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Full Length Research Paper

On solutions of nonlinear heat diffusion model for thermal energy storage problem

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An analysis is performed for an unsteady nonlinear heat diffusion problems modeling thermal energy storage in a medium with power law temperature-dependent heat capacity, thermal conductivity and heat source term and subjected to a convective heat transfer to the surrounding environment at the boundary through a variable heat transfer coefficient. Lie group theory is applied to determine symmetry reductions of the governing nonlinear partial differential equation (PDE) with the boundary conditions. The resulting nonlinear ordinary differential equation (ODE) with appropriate corresponding boundary conditions is solved using Adomian decomposition method (ADM) coupled with Padé approximation technique. The effects of material parameters on the thermal decay in the system are shown graphically and discussed quantitatively.

Key words: Unsteady heat diffusion, thermal energy storage, group method, decomposition method, nonlinear problem.

INTRODUCTION

For many years, considerable attention has been paid to the collection, storage and use of thermal energy to meet various energy demands. Thermal storage system is a specially designed energy saving device for temporary storage of heat energy. Recently, the use of solar energy to meet the thermal demands of industries, electronics devices, residential establishment, etc., is fast growing in many countries of the world (Duffie and Becham, 1980). Solar energy is provided by the light energy that comes from the sun. An important component of thermal systems designed for such purposes is a thermal energy storage unit. The medium in which the energy is stored may be fluid or solid (Jahiria and Gupta, 1982). In middle and low temperature solar energy systems, water and stones are the best and cheapest storing energy medium (Hawlader and Brinkworth, 1981). For instance, potable water is usually heated by a device known as a water heater or geysers for domestic and industrial usage. Most commonly, human-induced heating processes, such as combustion or electric-resistance, are relied upon to heat the water, but solar energy or where possible, geothermal power may be used (Shin et al., 2004). The effectiveness of a liquid thermal storage system is determined by how temperature of the system decays as a result of heat losses by convection to the environment (Davies, 1985). Thermal energy storage problem in a medium with temperature-dependent heat capacity and thermal conductivity constitutes an unsteady nonlinear heat diffusion problem and the solutions in space and time may reveal the appearance of thermal decay in the system. In order to predict the occurrence of such phenomena, it is necessary to analyze a simplified mathematical model from which insight might be gleaned into an inherently complex physical mechanism. Meanwhile, the solution of unsteady nonlinear heat diffusion equations in rectangular, cylindrical and spherical coordinates remains a very important problem of practical relevance in the engineering sciences (Badran and Abd-el-Malek, 1995). Recently, the ideas of hybrid analytical-numerical schemes for solving nonlinear differential equations have experienced a revival. One such trend is related to the combination of group theoretic approach and Adomian decomposition method (Adomian, 1994; Makinde, 2009; Moitsheki and Makinde, 2009).

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This hybrid analytical-numerical approach is also extremely useful in the validation of purely numerical scheme.

In the present work, we studied an unsteady nonlinear heat diffusion problems modeling thermal energy storage in a medium with power law temperature-dependent heat capacity, thermal conductivity and heat source term and subjected it to a convective heat transfer to the surrounding environment at the boundary.

MATHEMATICAL FORMULATION

Consider an unsteady thermal storage problem in a medium whose surface is subjected to heat transfer by convection to an external environment having a heat transfer coefficient that varies with respect to the time. The energy equation in a rectangular, cylindrical or spherical coordinate system with heat source term can be used to find the temperature distribution through a region defined in an interval $0 < r < a$. The unsteady heat conduction problem can be described by the following governing equation (Badran and Abd-el-Malek, 1995);

$$\rho C(T)\frac{\partial T}{\partial t} = \frac{1}{r^m}\frac{\partial}{\partial r}\left(K(T)r^m\frac{\partial T}{\partial r}\right) + S(T) \qquad (1)$$

with the initial condition:

$$T(r,t) = T_0 \quad \text{at } t = 0, \qquad (2)$$

and the following boundary conditions:

$$\frac{\partial T}{\partial r} = 0, \text{ at } r = 0, \qquad (3)$$

$$K(T)\frac{\partial T}{\partial r} = -h(t)(T - T_\infty), \text{ at } r = a, \qquad (4)$$

Where; T is the temperature, t is the time, ρ is the density, $S(T)$ is the temperature-dependent heat source term and $h(t) = h_0 f(t)$ is the time-dependent heat transfer coefficient.

