(f \wedge_{0.5} g \leq f \circ_{0.5} g\right)$, where f and g are any $(\in, \in \vee q)$ fuzzy bi-ideals.

(v) $f \wedge_{0.5} g = f \circ_{0.5} g \quad \left(f \wedge_{0.5} g \leq f \circ_{0.5} g\right)$, where f and g are any $(\in, \in \vee q)$ fuzzy generalized bi-ideals.

Proof

$(i) \Rightarrow (v)$. Let f and g be $(\in, \in \vee q)$ -fuzzy generalized bi-ideals of S . Then by theorem 5, f and g become $(\in, \in \vee q)$ -fuzzy ideals of S . Now for each a in S , we have $a = (y(xa))a$. Therefore;

$$\begin{aligned}
\left(f \circ_{0.5} g\right)(a) &= \bigvee_{a=pq} \left\{f(p) \wedge g(q) \wedge 0.5\right\} \\
&\geq f\left(y(xa)\right) \wedge g(a) \wedge 0.5 \\
&\geq f(a) \wedge g(a) \wedge 0.5 = f \wedge_{0.5} g(a)
\end{aligned}$$

Thus $f \circ_{0.5} g \geq f \wedge_{0.5} g$ and obviously $f \circ_{0.5} g \leq f \wedge_{0.5} g$. Hence $f \circ_{0.5} g(a) = f \wedge_{0.5} g(a).$

$(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (ii)$.

By hypothesis and lemma 2, we get

$$\begin{aligned}
(C_{BL})_{0.5}(a) &= (C_B \circ_{0.5} C_L)(a) = (C_B \wedge_{0.5} C_L)(a) \\
&= \left(C_{B \cap L}\right)_{0.5}(a) \geq 0.5.
\end{aligned}$$

Thus $BL = B \cap L$.

$(ii) \Rightarrow (i)$

Obviously $a \in Sa \cap Sa$. Now using (ii), we get $a \in Sa \cap Sa = (Sa)(Sa) = Sa^2 = (Sa^2)S.$ Hence S is intra-regular.

Lemma 5

If I is an ideal of an intra-regular AG-groupoid S with left identity, then $I = I^2$.

Proof

It is easy.

Theorem 8

For an intra-regular AG-groupoid S with left identity the following statements are equivalent (Madad and Naveed, 2010);

(i) A is a left ideal of S .

(ii) A is a right ideal of S .

(iii) A is an ideal of S .

(iv) A is a bi-ideal of S .

(v) A is a generalized bi-ideal of S .

(vi) A is an interior ideal of S .

(vii) A is a quasi-ideal of S .

(viii) $AS = A$ and $SA = A$.

Theorem 9

Let S be an AG-groupoid with left identity. Then the following are equivalent;

(i) S is intra-regular.

(ii) $A \cap B = AB$, for every left ideal A and every bi-ideal B of S .

(iii) $A \cap B = AB$, for every bi-ideal A and B of S .

(iv) $A \cap B = AB$, for every generalized bi-ideal A and B of S .

Proof

$(i) \Rightarrow (iv)$ Let A and B be generalized bi-ideals of S . Then by theorem 8, A, B are ideals of S . It is easy to see that $AB \subseteq A \cap B$. Now using lemma 5, we get $A \cap B = (A \cap B)^2 \subseteq AB$.

$(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$

It is same as the $(ii) \Rightarrow (i)$ of Theorem 6.

Theorem 10

If S is an AG-groupoid with left identity. Then the following are equivalent

(i) S is intra-regular
(ii) $f \wedge_{0.5} g = f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy left ideal f and every $(\in,\in \vee q)$ -fuzzy bi-ideal g of S.
(iii) $f \wedge_{0.5} g = f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy bi-ideals f and g of S .
(iv) $f \wedge_{0.5} g = f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy generalized bi-ideals f and g of S .

Proof

$(i) \Rightarrow (iv)$ Let f and g be $(\in,\in \vee q)$ -fuzzy generalized bi-ideals of S . Then by theorem 5, f and g become $(\in,\in \vee q)$ -fuzzy ideals of S . Now

$$\begin{aligned} f \circ_{0.5} g(a) &= \bigvee_{a=bc} f(b) \wedge g(c) \wedge 0.5 \\ &= \bigvee_{a=bc} (f(b) \wedge 0.5) \wedge (g(c) \wedge 0.5) \wedge 0.5 \\ &\leq \bigvee_{a=bc} f(bc) \wedge g(bc) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 = f \wedge_{0.5} g(a). \end{aligned}$$

