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Full Length Research Paper

Homotopy analysis method for fractional partial differential equations

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This paper applies the homotopy analysis method (HAM) to obtain analytical solutions of fractional heat- and wave-like equations with variable coefficients. The applications of the homotopy analysis method were extended to derive analytical solutions in the form of a series with easily computed terms for these generalized fractional equation. Some examples are presented to show the efficiency and simplicity of the method.

Key words: Homotopy analysis method, fractional calculus, fractional heat- and wave-like equations.

INTRODUCTION

The fractional partial differential equations appear very frequently in physical sciences. Number of physical phenomena are goverened by such equations (Podlubny, 1999; Rossikhin and Shitikova, 1997; Mohyud-Din and Noor 2008; Mohyud-Din et al., 2009, 2010). Several techniques including decomposition, variational iteration, homotopy analysis and variation of parameters have been applied to solve such problems (Rossikhin and Shitikova, 1997; Mohyud-Din and Noor 2008; Mohyud-Din et al., 2009, 2010). In this paper, we will consider the fractional heat-like and wave-like equations of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,y,z)u_{xx} + g(x,y,x)u_{yy} + h(x,y,x)u_{zz},$$
$$0 < x < a,\ 0 < y < b,\ 0 < z < c,\ t > 0, \quad (1)$$

subject to the Neumann boundary conditions:

$$u_x(0,y,z,t) = f_1(y,z,t),\ u_x(a,y,z,t) = f_2(y,z,t),$$
$$u_y(x,0,z,t) = g_1(x,z,t),\ u_y(x,b,z,t) = g_2(x,z,t), \quad (2)$$
$$u_z(x,y,0,t) = h_1(x,z,t),\ u_z(x,y,c,t) = h_2(x,z,t),$$

and initial conditions:

$$u(x,y,z,0) = \psi(x,y,z),\ u_t(x,y,z,0) = \theta(x,y,z), \quad (3)$$

where α is a parameter describing the fractional derivative. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $0 < \alpha \le 1$, then Equation (1) reduces to a fractional heat-like equation with variable coefficients, and to a wave-like equation with variable coefficients for $1 < \alpha \le 2$. In this paper, the homotopy analysis method (Liao, 1997, 1999, 2003, 2004; Hayat et al., 2004; Abbasband, 2007; Yildirim and Mohyud-Din, 2010a) is applied to solve fractional heat- and wave-like equations.

A new approach for solving the fractional heat- and wave-like equations is established. It is expected the proposed techniques can be further applied to derive solutions for other partial differential equations with fractional order. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Unlike,

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other numerical methods are given low degree of accuracy for large values of t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

Fractional calculus

We give some basic definitions and properties of the fractional calculus theory (Caputo, 1967) which are used further in this paper.

Definition 1

A real function $f(x), x>0,$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x)=x^p f_1(x),$ where $f_1(x) \in C[0,\infty),$ and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N.$

Definition 2

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0,$ of a function $f \in C_\mu, \mu \geq -1,$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in (Caputo, 1967), we mention only the following. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta = J^{\alpha+\beta} f(x),$
2. $J^\alpha J^\beta = J^\beta J^\alpha f(x),$
3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

The Riemann–Liouville derivative has certain disadvantages when trying to model realworld phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo (1967) in his work on the theory of viscoelasticity.

Definition 3

The fractional derivative $f(x)$ in the Caputo sense is defined as:

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{4}$$

for $m-1 < \alpha \leq m, \quad m \in N, \; x > 0, \; f \in C_{-1}^m.$

Also, we need here two of its basic properties.

Lemma 1

If $m-1 < \alpha \leq m, \quad m \in N$ and $f \in C_\mu^m, \mu \geq -1,$ then

$$D^\alpha J^\alpha f(x) = f(x), \qquad \text{and,}$$
$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the fractional heat- and wave-like equation (1), where the unknown function *u* = *u(x, t)* is a assumed to be a causal function of time and space, and the fractional derivatives are taken in Caputo sense as follows.