Following Makinde (2007), the power law temperature-dependent thermal conductivity and heat capacity are taken as

$K(T) = K_0(T - T_\infty)^n/(T_i - T_\infty)^n$ and $C(T) = C_0(T - T_\infty)^b/(T_0 - T_\infty)^b$, where K_0, C_0, n and b are constants, T_0 is the initial temperature of the body; T_∞ is the temperature of the surrounding environment. The geometry of the body is specified by m = 0, 1, 2 representing rectangular, cylindrical and spherical coordinates respectively.

Equations 1 - 4 are made dimensionless by introducing the following quantities:

$$\bar{r} = \frac{r}{a},\ \bar{t} = \frac{K_0 t}{\rho C_0 a^2},\ \bar{T} = \frac{T - T_\infty}{T_0 - T_\infty},\ \bar{S} = \frac{S}{\rho C_0},\ Bi = \frac{a h_0}{K_0} \qquad (5)$$

The dimensionless governing equations become;

$$\bar{T}^b\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{1}{\bar{r}^m}\frac{\partial}{\partial \bar{r}}\left(\bar{T}^n\bar{r}^m\frac{\partial \bar{T}}{\partial \bar{r}}\right) + \bar{S}(\bar{T}), \qquad (6)$$

$$\frac{\partial \bar{T}}{\partial \bar{r}} = 0, \text{ at } \bar{r} = 0, \qquad (7)$$

$$\bar{T}^n\frac{\partial \bar{T}}{\partial \bar{r}} = -Bi f(\bar{t})\bar{T}, \text{ at } \bar{r} = 1, \qquad (8)$$

Where; Bi is the Biot number.

SOLUTION OF THE PROBLEM

In the following sections, we shall neglect the bar symbol in the dimensionless governing equations 5 - 8 for clarity. Firstly, we reduce the system of PDEs in Equations 6 - 8 to a system of ODEs using the Lie group methods (Stephani, 1989). This will be followed by the application of Adomian decomposition method (Adomian, 1994; Makinde, 2009), in order to obtain a semi-analytical non-perturbative approximate solution to the problem.

Lie point symmetry analysis

In brief, symmetry of a differential equation is an invertible transformation of the dependent and independent variables that does not change the equation. Symmetries depend continuously on a parameter and form a group; the one-parameter group of transformations. This group can be determined algorithmically. The theory and applications of Lie groups may be obtained in excellent text such as those of Bluman and Anco (2002, 1986), Stephani (1989), Olver (1986), Ovsianikov (1982) and Ibragimov (1999). In essence, determining symmetries for the governing Equation 6, implies seeking transformations of the for;

$$\begin{aligned} r_* &= r + \varepsilon\xi(t,r,T) + O(\varepsilon^2); \\ t_* &= t + \varepsilon\tau(t,r,T) + O(\varepsilon^2); \\ T_* &= T + \varepsilon\eta(t,r,T) + O(\varepsilon^2); \end{aligned} \qquad (9)$$

Generated by the vector field.

$$X = \tau(t,r,T)\frac{\partial}{\partial t} + \xi(t,r,T)\frac{\partial}{\partial r} + \eta(t,r,T)\frac{\partial}{\partial T}, \qquad (10)$$

