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Mannheim partner curves in the special linear group

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In this study, we state some basic results about the geometry of the special linear group SL(2,R), seen as a subset of $\mathbb{R}^4_2$, in terms of the left invariant fields, such as bracketing, Levi Civita connection ∇ and Riemann curvature tensor R, we give some basic theorems for Mannheim partner curves in the special linear group. We also find the relations between the curvatures and torsions of these associated curves and we give necessary and sufficient conditions for a given curve to be a Mannheim partner curve of another given curve through a relation between its curvature and torsion.

Key words: Special linear group, Mannheim curves, Mannheim, partner curves.

INTRODUCTION

Over the years, many mathematicians have studied surfaces isometrically immersed and special types of curves in tridimensional spaces; in particular, in the study the fundamental theory of space curves, a way to characterize or classify these curves is sought. The interesting classes of space curves are those that are related by some geometric property generating interesting facts and important problems to be addressed.

There are some examples of curves such as Bertrand and Natural mate curve whose the Frenet frame satisfy some geometric conditions. Other types of associated curves are Mannheim curves, discovered by Mannheim (1878): “if the normal vector field of a given curve coincides with the bi-normal vector field of another curve, at corresponding points, we say that the curve is a Mannheim curve and the pair of curves is called Mannheim pairs”.

Blum (1966) gives some theorems to Mannheim curves in $\mathbb{E}^3$ by means of the Riccati equations. In Liu and Wang (2008) study, Mannheim curves in 3-Euclidean space and in the 3-Minkowski space presenting a relation between their curvatures and torsions. After these papers several studies were performed on Mannheim curves with additional conditions and in various spaces. Turhan and Körpinar (2010) study Mannheim curves in the Lorentzian Heisenberg group Heis^3. Gok et al. (2014) define Mannheim partner curves in the 3-dimensional Lie Group with a bi-invariant metric. Kaymaz and Aksoyak (2017) give conditions on a curvature and the torsion for a Mannheim partner curve to be a general helix and a rectifying curve. Okuyucu and Yazici (2022), give a new approach for Bertrand and Mannheim curves in 3*D* Lie groups with bi-invariant metrics. Among others recent studies are Orbay and Kasap (2009), Tul and Sarioğlugil (2011), Ceylan and Ergin (2016), Senyurt et al. (2017a, b), Senyurt (2012) and Has and Yilmaz (2021).

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With the exception of the 3-dimensional hyperbolic space, the 3-dimensional homogeneous spaces whose isometry group dimension is 4 or 6, provided with a metric depending on two real parameters, form a family of spaces called Bianchi-Cartan-Vranceanu spaces, among them is the special linear group $SL(2,\mathbb{R})$, being extensively studied, for example in Lang (1985). Surfaces isometrically immersed in the linear special group are also studied, for example, in Montaldo et al. (2016) and Belkhelfa et al. (2022) classify the constant angle surfaces and parallel surfaces, respectively.

In this paper, we define and study the Mannheim curves, Mannheim partner curves and Mannheim pair in the special linear group, finding the relation between the curvatures and torsions and some consequences.

PRELIMINARIES

The special linear group $SL(2,\mathbb{R})$ is the group of 2×2 real matrices with determinant one:

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

Let $\mathbb{R}^4_2$ denote the 4-dimensional pseudo-Euclidean space endowed with the semi definite inner product of signature (2, 2) given by

$$\langle v,w \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3 - a_4 b_4, \quad v,w \in \mathbb{R}^4,$$

we can see the space $SL(2,\mathbb{R})$ as a subset of $\mathbb{R}^4_2$:

$$SL(2,\mathbb{R}) = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2 + y^2 - z^2 - w^2 = 1\} \subset \mathbb{R}^4_2 .$$

The next lemma shows that the geometry of this space can be described in terms of left invariant vector fields.