Therefore $f \circ_{0.5} g \leq f \wedge_{0.5} g$. Also

$$\begin{aligned} (f \circ_{0.5} g)(a) &= \bigvee_{a=pq} \{f(y(xa)) \wedge g(a) \wedge 0.5\} \\ &= \bigvee_{a=a=(y(xa))a} \{f(y(xa)) \wedge g(a) \wedge 0.5\} \\ &\geq f(y(xa)) \wedge g(a) \wedge 0.5 \\ &\geq f(a) \wedge g(a) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 = (f \wedge_{0.5} g)(a) \end{aligned}$$

Therefore $f \circ_{0.5} g \geq f \wedge_{0.5} g$. Hence $f \circ_{0.5} g = f \wedge_{0.5} g$.
$(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.
$(ii) \Rightarrow (i)$

Let A be left and B be a bi-ideal of S . Then by lemma 2, C_A and C_B are $(\in,\in \vee q)$ -fuzzy left and $(\in,\in \vee q)$ -fuzzy bi-ideals of S and by (ii) $C_A \wedge_{0.5} C_B = C_B \circ_{0.5} C_A$. Now using lemma 2, we get $(C_{A\cap B})_{0.5} = C_A \wedge_{0.5} C_B = C_A \circ_{0.5} C_B = (C_{AB})_{0.5}$.

Therefore $A \cap B = AB$. Hence by theorem 9, S is intra-regular.

Theorem 11

For an AG-groupoid with left identity, the following conditions are equivalent.

(i) S is intra-regular.
(ii) $A \cap B \subseteq AB$, for every bi-ideal A and left ideal B of S .
(iii) $f \wedge_{0.5} g \leq f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy bi-ideal f and every $(\in,\in \vee q)$ -fuzzy left ideal g .
(iv) $f \wedge_{0.5} g \leq f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy bi-ideals f and g .
(v) $f \wedge_{0.5} g \leq f \circ_{0.5} g$, for every $(\in,\in \vee q)$ -fuzzy generalized bi-ideals f and g .

Proof

$(i) \Rightarrow (iv)$ Let f and g be $(\in,\in \vee q)$ -fuzzy generalized bi-ideals of S. Then by Theorem 5, f and g become $(\in,\in \vee q)$ -fuzzy ideals of S . Now since S is intra-regular. Therefore for $a \in S$ there exist x, y in S such that $a = (xa^2)y$ which yields that $a = (a(vu)(ax)$. Then

$$\begin{aligned} (f \circ_{0.5} g)(a) &= \bigvee_{a=pq} \{f(p) \wedge g(q) \wedge 0.5\} = \bigvee_{a=(a(vu)(ax)} \{f(a(vu) \wedge g(ax) \wedge 0.5\} \\ &\geq f(a(vu) \wedge g(ax) \wedge 0.5 = f(a) \wedge g(a) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 = (f \wedge_{0.5} g)(a). \end{aligned}$$

Thus $f \circ_{0.5} g \geq f \wedge_{0.5} g$.
$(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (ii)$

Let A and B be bi and left ideals of S . Then by Lemma 2, C_A and C_B are $(\in,\in \vee q)$ -fuzzy quasi and $(\in,\in \vee q)$ -fuzzy left ideals of S and by (ii) , we get $(C_{A\cap B})_{0.5} = C_A \wedge_{0.5} C_B \leq C_A \circ_{0.5} C_B = (C_{AB})_{0.5}$.

Therefore $A \cap B \subseteq AB$.
$(ii) \Rightarrow (i)$
It is same as the $(ii) \Rightarrow (i)$ of Theorem 6.

Theorem 12

For an AG-groupoid S with left identity, the following conditions are equivalent.

(i) S is intra-regular.
(ii) $(f \wedge_{0.5} g) \wedge_{0.5} h \leq (f \circ_{0.5} g) \circ_{0.5} h$, for all $(\in, \in \vee q)$-fuzzy bi-ideals f, g and $(\in, \in \vee q)$-fuzzy left ideal h of S.
(iii) $(f \wedge_{0.5} g) \wedge_{0.5} h \leq (f \circ_{0.5} g) \circ_{0.5} h$, for all $(\in, \in \vee q)$-fuzzy bi-ideals f, g and h of S.
(iv) $(f \wedge_{0.5} g) \wedge_{0.5} h \leq (f \circ_{0.5} g) \circ_{0.5} h$, for all $(\in, \in \vee q)$-fuzzy generalized bi-ideals f, g and h of S.