Which leave the governing Equation 6 invariant. It is possible to find all possible functions or cases for the source term $S(T)$ such that extra symmetries are admitted by Equation 6. Determination of such cases and symmetries admitted is called group or symmetry classification. In this work we restrict our analysis to one case for application purpose. Full symmetry classification and further investigations will be dealt with elsewhere. Note that we seek point symmetries that leave a single Equation 6 invariant rather than the entire boundary value problem (BVP), and apply boundary conditions onto the obtained invariant solutions. It is a well known fact that the symmetry algebra may be reduced if invariance is sought for the entire BVP. In the initial symmetry analysis of Equation 6, we obtained nothing more than the translation in *t.* Extra symmetries may be obtained for various choices of *S(T)* via symmetry classification. However, in this work we restrict analysis to the case $n = b$ and $S = S_0 T^{n+1}$, where $S_0 = \frac{1}{n+1}$. With this case of the source term, Equation 6 admits a finite four-dimensional Lie algebra spanned by;

$$\begin{aligned} X_1 &= S_0\left[t^2 - \frac{r^2}{4} - \frac{(m+1)}{2}t\right]T\frac{\partial}{\partial T} + rt\frac{\partial}{\partial r} + 4t^2\frac{\partial}{\partial t}; \\ X_2 &= 2S_0 tT\frac{\partial}{\partial T} + r\frac{\partial}{\partial r} + 2t\frac{\partial}{\partial t}; \\ X_3 &= \frac{\partial}{\partial t}; \\ X_4 &= -nS_0T\frac{\partial}{\partial T} \end{aligned} \quad (11)$$

and the infinite symmetry algebra, namely:

$$X_5 = -H(r,t)T^{-n}\frac{\partial}{\partial T}. \quad (12)$$

H is an arbitrary function of the independent variables *r* and *t*. The obtained finite symmetry algebra may result in reduction of the PDE in Equation 6 to ODE, using any linear combination. Furthermore, group invariant solutions may be classified according to the set of in equivalent subalgebras (Olver, 1986).

Symmetry reduction

Reduction by one of the variables of the governing equation is performed using a linear combination of the symmetries X_1 and X_4. Letting $\beta = -\left(\frac{n}{n+1}\right)$ and using $X_3 + X_4$ yield:

$$T(r,t) = \theta(r)e^{\beta t}. \quad (13)$$

Note that one may obtain this solution using method of separation of variables. However, this has been exposed by symmetry classification. Also, symmetry analysis is a systematical method that may lead to extra solutions. The time dependent heat transfer coefficient may be represented $h(t) = h_0 e^{n\beta t}$. With $S(r,t) = S_0\theta(r)^{n+1}e^{(n+1)\beta t}$ and $b = n$, Equations 6 - 8 reduce to:

$$\frac{1}{r^m}\frac{d}{dr}\left(r^m\frac{d\theta}{dr}\right) + \frac{n}{\theta}\left(\frac{d\theta}{dr}\right)^2 + \theta = 0, \quad (14)$$

$$\frac{d\theta}{dr} = 0, \text{ at } r = 0, \quad (15a)$$

$$\frac{d\theta}{dr} = -Bi\,\theta^{1-n}, \text{ at } r = 1 \quad (15b)$$

Computational method

In this section, we employed Adomian decomposition technique in order to explicitly construct an approximate non-perturbative solution for the nonlinear ordinary differential equations above. The advantage of this method is that it provides a direct scheme for solving the problem, that is, without the need for linearization, perturbation, massive computation and any further transformation.

Following Adomian (1994), we rewrite Equation 14 with respect to Equation 15a in the form;

$$L_r\theta = -\frac{n(\theta_r)^2}{\theta} - \theta, \quad (16)$$

Where the subscript *r* represents derivative with respect to *r* and the differential operator employs the first two derivatives in the form:

$$L_r = \frac{1}{r^m}\frac{d}{dr}\left(r^m\frac{d}{dr}\right), \quad (17)$$

In order to overcome the singularity behaviour at the point $r = 0$. In view of Equation 17, the inverse operator L_r^{-1} is considered a twofold integral operator defined by:

$$L_r^{-1} = \int_0^r r^{-m}\int_0^r r^m(.)drdr. \quad (18)$$

Applying L_r^{-1} to both sides of Equation 16, using the boundary conditions in Equation 15, we obtain;

$$\theta(r) = \theta(0) - nL_r^{-1}\left(\frac{(\theta_r)^2}{\theta}\right) - L_r^{-1}(\theta). \quad (19)$$