Definition 4

For *m* to be the smallest integer that exceeds *α*, the Caputo time-fractional derivative operator of order *α* > 0 is defined as:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & for\ m-1<\alpha<m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & for\ \alpha = m \in N \end{cases} \tag{5}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

HOMOTOPY ANALYSIS METHOD (HAM)

We apply the HAM (Liao, 1997, 1999, 2003, 2004; Hayat et al., 2004; Abbasband, 2007; Yildirim and Mohyud-Din, 2010b) to the fractional heat- and wave-like equation (1). We consider the following differential equation:

$$FD[u(x,t)]=0, \tag{6}$$

where *FD* is a nonlinear operator for this problem, *x* and *t* denote an independent variables, $u(x,t)$ is an unknown function.

In the frame of HAM (Liao, 1997, 1999, 2003, 2004; Hayat et al., 2004; Abbasband, 2007; Yildirim and Mohyud-Din, 2010a), we can construct the following zeroth-order deformation:

$$(1-q)L(U(x,t;q)-u_0(x,t))=q\hbar H(x,t)FD(U(x,t;q)), \tag{7}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, *L* is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function on the independent variables *x, t* and *q*.

Obviously, when $q=0$ and $q=1$, it holds:

$$U(x,t;0)=u_0(x,t),\ U(x,t;1)=u(x,t), \tag{8}$$

respectively. Using the parameter q, we expand $U(x,t;q)$ in Taylor series as follows:

$$U(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{9}$$

where: $$u_m=\frac{1}{m!}\frac{\partial^m U(t;q)}{\partial^m q}\bigg|_{q=0} \tag{10}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (Equation 9) is convergent at $q=1$, then due to Equation (8) we have:

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t) \tag{11}$$

Let us define the vector:

$$\vec{u}_n(x,t)=\{u_0(x,t),u_1(x,t),\ldots,u_n(x,t)\} \tag{12}$$

Differentiating Equation (7) m times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation:

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar H(x,t)R_m(\vec{u}_{m-1}), \tag{13}$$

where:

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}FD(U(t;q))}{\partial^{m-1}q}\bigg|_{q=0} \tag{14}$$

and

$$\chi_m=\begin{cases}0 & m\leq 1,\\ 1 & m>1.\end{cases} \tag{15}$$

Finally, for the purpose of computation, we will approximate the HAM solution (11) by the following truncated series:

$$\phi_m(t)=\sum_{k=0}^{m-1}u_k(t). \tag{16}$$

Examples

Example 1

In this example we consider the following one-dimensional fractional heat-like problem:

$$D_t^{\alpha}=\frac{1}{2}x^2u_{xx},\ 0<x<1,\ 0<\alpha\leq 1,\ t>0, \tag{17}$$

subject to the boundary conditions:

$u(0,t)=0$, $u(1,t)=e^t$,

and the initial condition $u(x,0)=x^2$.

The exact solution, for the special case $\alpha=1$, is given by:

$$u(x,t)=x^2e^t \tag{18}$$

According to Equation (7), the zeroth-order deformation can be given by:

$$(1-q)L(U(x,t;q)-u_0(x,t))=q\hbar H(x,t)\left(D_\alpha^t U(x,t;q)-\frac{1}{2}x^2\frac{\partial^2 U(x,t;q)}{\partial x^2}\right) \tag{19}$$