Lemma 1

The following vector fields form an orthonormal frame on $SL(2,\mathbb{R})$ (Lang, 1985):

$$E_1 = \frac{1}{2}\left(-w\,\partial_x - z\partial_y - y\partial_z - x\partial_w\right),$$
$$E_2 = \frac{1}{2}\left(-z\,\partial_x + w\partial_y - x\partial_z + y\partial_w\right),$$
$$E_3 = \frac{1}{2}\left(-y\,\partial_x + x\partial_y - w\partial_z + z\partial_w\right).$$

The geometry of the $SL(2,\mathbb{R})$ can be described in terms of this frame as follows:

(i) These vector fields satisfy the commutation relations:

$$[E_1,E_2] = -E_3, \quad [E_2,E_3] = E_1, \quad [E_3,E_1] = E_2.$$

(ii) The Levi Civita connection ∇ of $SL(2,\mathbb{R})$ is given by

$$\nabla_{E_1}E_1 = 0, \qquad \nabla_{E_2}E_1 = \frac{1}{2}E_3, \qquad \nabla_{E_3}E_1 = \frac{3}{2}E_2,$$

$$\nabla_{E_1}E_2 = -\frac{1}{2}E_3, \quad \nabla_{E_2}E_2 = 0, \qquad \nabla_{E_3}E_2 = -\frac{3}{2}E_1,$$

$$\nabla_{E_1}E_3 = \frac{1}{2}E_2, \qquad \nabla_{E_2}E_3 = -\frac{1}{2}E_1, \quad \nabla_{E_3}E_3 = 0.$$

(iii) The Riemannian curvature tensor R of $SL(2,\mathbb{R})$ is determined by

$$R(E_1,E_2)E_3 = R(E_1,E_3)E_2 = R(E_2,E_3)E_1 = 0,$$

$$R(E_1,E_2)E_1 = -\frac{7}{4}E_2, \quad R(E_1,E_2)E_2 = \frac{7}{4}E_1, \quad R(E_1,E_3)E_1 = \frac{1}{4}E_3,$$

$$R(E_1,E_3)E_3 = -\frac{1}{4}E_1, \quad R(E_2,E_3)E_2 = \frac{1}{4}E_3, \quad R(E_2,E_3)E_3 = -\frac{1}{4}E_2.$$

MANNHEIM PARTNER CURVES IN *SL*(2, $\mathbb{R}$)

Let $\alpha : I \to SL(2,\mathbb{R})$ be a differentiable curve in special linear group defined on an open interval I, parametrized by arc length and let $\{t = \alpha', n, b\}$ be Frenet frame satisfying:

$$\begin{aligned} \nabla_t t &= \kappa\, n, \\ \nabla_t n &= -\kappa\, t - \tau\, b, \\ \nabla_t b &= \tau\, n. \end{aligned} \tag{1}$$

where κ e τ are differentiable functions on I called the *curvature* and the *torsion* of α, respectively.

Definition 1

Let $\alpha : \mathrm{I} \to SL(2,\mathbb{R})$ and $\beta : \mathrm{J} \to SL(2,\mathbb{R})$ be two curves in the special linear group defined on an open interval I and J, respectively. If there exists a corresponding relationship between α and β such that, at the corresponding points of the curves, the principal normal vector of α coincides with the binormal lines of

β, then α is called a Mannheim curve, β is called Mannheim partner curve of α and (α, β) is called Mannheim pair.

Let (α, β) a Mannheim pair parametrized by arc length *s* and $\bar{s}$, respectively, with Frenet frame $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$, respectively.

From the above mentioned definition, we obtain:

$$\alpha(s) = \beta(\bar{s}) - \lambda(\bar{s})\bar{b}(\bar{s}), \tag{2}$$

where $\lambda: J \rightarrow \mathbb{R}$.

The next result is basic to what comes next.

Theorem 1 (Theorem 2)

Let (α, β) be a Mannheim pair in *SL*(2, $\mathbb{R}$). The distance between corresponding points of α and β is constant, that is, λ is nonzero constant (Ceylan and Ergin, 2016).