Proof

$(i) \Rightarrow (iv)$ Let f and g be $(\in, \in \vee q)$-fuzzy generalized bi-ideals and h be an $(\in, \in \vee q)$-fuzzy left ideal of S. Now using (1), left invertive law, paramedial and medial laws, we get

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((xa^2)(xy^2))a \\ &= ((y^2x)(a^2x))a = (a^2((y^2x)x))a = ((a(y^2x))(ax))a. \end{aligned}$$

Then

$$\begin{aligned} ((f \circ_{0.5} g) \circ_{0.5} h)(a) &= \bigvee_{a=pq} (f \circ_{0.5} g)(p) \wedge h(q) \wedge 0.5 \\ &= \bigvee_{a=((a(y^2x))(ax))a} (f \circ_{0.5} g)((a(y^2x))(ax)) \wedge h(a) \wedge 0.5 \\ &\geq (f \circ_{0.5} g)((a(y^2x))(ax)) \wedge h(a) \wedge 0.5 \\ &= \bigvee_{(a(y^2x))(ax)=cd} f(c) \wedge g(d) \wedge h(a) \wedge 0.5 \\ &\geq f(a(y^2x)) \wedge g(ax) \wedge h(a) \wedge 0.5 \\ &\geq f(a) \wedge g(a) \wedge h(a) \wedge 0.5 \\ &= (f \wedge_{0.5} g \wedge_{0.5} h)(a). \end{aligned}$$

$(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.
$(ii) \Rightarrow (i)$

Let f and g be $(\in, \in \vee q)$-fuzzy bi and $(\in, \in \vee q)$-fuzzy left ideals of S. Then

$$\begin{aligned} ((C_S \wedge_{0.5} f) \wedge_{0.5} g)(a) &= (C_S \wedge_{0.5} f)(a) \wedge g(a) \wedge 0.5 \\ &= C_S(a) \wedge f(a) \wedge 0.5 \wedge g(a) \wedge 0.5 \\ &= 1 \wedge f(a) \wedge g(a) \wedge 0.5 = f \wedge_{0.5} g(a) \end{aligned}$$

Therefore $((C_S \wedge_{0.5} f) \wedge_{0.5} g) = f \wedge_{0.5} g$. Also

$$\begin{aligned} C_S \circ_{0.5} f(a) &= \bigvee_{a=pq} C_S(p) \wedge f(q) \wedge 0.5 \\ &= \bigvee_{a=ea} C_S(e) \wedge f(a) \wedge 0.5 \\ &= f(a) \wedge 0.5 \leq f(a). \end{aligned}$$

Thus $C_S \circ_{0.5} g \leq f$. Now using (ii), we get $(f \wedge_{0.5} g)(a) = ((C_S \wedge_{0.5} f) \wedge_{0.5} g)(a) \leq ((C_S \circ_{0.5} f) \circ_{0.5} g)(a) \leq f \circ_{0.5} g(a).$

Therefore by Theorem 11, S is intra-regular.

Theorem 13

For an AG-groupoid with left identity, the following conditions are equivalent;

(i) S is intra-regular.
(ii) $B \cap L \subseteq LB$, for every bi-ideal B and left ideal L of S.
(iii) $f \wedge_{0.5} g \leq g \circ_{0.5} f$, for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g.
(iv) $f \wedge_{0.5} g \leq g \circ_{0.5} f$, for every $(\in, \in \vee q)$-fuzzy bi-ideals f and g.
(v) $f \wedge_{0.5} g \leq g \circ_{0.5} f$, for every $(\in, \in \vee q)$-fuzzy generalized bi-ideals f and g.

Proof

$(i) \Rightarrow (v)$ Let f and g be $(\in, \in \vee q)$-fuzzy generalized bi-ideals of S. Then by Theorem 5, f and g become $(\in, \in \vee q)$-fuzzy ideals of S. Now

$$\begin{aligned} (g \circ_{0.5} f)(a) &= \bigvee_{a=pq} \{g(p) \wedge f(q) \wedge 0.5\} = \bigvee_{a=(y(xa))a} \{g(y(xa)) \wedge f(a) \wedge 0.5\} \\ &\geq g(y(xa)) \wedge f(a) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 = (f \wedge_{0.5} g)(a). \end{aligned}$$

Thus $g \circ_{0.5} f \geq f \wedge_{0.5} g$.
$(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are obvious.
$(iii) \Rightarrow (ii)$

Let B and L be bi and left ideals of S. Then by lemma 2, C_A and C_B are $(\in, \in \vee q)$-fuzzy bi and $(\in, \in \vee q)$-fuzzy left ideals of S and by (ii), we get $(C_{B \cap L})_{0.5} = C_B \wedge_{0.5} C_L \leq C_L \circ_{0.5} C_B = (C_{LB})_{0.5}$.

Therefore $B \cap L \subseteq LB$.

$(ii) \Rightarrow (i)$

It is same as the $(ii) \Rightarrow (i)$ of Theorem 6.

Conclusions

In this paper, we have characterized intra-regular AG-groupoids using both classical bi-ideals and generalized fuzzy bi-ideals. In our future work, we will concentrate on other type of $(\in, \in \vee q)$-fuzzy ideals.

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