We can start with an initial approximation $u_0(x,t)=x^2$

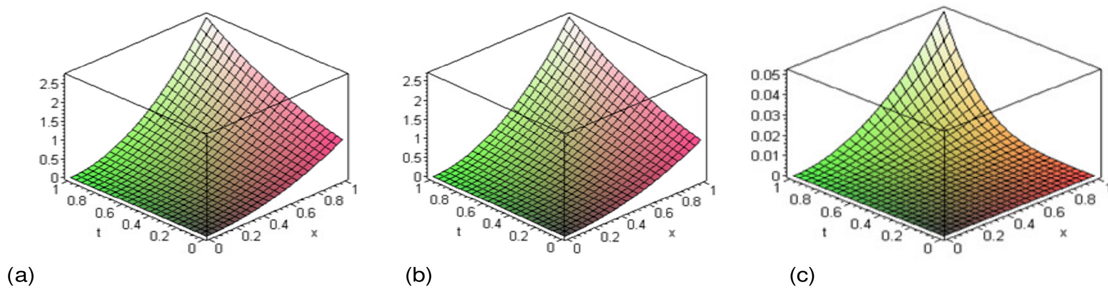


Figure 1. The surface shows the solution $u(x, t)$ for Equation (17) when α = 1: (a) exact solution (23) (b) approximate solution (22) and (c) $\left|u_{ex} - u_{app}\right|$.

and we choose the auxiliary linear operator:

$$L(U(x,t;q)) = D_\alpha^t U(x,t;q),$$

with the property $L(C) = 0,$ where C is an integral constant. We also choose the auxiliary function to be:

$$H(x,t) = 1.$$

Hence, the *m*th-order deformation can be given by:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}),$$

where:

$$R_m(\vec{u}_{m-1}) = D_\alpha^t(u_{m-1}) - \frac{1}{2}x^2 \frac{\partial^2(u_{m-1})}{\partial x^2} \tag{20}$$

Now the solution of the *m*th-order deformation equations (20) for $m \geq 1$ become:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{21}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = x^2,$$

$$u_1(x,t) = -\hbar x^2 \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = -\hbar x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2 x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\vdots$$

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$
$$= x^2\left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right] \tag{22}$$

For the special case $\alpha = 1,$ we obtain from (22):

$$u(x,t) = x^2 e^t \tag{23}$$

The evolution results for the exact solution (23) and the approximate solution obtained using the homotopy analysis method, for the special case α = 1, are shown in Figure 1. It can be seen from Figure 1 that the solution obtained by the present method is nearly identical with the exact solution. Figure 2(a, b) show the approximate solutions when α = 0.5 and α = 0.25, respectively. It is to be noted that only the third-order term of the homotopy analysis solution was used in evaluating the approximate solutions for Figures 1 and 2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x, t)$ when the homotopy analysis method is used.

Example 2

In this example we consider the two-dimensional fractional heat-like equation:

$$D_t^\alpha u = u_{xx} + u_{yy}, 0 < x, y < 2\pi, 0 < \alpha \leq 1, \ t > 0, \tag{24}$$

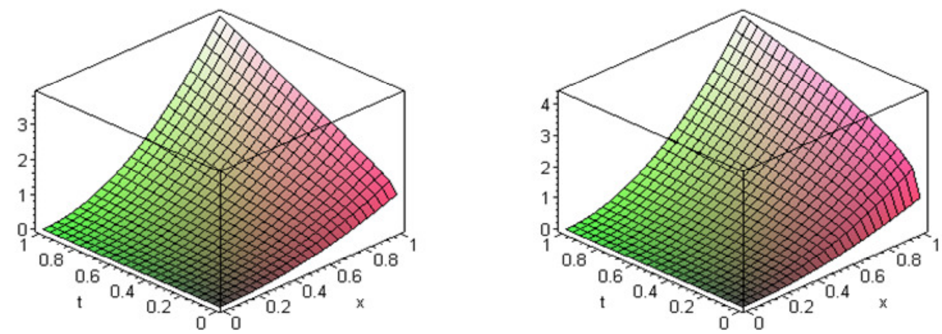
subject to the boundary conditions:

$$u(0, y, t) = 0, \qquad u(2\pi, y, t) = 0,$$
$$u(x, 0, t) = 0, \qquad u(x, 2\pi, t) = 0,$$

and the initial condition:

$$u(x, y, 0) = \sin x \sin y,$$

The exact solution, for the special case $\alpha = 1,$ is given



(a) (b)

