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Full Length Research Paper

A numeric-analytic method for approximating three-species food chain models

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This paper investigates the accuracy of the differential transformation method (DTM) for solving the three-species food chain models which is described as three-dimensional system of ODES with quadratic and rational nonlinearities. Numerical results are compared to those obtained by the fourth-order Runge-Kutta method to illustrate the preciseness and effectiveness of the proposed method. The direct symbolic-numeric scheme is indicated to be efficient and accurate.

Key words: Three-species food chain models, multi-step differential transformation method, attractors.

INTRODUCTION

In this paper, we consider two different three-species food chain model: Model with a Holling type II functional response (Hastings and Powell, 1991; Varriale and Gomes, 1998; Gomes et al., 2008) and model with a Beddington-DeAngelis functional response (Li et al., 2006; Cantrell and Consner 2001; Gakkhar and Naji, 2003; Naji and Balasim 2007; Hwang, 2003; Hwang, 2004; Zhang et al., 2006 and Zhao and Lv, 2009).

Model with a Holling type II functional response is described by the following differential equation system:

$$\frac{dX}{dT} = R_0 X\left(1-\frac{X}{K_0}\right) - C_1 F_1(X)Y,$$
$$\frac{dY}{dT} = F_1(X)Y - F_2(Y)Z - D_1 Y, \tag{1}$$
$$\frac{dZ}{dT} = C_2 F_2(Y)Z - D_2 Z,$$

where X, Y, Z represent, respectively, young tilapia (prey), developed tilapia (predator), tucunare fish (top-opredator); $F_i(U) = \frac{A_i U}{B_i + U}, i = 1,2$ are Holling type II functional responses (Holling, 1965) with B_i, the half – saturation constant, satisfying $[F_i(U)]_{U=B_i} = \frac{A_i}{2}, i = 1,2,$ that is, the value of prey density U at which the per capita removal rate of U is half maximal; R_0 and K_0 are, respectively, the intrinstic growth rate and carrying capacity of the environment of the fish farm for the prey species; C_1^{-1} and C_2 are conversion factors of prey-to-predator; D_1 are D_2 are the death rates for Y and Z, respectively.

Using the same change of variables as in Hastings and Powell (1991), the dimensionless version of the model becomes

$$\frac{dx}{dt} = x(1-x) - \frac{a_1 x}{1+b_1 x}y,$$
$$\frac{dy}{dt} = \frac{a_1 x}{1+b_1 x}y - \frac{a_2 y}{1+b_2 y}z - d_1 y, \tag{2}$$
$$\frac{dz}{dt} = \frac{a_2 y}{1+b_2 y}z - d_2 z,$$

where x, y and z are the dimensionless population variables; t is the dimensionless time variable;

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$a_1, a_2, b_2, d_1, d_2, b_1 = \frac{K_0}{B_1}$ are dimensionless parameters. Then, as in Hastings and Powell (1991), we take $a_1 = 5, a_2 = 0.1, b_2 = 2, d_1 = 0.4, d_2 = 0.01$ and $b_1 = 2.75$.

Model with a Beddington-DeAngelis functional response is described by following differential equation system:

$$\frac{dX}{dT} = \frac{R_0(G_0 - X)X}{K_1 - X} - F_1(X,Y)Y,$$
$$\frac{dY}{dT} = E_1F_1(X,Y)Y - F_2(Y,Z)Z - I_1Y, \tag{3}$$
$$\frac{dZ}{dT} = E_2F_2(Y,Z)Z - I_2Z$$

with

$$F_i(U,V) = \frac{A_iU}{B_iV + C_iU + D_i}, i = 1,2. \tag{4}$$

The functions $F_1(X,Y)$ and $F_2(Y,Z)$ present a functional response of Beddington-DeAngelis type. A_i, B_i and $C_i, i = 1,2$ are the saturating parameters of the two responses. $\frac{A_i}{D_i}$ is the maximum harvest rate of predator from prey U, $\frac{D_i}{B_i}$ and $\frac{D_i}{C_i}$ are the half saturation constants, $i = 1,2$. The parameters $R_0, G_0, G_1, A_i, B_i, C_i, D_i, E_i$ and $I_i, i = 1,2$ are all positive constants. R_0 is the intrinsic growth rate of species X and R_0G_0 is its carrying capacity. G_1 is the limiting value of resources. E_1 and E_2 are the conversion rates of prey to predator for species Y and Z, respectively; I_1 and I_2 are death rates of species Y and Z respectively.