As usual in Adomian decomposition method, the solution of Equation 19 is approximated as an infinite series;

$$\theta(r) = \sum_{j=0}^{\infty}\theta_j, \quad (20)$$

and the nonlinear terms are decomposed as follows:

$$\frac{(\theta_r)^2}{\theta} = \sum_{j=0}^{\infty}A_j, \quad (21)$$

Where; A_j, are polynomials (called Adomian polynomials) given by:

$$A_j = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[\frac{\left(\sum_{i=0}^{\infty}\theta_{ri}\lambda^i\right)^2}{\sum_{i=0}^{\infty}\theta_i\lambda^i}\right]_{\lambda=0}, \quad (22)$$

Thus, we can identify

$$\theta_0 = \theta(0),$$
$$\theta_{j+1} = -nL_r^{-1}(A_j) - L_r^{-1}(\theta_j), \text{ for } j \geq 0. \quad (23)$$

Using Equation 22, we compute some of the Adomian polynomials as follows:

$$A_0 = \frac{(\theta_{r0})^2}{\theta_0},\ A_1 = \frac{\theta_{r0}(2\theta_{r1}\theta_0 - \theta_{r0}\theta_1)}{\theta_0^2}, \dots \quad (24)$$

Substituting Equations 20 - 21 into Equation 19, and using Maple we obtained a few terms approximation to the solution as;

$$\psi_N = \sum_{n=0}^{N} \theta_n, \quad (25)$$

Where; $\theta(r) = \lim_{N\to\infty}(\psi_N)$. Using the above procedure, we obtain

$$\theta(r) = \theta_0 - \frac{r^2\theta_0}{2(m+1)} + \left(\frac{\theta_0}{8(m+1)(3+m)} - \frac{\theta_0(60n+12nm)}{48(3+m)(m+1)^2(5+m)}\right)r^4 + \left(-\frac{\theta_0}{48(m+1)(3+m)(5+m)} + \frac{\theta_0(320nm+48nm^2-112n)}{384(m+1)^3(3+m)(5+m)(7+m)}\right)r^6 + \frac{\theta_0(m^2+2m+1)r^8}{384(m+1)^3(3+m)(5+m)(7+m)} + O(r^{10}) \quad (26)$$

where other terms up to $O(r^{16})$ were derived. By applying the boundary conditions in Equation 15b to the expression in Equation 26, we obtain approximately the values for θ_0 as shown in the following section. Usually, the decomposition method yields rapidly convergent series solutions by using a few terms in the partial sum (Abboui and Cherruault, 1995). However, in this particular problem the convergence of the decomposition series partial sum in Equation 26 is enhanced using Padé approximation technique (Baker, 1975; Makinde, 2009).

Table 1. Computations showing the convergence of the ADM procedure (*Bi* = 1, *n* = 2, *m* = 0).

ψ_N	*θ(0)*
ψ_2	1.299038
ψ_3	1.327342
ψ_4	1.327667
ψ_6	1.327667

Table 2. Computations showing the core temperature at various parameter values (*n=2*).

Bi	*m*	*θ(0)*
1	0	1.327667
10	0	4.198452
15	0	5.142033
10	1	4.930336
10	2	5.924108

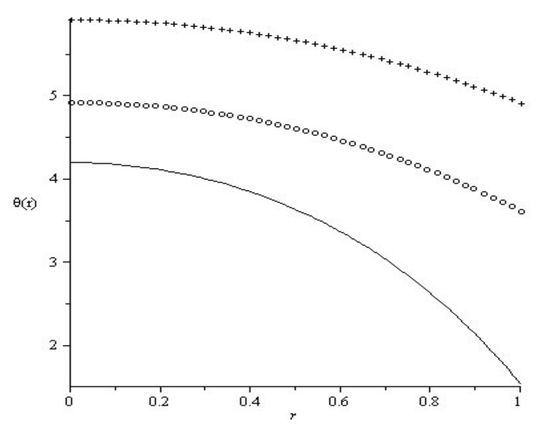

Figure 1. Temperature profile: *Bi* =10; *n* =2; _______ *m* = 0; ooooo *m* = 1; ++++ *m* = 2.