Figure 2. The surface shows the solution *u(x, t)* for Equation(17): (a) *α* = 0.5, (b) *α* = 0.25.

by:

$$u(x,y,t)=e^{-2t}\sin x\sin y. \tag{25}$$

According to Equation (7), the zeroth-order deformation can be given by:

$$(1-q)L[U(x,y,t;q)-u_0(x,y,t)]=q\hbar H(x,y,t)\left(D_\alpha^t U(x,y,t;q)-\frac{\partial^2 U(x,y,t;q)}{\partial x^2}-\frac{\partial^2 U(x,y,t;q)}{\partial y^2}\right) \tag{26}$$

We can start with an initial approximation $u_0(x,y,t)=\sin x\sin y$ and we choose the auxiliary linear operator:

$$L(U(x,y,t;q))=D_\alpha^t U(x,y,t;q),$$

with the property $L(C)=0,$

where C is an integral constant. We also choose the auxiliary function to be:

$$H(x,y,t)=1.$$

Hence, the *m*th-order deformation can be given by:

$$L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)]=\hbar H(x,y,t)R_m(\vec{u}_{m-1}),$$

where:

$$R_m(\vec{u}_{m-1})=D_\alpha^t(u_{m-1})-\frac{\partial^2(u_{m-1})}{\partial x^2}-\frac{\partial^2(u_{m-1})}{\partial y^2} \tag{27}$$

Now the solution of the *m*th-order deformation equations (27) for $m\geq 1$ become:

$$u_m(x,y,t)=\chi_m u_{m-1}(x,y,t)+\hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{28}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$\begin{aligned}
&u_0(x,y,t)=\sin x\sin y,\\
&u_1(x,y,t)=2h\sin x\sin y\frac{t^\alpha}{\Gamma(\alpha+1)},\\
&u_2(x,y,t)=2\hbar\sin x\sin y\frac{t^\alpha}{\Gamma(\alpha+1)}+2\hbar^2\sin x\sin y\frac{t^\alpha}{\Gamma(\alpha+1)}+4\hbar^2\sin x\sin y\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\vdots
\end{aligned} \tag{29}$$

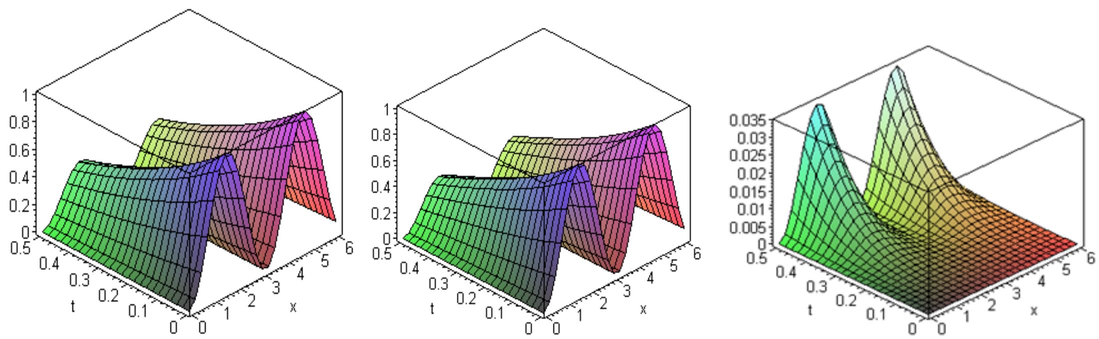
Hence, the HAM series solution (for $\hbar=-1$) is:

$$\begin{aligned}
u(x,t)&=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+.....\\
&=\sin x\sin y\left[1-2\frac{t^\alpha}{\Gamma(\alpha+1)}+4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+.....\right]
\end{aligned} \tag{30}$$