Proof

By taking the derivative of Equation 2 with respect to $\bar{s}$ and using Equation 1, we obtain

$$t\frac{ds}{d\bar{s}} = \bar{t} - \lambda' \bar{b} - \lambda\bar{\tau}\,\bar{n}.$$

Since n and $\bar{b}$ are linearly dependent,

$$0 = \langle t\frac{ds}{d\bar{s}}, \bar{b}\rangle = \langle \bar{t} - \lambda'\bar{b} - \lambda\bar{\tau}\,\bar{n}, \bar{b}\rangle = -\lambda'.$$

Thus, λ is a nonzero constant. From the distance function between two points, we have

$$d\big(\beta(\bar{s}), \alpha(s)\big) = \|\beta(\bar{s}) - \alpha(s)\| = \|\lambda\,\bar{b}\| = |\lambda|$$

is constant.

The next Corollary is an easy consequence of the Theorem 1 and guarantees the existence of the Mannheim pair.

Corollary 1

For a curve α in $SL(2, \mathbb{R})$, there is a curve β so that (α, β) is a Mahheim pair.

Proof

Since n and $\bar{b}$ are linearly dependent and from Equation 2, we can write:

$$\beta(\bar{s}) = \alpha(s) + \lambda\, n(s). \tag{3}$$

Now that λ is a nonzero constant, there is a curve β for all values of λ.

Also, applying the Equation 2, we have how to determine the torsion of a Mannheim partner curve by means of the curvature and torsion of the Mannheim curve.

Theorem 2

Let $\alpha: I \rightarrow SL(2, \mathbb{R})$ be a Mannheim curve with curvature κ and torsion τ. The torsion $\bar{\tau}$ of the Mannheim partner curve β of α is

$$\bar{\tau} = -\frac{\kappa}{\lambda\,\tau}.$$

Proof

By taking the derivative of Equation 2 with respect to $\bar{s}$ and using Equation 1, we get:

$$t = \frac{d\bar{s}}{ds}\bar{t} - \lambda\bar{\tau}\frac{d\bar{s}}{ds}\bar{n}. \tag{4}$$

Let ϑ be the angle between t and $\bar{t}$, then we obtain:

$$t = \cos\vartheta\ \bar{t} + \sin\vartheta\ \bar{n},$$

$$b = \sin\vartheta\ \bar{t} - \cos\vartheta\ \bar{n}. \tag{5}$$

By taking into consideration Equations 4 and 5, we get:

$$\cos\vartheta = \frac{d\bar{s}}{ds} \text{ and } \sin\vartheta = -\lambda\bar{\tau}\frac{d\bar{s}}{ds}. \tag{6}$$

By taking the derivative of Equation 3 with respect to *s* and using Equation 1, we have

$$\bar{t} = (1 + \lambda\kappa)\frac{ds}{d\bar{s}}t + \lambda\tau\frac{ds}{d\bar{s}}b. \tag{7}$$

From system (Equation 5), we get:

$$\bar{t} = \cos\vartheta\, t + \sin\vartheta\, b,$$
$$\bar{n} = \sin\vartheta\, t - \cos\vartheta\ b. \tag{8}$$

By applying Equation (7) and system (Equation 8), we get

$$\cos\vartheta = (1+\lambda\kappa)\frac{d\,s}{d\,\bar{s}} \quad \text{and} \quad \sin\vartheta = \lambda\tau\frac{d\,s}{d\,\bar{s}} \qquad (9)$$

Thus, from both values of $\cos\theta$ and $\sin\theta$, we obtain:

$$cos^2\vartheta = 1+\lambda\kappa \;\; and \;\; sin^2\vartheta = \lambda^2\tau\bar{\tau}.$$

Since $sin^2\vartheta + cos^2\vartheta = 1$, then

$$\bar{\tau} = -\frac{\kappa}{\lambda\,\tau}.$$

Corollary 2

Let (α,β) be a Mannheim pair in $SL(2,\mathbb{R})$. Between the curvature and the torsion of the curve α there is the relationship:

$$\lambda(\cot\vartheta\;\tau - \kappa) = 1,$$

where λ is a nonzero constant and ϑ is the angle between the vectors t and $\bar{t}$.