RESULTS AND DISCUSSION

In this investigation, symmetry analysis resulted in a large Lie symmetry algebra being admitted. More invariant solutions may be constructed using this algebra. Approximate solution obtained using Adomian decomposition method (ADM) in Equation 26 coupled with Padé approximation technique is valid for energy storage systems in an interval 0 < *r* < 1. For the numerical validation of our results we have chosen physically meaningful values of the parameters for the problem. Unless otherwise stated we have taken: *Bi* = 0, 1, 10, 100; *n* = 2, *m* = 0, 1, 2. In Table 1, we observed that the convergence of our computational procedure (ADM), improves with gradual increase in the number of decomposition series coefficients utilized. The effects of material geometry and convective heat exchange at the surface on the energy storage are demonstrated in Table 2 and Figures 1 and 2.

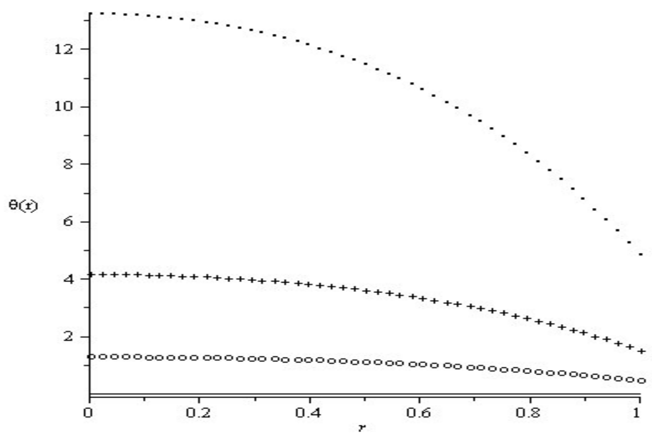

Figure 2. Temperature profile: *m* = 0; *n* = 2; _____*Bi* = 0; ooooo *Bi* = 1; ++++ *Bi* = 10; ……..*Bi* = 100.

It is interesting to note that the core temperature of the energy storage medium is higher than that of the material surface for all geometry due to convective heat transfer to the surrounding Bi > 0. However, in the absence of convective heat exchange at the surface (Bi = 0), a uniform temperature is observed in the energy storage system, since the system is insulated.

Moreover, it is noteworthy that the core temperature of the energy storage system is lowest when the material geometry is rectangular (m = 0) and highest when the material geometry is spherical (m = 2) as illustrated in Table 2 and Figure 1.

Hence, the energy storage capacity of materials in cylindrical and spherical geometry is higher than that of the material in rectangular geometry. Practical application of this result can be easily seen in the design of hot water storage systems used in many houses which are normally cylindrical in shape. The thermal decay at the surface of the material due to heat losses by convection to the surrounding environment is demonstrated in Figure 2. Although the core temperature is high at large Biot number, a rapid decrease in the material surface temperature is observed. When the Biot number is low, the surface heat loss is also very low.

Conclusion

We have solved the nonlinear heat diffusion problem for energy storage in a medium with power law temperature-dependent heat capacity and thermal conductivity using both the group theoretical and decomposition methods. The numerical results demonstrate that Adomian-Padé technique gives the approximate solution with faster convergence rate and higher accuracy. Our results show that a storage device with cylindrical and spherical configuration conserve more energy than the rectangular one, and the thermal decay at the material surface due to convective heat transfer to the surrounding environment at the boundary can be reduced by decreasing the Biot number. Symmetry analysis resulted in a large Lie symmetry algebra being admitted. More invariant solutions may be constructed using this algebra. Full symmetry classification is currently under investigation in our future work.

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