For the special case (when $\alpha=1$), we can reproduce the series solution of (24), and the solution in a closed form:

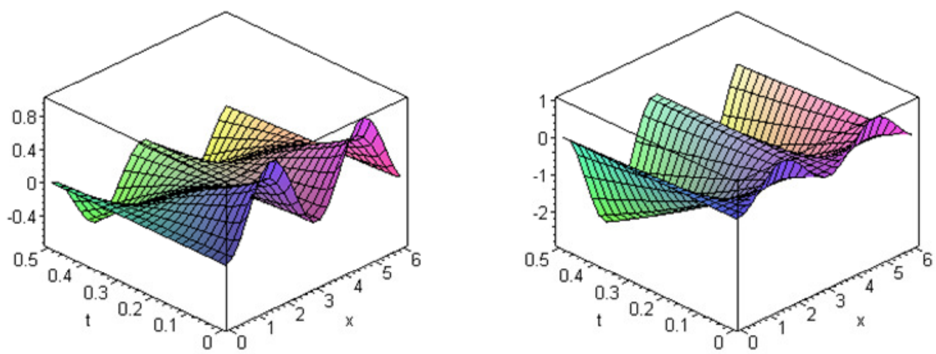
$$u(x,y,t)=e^{-2t}\sin x\sin y, \tag{31}$$

follows immediately. Figure 3(a,b,c) show the exact solution (31) and the approximate solution obtained using the homotopy analysis method, for the special case *α* = 1. Figure 4 (a,b) show the approximate solutions when *α* = 0.5 and *α* = 0.25, respectively. It is to be noted that only the third-order term of the homotopy analysis solution was used in evaluating the approximate solutions for Figures 3 and 4. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of *u(x*,y, *t)* when the homotopy analysis method is used.



(a) (b) (c)

Figure 3. The surface shows the solution *u(x*,y, *t)* for Equation (24) when *α* = 1 and x = y. (a) exact solution (31) (b) approximate solution (30) (c) $\left|u_{ex} - u_{app}\right|$.



(a) (b)

Figure 4. The surface shows the solution *u(x*,y, *t)* for Equation (24): (a) *α* = 0.5, (b) *α* = 0.25.

Example 3

Consider the three-dimensional inhomogeneous fractional heat-like equation:

$$D_t^{\alpha} u = x^4 y^4 z^4 + \frac{1}{36}\left[x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right],$$
$$0 < x, y, z < 1,\ 0 < \alpha \le 1, \quad t > 0, \tag{32}$$

subject to the boundary conditions:

$$u(0, y, z, t) = 0,\ u(1, y, z, t) = y^4 z^4 \left(e^t - 1\right),$$
$$u(x, 0, z, t) = 0,\ u(x, 1, z, t) = x^4 z^4 \left(e^t - 1\right),$$
$$u(x, y, 0, t) = 0,\ u(x, y, 1, t) = y^4 z^4 \left(e^t - 1\right),$$

and the initial condition:

$$u(x, y, z, 0) = 0.$$

The exact solution of Equation (31) when $\alpha = 1$ is:

$$u(x, y, z, t) = x^4 y^4 z^4 \left(e^t - 1\right).$$

According to Equation (7), the zeroth-order deformation can be given by:

$$(1-q)L\left(U(x,y,z,t;q) - u_0(x,y,z,t)\right) = q\hbar H(x,y,z,t)\begin{pmatrix} D_\alpha^t U - x^4 y^4 z^4 \\ -\dfrac{1}{36}\left[x^2 (U)_{xx} + y^2 (U)_{yy} + z^2 (U)_{zz}\right] \end{pmatrix} \tag{33}$$