Proof

From Equation 9, we have:

$$\frac{cos\;\vartheta}{1+\lambda\kappa} = \frac{sin\;\vartheta}{\lambda\tau}.$$

Thus, we obtain

$$\lambda(\cot\vartheta\;\tau - \kappa) = 1.$$

Theorem 3

Let (α,β) be a Mannheim pair in $SL(2,\mathbb{R})$. Following equations are for the curvatures and torsions of the curves α and β:

$$(i)\;\; \bar{\kappa} = -\frac{d\,\vartheta}{d\,\bar{s}},$$
$$(ii)\;\; \bar{\tau} = -\kappa\sin\vartheta\frac{d\,s}{d\,\bar{s}} + \tau\cos\vartheta\frac{d\,s}{d\,\bar{s}},$$
$$(iii)\;\; \kappa = -\bar{\tau}\sin\vartheta\frac{d\,\bar{s}}{d\,s},$$
$$(iv)\;\tau = \bar{\tau}\cos\vartheta\frac{d\,\bar{s}}{d\,s}.$$

Proof

By taking the derivative of equations $\langle t,\bar{t}\rangle = \cos\vartheta, \langle n,\bar{n}\rangle = 0, \langle t,\bar{b}\rangle = 0$ and $\langle b,\bar{b}\rangle = 0$ with respect to $\bar{s}$, using Equation 8 and considering n and $\bar{b}$ linearly dependent, we have:

$$\langle \kappa\, n\frac{d\,s}{d\,\bar{s}},\bar{t}\rangle + \langle t,\bar{\kappa}\,(\sin\vartheta\, t - \cos\vartheta\; b)\rangle = -\sin\vartheta\frac{d\,\vartheta}{d\,\bar{s}} \Rightarrow \bar{\kappa} = -\frac{d\,\vartheta}{d\,\bar{s}},$$

$$0 = \langle(-\kappa\, t - \tau\, b)\frac{d\,s}{d\,\bar{s}},\bar{n}\rangle + \langle n, -\bar{\kappa}\bar{t} - \bar{\tau}\bar{b}\rangle \Rightarrow \bar{\tau} = -\kappa\sin\vartheta\frac{d\,s}{d\,\bar{s}} + \tau\cos\vartheta\frac{d\,s}{d\,\bar{s}},$$

$$0 = \langle \kappa\, n\frac{d\,s}{d\,\bar{s}},\bar{b}\rangle + \langle t,\bar{\tau}\bar{n}\rangle \Rightarrow \kappa = -\bar{\tau}\sin\vartheta\frac{d\,\bar{s}}{d\,s},$$
$$0 = \langle \tau\, n\frac{d\,s}{d\,\bar{s}},\bar{b}\rangle + \langle b,\bar{\tau}\bar{n}\rangle \Rightarrow \tau = \bar{\tau}\cos\vartheta\frac{d\,\bar{s}}{d\,s}.$$

Remark 1

From Equations (iii) and (iv) of the Theorem 3, we obtain:

$$\kappa^2 + \tau^2 = \bar{\tau}^2\left(\frac{d\,\bar{s}}{d\,s}\right)^2 \quad and \quad \frac{\tau}{\kappa} = -\cot\vartheta.$$

Corollary 3

Let α be a curve in $SL(2,\mathbb{R})$ with curvature $\kappa \neq 0$ and torsion τ. Let β be its Mannheim partner curve. If

$$\frac{\tau}{\kappa} = constant$$

then β is a straight line.

Corollary 4

Let be a Mannheim partner curve of . If is a planar curve then is a straight line. The next result gives a necessary and sufficient condition for a curve to be a Mannheim partner curve in the special linear group.

Theorem 4

Let α be a Mannheim curve in $SL(2,\mathbb{R})$ parametrized by arc lenght s. Then $\beta(\bar{s})$ is the Mannheim partner curve of α if and only if the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of β satisfy the following equation:

$$\frac{d\,\bar{\tau}}{d\,\bar{s}} = \frac{\bar{\kappa}}{\lambda}(1+\lambda^2\bar{\tau}^2),$$

for some nonzero constant λ.

Proof

Let $\beta : J \to SL(2,\mathbb{R})$ be a Mannheim partner curve of $\alpha: I \to SL(2,\mathbb{R})$. From Equation 6, we have

$$\lambda\,\bar{\tau} = -\tan\vartheta.$$

By taking the derivative of this equation and applying (i) of the Theorem 3, we get:

$$\lambda\,\frac{d\,\bar{\tau}}{d\,\bar{s}} = \frac{\bar{\kappa}}{cos^2\vartheta}.$$

From Equation 6, we get:

$$cos^2\vartheta + sin^2\vartheta = 1 \Rightarrow sec^2\vartheta = 1 + \lambda^2\,\bar{\tau}^2.$$

Thus,

$$\frac{d\,\bar{\tau}}{d\,\bar{s}} = \frac{\bar{\kappa}}{\lambda}(1+\lambda^2\bar{\tau}^2).$$

Conversely, if the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of the curve β satisfy

$$\lambda\frac{d\,\bar{\tau}}{d\,\bar{s}} = \bar{\kappa}(1+\lambda^2\bar{\tau}^2),$$

for some nonzero constant λ, then a curve α by

$$\alpha(s) = \beta(\bar{s}) - \lambda\bar{b}(\bar{s}) \tag{10}$$

By taking the derivative of Equation 10 with respect to $\bar{s}$ twice, we get

$$t\,\frac{d\,s}{d\,\bar{s}} = \bar{t} - \lambda\,\bar{\tau}\,\bar{n} \tag{11}$$

$$\kappa\left(\frac{d\,s}{d\,\bar{s}}\right)^2 n + \frac{d^2\,s}{d\,\bar{s}^2}\,t = \lambda\,\bar{\kappa}\,\bar{\tau}\,\bar{t} + \left(\bar{\kappa} - \lambda\,\frac{d\,\bar{\tau}}{d\,\bar{s}}\right)\bar{n} + \lambda\,\bar{\tau}^2\,\bar{b} \tag{12}$$

By taking the cross product of Equations 11 with 12, we obtain:

$$\kappa\left(\frac{d\,s}{d\,\bar{s}}\right)^3 b = \lambda^2\,\bar{\tau}^3\,\bar{t} + \lambda\,\bar{\tau}^2\bar{n} \tag{13}$$

By taking the cross product of Equation 11 with Equation 13, we obtain:

$$\kappa\left(\frac{d\,s}{d\,\bar{s}}\right)^4 n = -\lambda\,\bar{\tau}^2(1+\lambda^2\,\bar{\tau}^2)\bar{b}$$

This means that the principal normal direction n of α coincides with the bi-normal direction $\bar{b}$ of β. Hence, α is a Mannheim curve and β is its Mannheim partner curve.

Conclusion

Recent studies on Mannheim partner curves are being done by many authors. In this paper we introduce the notion of Mannheim curves, Mannheim partner curves and Mannheim pairs in the special linear group $SL(2,\mathbb{R})$, objects that can be useful for future studies in geometry. We give necessary and sufficient conditions for a given curve to be a Mannheim partner curve of another given curve through its curvature and torsion; we also relate the Frenet frame by means of a rotation matrix establishing conditions for a Mannheim curve and a Mannheim partner curve to be straight lines.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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