We can start with an initial approximation $u_0(x, y, z, t) = 0$ and we choose the auxiliary linear operator:

$$L\left(U(x, y, z, t; q)\right) = D_\alpha^t U(x, y, z, t; q),$$

with the property:

$$L(C) = 0,$$

where C is an integral constant. We also choose the auxiliary function to be:

$$H(x, y, z, t) = 1.$$

Hence, the mth-order deformation can be given by:

$$L[u_m(x, y, z, t) - \chi_m u_{m-1}(x, y, z, t)] = \hbar H(x, y, z, t) R_m(\vec{u}_{m-1}),$$

where:

$$R_1(\vec{u}_0) = D_\alpha^t(u_0) - x^4 y^4 z^4 - \frac{1}{36}\left[x^2(u_0)_{xx} + y^2(u_0)_{yy} + z^2(u_0)zz\right], \quad (34)$$

$$R_2(\vec{u}_1) = D_\alpha^t(u_1) - \frac{1}{36}\left[x^2(u_1)_{xx} + y^2(u_1)_{yy} + z^2(u_1)zz\right], \quad (35)$$

$$R_3(\vec{u}_2) = D_\alpha^t(u_2) - \frac{1}{36}\left[x^2(u_2)_{xx} + y^2(u_2)_{yy} + z^2(u_2)zz\right], \quad (36)$$

$\vdots$

Now the solution of the mth-order deformation equations (34 to 36) for $m \geq 1$ become:

$$u_m(x, y, z, t) = \chi_m u_{m-1}(x, y, z, t) + \hbar L^{-1}\left[R_m(\vec{u}_{m-1})\right]. \quad (37)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x, y, z, t) = 0,$$

$$u_1(x, y, z, t) = -\hbar x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x, y, z, t) = -\hbar x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2 x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2 x^4 y^4 z^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$\vdots$

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,t) = u_0(x, y, z, t) + u_1(x, y, z, t) + u_2(x, y, z, t) + u_3(x, y, z, t) + \ldots.$$

$$= x^4 y^4 z^4 \left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\ldots\right] \quad (38)$$

For the special case (when $\alpha = 1$), we can reproduce the series solution of (32), and the solution in a closed form:

$$u(x, y, z, t) = x^4 y^4 z^4 \left(e^t - 1\right) \quad (39)$$

follows immediately. Figure 5(a,b,c) show the exact solution (39) and the approximate solution obtained using the homotopy analysis method, for the special case α = 1. Figure 6(a,b) show the approximate solutions when α = 0.5 and α = 0.25, respectively. It is to be noted that only the third-order term of the homotopy analysis solution was used in evaluating the approximate solutions for Figures 5 and 6. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x,y,z\ t)$ when the homotopy analysis method is used.

Example 4

Consider the one-dimensional fractional wave-like equation:

$$D_t^\alpha = \frac{1}{2} x^2 u_{xx}, \quad 0 < x < 1, \; 1 < \alpha \leq 2, \; t > 0, \quad (40)$$

subject to the boundary conditions:

$$u(0,t) = 0, \qquad u(1,t) = 1 + \sinh t,$$

and the initial condition:

$$u(x,0) = x, \qquad u_t(x,0) = x^2$$

The exact solution, for the special case $\alpha = 2$, is given by:

$$u(x,t) = x + x^2 \sinh t \quad (41)$$

We can start with an initial approximation:

$$u_0(x,t) = u(x,0) + t u_t(x,0) = x + x^2 t,$$

We can use similar procedures which was used in Example 1 and the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = x + x^2 t,$$

$$u_1(x,t) = -\hbar x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$u_2(x,t) = -\hbar x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar^2 x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \hbar^2 x^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$\vdots$

Hence, the HAM series solution (for $\hbar = -1$) is:

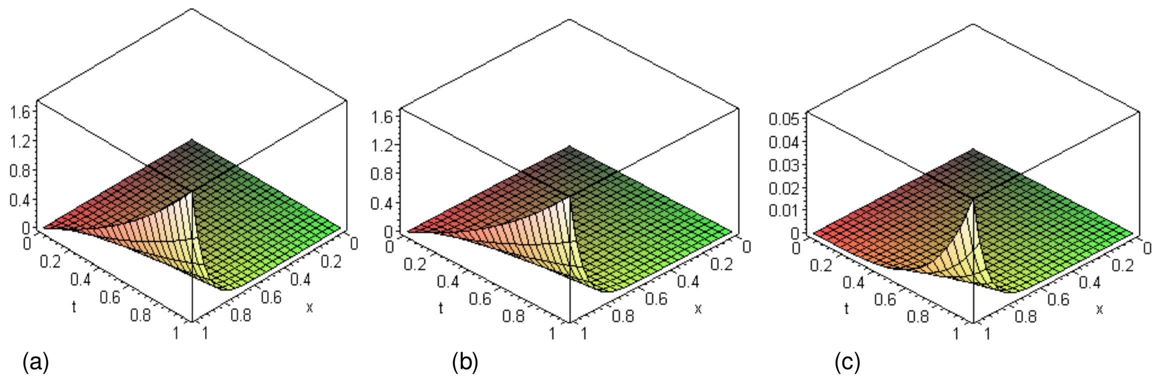


Figure 5. The surface shows the solution *u(x*,y,z, *t)* for Eq.(32) when α = 1 and x = y = z. (a) exact solution (39), (b) approximate solution (38) and (c) $\left|u_{ex} - u_{app}\right|$.

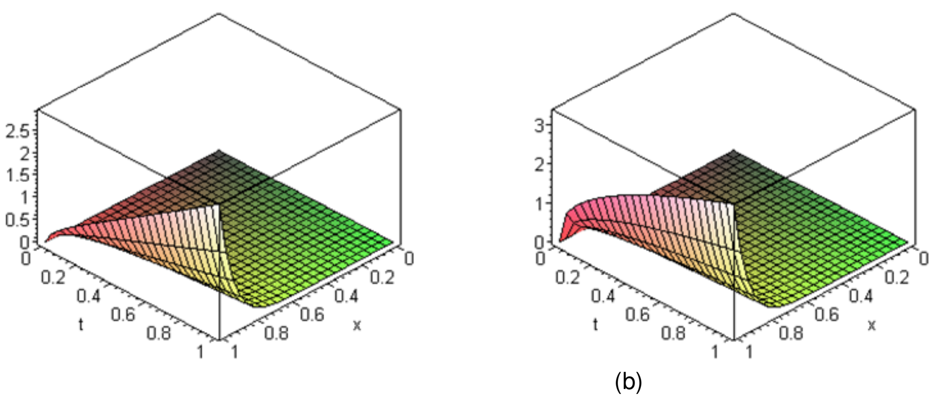


Figure 6. The surface shows the solution *u(x*,y, z,*t)* for equation (32): (a) α = 0.5, (b) α = 0.25.

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots..$$
$$= x + x^2\left[t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \ldots..\right] \quad (42)$$

For the special case (when $\alpha = 2$), we can reproduce the series solution of (40), and the solution in a closed form:

$$u(x,t) = x + x^2 \sinh t \quad (43)$$

follows immediately

Example 5

Next, we consider the two-dimensional fractional wave-like equation:

$$D_t^{\alpha} u = \frac{1}{12}\left[x^2 u_{xx} + y^2 u_{yy}\right], \quad 0 < x, y < 1, \quad 1 < \alpha \le 2,$$
$$t > 0, \quad (44)$$

subject to the boundary conditions:

$$u(0, y, t) = 0,\ u(1, y, t) = 4\cosh t,$$
$$u(x, 0, t) = 0,\ u(x, 1, t) = 4\sinh t,$$

and the initial condition:

$$u(x, y, 0) = x^4,\ u_t(x, y, 0) = y^4.$$

The exact solution, for the special case $\alpha = 2$, is given by:

$$u(x, y, t) = x^4 \cosh t + y^4 \sinh t \quad (45)$$

We can start with an initial approximation:

$$u_0(x,y,t) = u(x,y,0) + t u_t(x,y,0) = x^4 + y^4 t,$$

We can use similar procedures which was used in Example 2 and the first few terms of the HAM series solution are as follows:

$$u_0(x,y,t) = x^4 + y^4 t,$$
$$u_1(x,y,t) = -\hbar\left(x^4 \frac{t^\alpha}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right),$$
$$u_2(x,y,t) = -\hbar\left(x^4 \frac{t^\alpha}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right) - \hbar^2\left(x^4 \frac{t^\alpha}{\Gamma(\alpha+2)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right)$$
$$+\hbar^2\left(x^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + y^4 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right),$$
$\vdots$

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + ...$$
$$= x^4\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) + y^4\left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + ...\right) \quad (46)$$

For the special case (when $\alpha = 2$), we can reproduce the series solution of (44), and the solution in a closed form:

$$u(x,y,t) = x^4 \cosh t + y^4 \sinh t \quad (47)$$

follows immediately.

Example 6

Finally, we consider the three-dimensional fractional wave-like equation of the form:

$$D_t^\alpha u = x^2 + y^2 + z^2 + \frac{1}{36}\left[x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right],$$
$$0 < x, y, z < 1,\ 1 < \alpha \le 2,\ \ t > 0, \quad (48)$$

subject to the boundary conditions:

$$u(0,y,z,t) = y^2(e^t - 1) + z^2(e^{-t} - 1),$$
$$u(1,y,z,t) = (1 + y^2)(e^t - 1) + z^2(e^{-t} - 1),$$
$$u(x,0,z,t) = x^2(e^t - 1) + z^2(e^{-t} - 1),$$
$$u(x,1,z,t) = (1 + x^2)(e^t - 1) + z^2(e^{-t} - 1),$$
$$u(x,y,0,t) = (x^2 + y^2)(e^t - 1),$$
$$u(x,y,1,t) = (x^2 + y^2)(e^t - 1) + (e^{-t} + 1),$$

and the initial condition:

$$u(x,y,z,0) = 0, \qquad u_t(x,y,z,0) = x^2 + y^2 - z^2,$$

The exact solution, for the special case $\alpha = 2,$ is given by:

$$u(x,y,z,t) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^t + z^2 e^{-t} \quad (49)$$

We can start with an initial approximation:

$$u_0(x,y,z,t) = u(x,y,z,0) + t u_t(x,y,z,0) = (x^2 + y^2 - z^2)t$$

We can use similar procedures which was used in Example 3 and the first few terms of the HAM series solution are as follows:

$$u_0(x,y,z,t) = (x^2 + y^2 - z^2)t,$$
$$u_1(x,y,z,t) = -\hbar(x^2 + y^2 + z^2)\frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar(x^2 + y^2 - z^2)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$
,
$$u_2(x,y,z,t) = u_1(x,y,z,t)(1+\hbar) + \hbar^2(x^2 + y^2 - z^2)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \hbar^2(x^2 + y^2 + z^2)\frac{t^{2\varepsilon}}{\Gamma(2\alpha+1)},$$
$\vdots$

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + ...$$
$$= (x^2 + y^2)\left[t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ...\right] + \quad (50)$$
$$z^2\left[-t + \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - .....\right]$$

For the special case (when $\alpha = 2$), we can reproduce the series solution of (48), and the solution in a closed form:

$$u(x,y,z,t) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^t + z^2 e^{-t} \quad (51)$$

follows immediately.

CONCLUSION

The applications of the homotopy analysis method (HAM) were extended successfully for solving the fractional heat-like and wave-like equations with variable coefficients. The homotopy analysis method was clearly very efficient and powerful technique in finding the solutions of the proposed equations. The procedure presented to solve the fractional heat-like and wave-like equations is the same as those for standard heat-like and wave-like equations, and in special cases of $\alpha = 1$ and 2, the general solution reduces to the heat-like and wave-like solutions.

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