Using the same change of variables as in Zhao and Lv (2009), the dimensionless version of the model becomes

$$\frac{dx}{dt} = \frac{r_0(K_0 - x)x}{K_1 - x} - \frac{a_1xy}{a_2y + a_3x + 1},$$
$$\frac{dy}{dt} = \frac{a_4xy}{a_2y + a_3x + 1} - \frac{a_5yz}{a_6z + a_7y + a_8} - y \tag{5}$$
$$\frac{dz}{dt} = \frac{a_9yz}{a_6z + a_7y + a_8} - a_{10}z$$

where

$$r_0 = \frac{R_0}{I_1}, K_0 = \frac{G_0}{D_1}, K_1 = \frac{G_1}{D_1}, a_1 = \frac{A_1}{I_1}, a_2 = B_1, a_3 = C_1$$

$$a_4 = \frac{A_1E_1}{I_1}, a_5 = \frac{A_2}{I_1}, a_6 = B_2, a_7 = C_2, a_8 = \frac{D_2}{D_1}, a_9 = \frac{A_2E_2}{I_1}, a_{10} = \frac{I_2}{I_1}.$$

Then, as in Zhao and Lv, 2009, we take $a_1 = 5, a_2 = 0.1, b_2 = 2, d_1 = 0.4, d_2 = 0.01$ and $b_1 = 2.75$.

When dealing with nonlinear systems of ordinary differential equations, such as the chaotic three-species food chain models, it is often difficult to obtain a closed form of the analytic solution. In the absence of such a solution, the accuracy of the DTM method is then tested against classical numerical methods, such as the Runge–Kutta method (RK4). RK4 has been widely and commonly used for simulating solutions for chaotic systems (Lu et al., 2002; Yassen, 2003; Park, 2006a, b).

The goal of this paper is to extend application to classical DTM and multi-step DTM for obtained approximant analytical solution of the aboved mentioned three-species food chain models. The differential transform method (DTM) was first proposed by Zhou (1986). See the references, Ayaz (2004a), Ayaz (2004b), Kanth and Aruna (2009), Odibat et al. (2010) and Al-sawalha et al. (2009), for development of DTM. This technique has been employed to solve a large variety of linear and nonlinear problems. For more applications of the differential transformation method and other semi-analytical methods in various problems of physics and engineering see the following references (Rashidi et al., 2010, 2009; Yeh et al., 2006; Kuo, 2005; Ebaid, 2010; Yalcin et al., 2009; Bert and Zeng, 2004; Arenas et al., 2009; Noor and Mohyud, 2008; Mohyud-Din et al., 2009a; Mohyid-Din and Noor, 2009b; Mohyud-Din, 2009c; Yildirim et al., 2010; Merdan and Gokdogan, 2011; Gökdogan and Merdan, 2010).

METHODOLOGY

Differential transformation method

Consider a general system of first-order ODES

$$\frac{dx_1}{dt} + h_1(t, x_1, x_2, \dots, x_m) = g_1(t),$$
$$\frac{dx_2}{dt} + h_2(t, x_1, x_2, \dots, x_m) = g_2(t), \tag{6}$$

subject to the initial conditions

$$x_1(t_0) = d_1, \quad x_2(t_0) = d_2, \cdots, x_m(t_0) = d_m. \tag{7}$$

To illustrate the differential transformation method (DTM) for solving differential equations systems, the basic definitions of differential transformation are introduced as follows. Let $x(t)$ be analytic in a domain D and let $t = t_0$ represent any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_0. The differential transformation of the *k* th derivative of a function $x(t)$ is defined as follows:

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \forall t \in D \tag{8}$$

In (7), $x(t)$ is the original function and $X(k)$ is the transformed function. As in Zhou (1986), Ayaz (2004a), Ayaz (2004b), Kanth and Aruna (2009), Odibat et al. (2010) and Al-sawalha et al. (2009) the differential inverse transformation of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_i)^k, \forall t \in D \tag{9}$$

From (7) and (8), we obtain

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \forall t \in D \tag{10}$$

The fundamental theorems of the one-dimensional differential transform are:

Theorem 1: If $z(t) = x(t) \pm y(t)$, then $Z(k) = X(k) \pm Y(k)$.

Theorem 2: If $z(t) = cy(t)$, then $Z(k) = cY(k)$.

Theorem 3: If $z(t) = \frac{dy(t)}{dt}$, then $Z(k) = (k+1)Y(k+1)$.

Theorem 4: If $z(t) = \frac{d^n y(t)}{dt}$, then $Z(k) = \frac{(k+n)!}{k!}Y(k)$.

Theorem 5: If $z(t) = x(t)\,y(t)$, then $Z(k) = \sum_{k_1=0}^{k} X(k_1)Y(k-k_1)$.

Theorem 6: If $z(t) = t^n$, then $Z(k) = \delta(k-n) = \begin{cases}1 & k = n\\ 0 & k \neq n\end{cases}$.

Theorem 7: If $z(t) = \frac{F(x(t),y(t))}{ax(t)+by(t)+c}$, then

$$Z(k) = \frac{F(X(k),Y(k)) - a\sum_{k_1=1}^{k} X(k_1)Z(k-k1) - b\sum_{k_1=1}^{k} Y(k_1)Z(k-k1)}{(aX(0)+bY(0)+c)}.$$

Theorem 8: If $z'(t) = \frac{F(x(t),y(t))}{ax(t)+by(t)+c}$, then

$$Z(k+1) = \frac{F(X(k),Y(k)) - a\sum_{k_1=1}^{k}(k-k1+1)X(k_1)Z(k-k1+1) - b\sum_{k_1=1}^{k}(k-k1+1)Y(k_1)Z(k-k1+1)}{(aX(0)+bY(0)+c)(k+1)}$$

In real applications, the function $x(t)$ is expressed by a finite series and (9) can be written as

$$x(t) = \sum_{k=0}^{N} X(k)(t-t_i)^k, \forall t \in D \tag{11}$$

Equation (10) implies that

$$\sum_{k=N+1}^{\infty} X(k)(t-t_i)^k$$

is negligibly small.

According to DTM, by taking differential transforms, both sides of the systems of equations given Equations (6) and (7) is transformed as follows:

$$(k+1)X_1(k+1) + H_1(k) = G_1(k),$$
$$(k+1)X_2(k+1) + H_2(k) = G_2(k), \tag{12}$$
$$\vdots$$
$$(k+1)X_m(k+1) + H_m(k) = G_m(k). \tag{13}$$

Therefore, according to DTM the n –term approximations for (1) can be expressed as

$$\varphi_{1,n}(t) = x_1(t) = \sum_{k=1}^{n} X_1(k)t^k,$$
$$\varphi_{2,n}(t) = x_2(t) = \sum_{k=1}^{n} X_2(k)t^k, \tag{14}$$
$$\vdots$$
$$\varphi_{m,n}(t) = x_m(t) = \sum_{k=1}^{n} X_m(k)t^k.$$

Multi-step differential transformation method

The approximate solutions (5) are generally, as will be shown in the numerical experiments of this paper, not valid for large t. A simple way of ensuring validity of the approximations for large t is to treat (11)-(12) as an algorithm for approximating the solutions of (5)-(6) in a sequence of intervals choosing the initial approximations as

$$x_{1,0}(t) = x_1(t^*) = d_1^*,$$
$$x_{2,0}(t) = x_2(t^*) = d_2^*,$$
$$\vdots$$
$$x_{m,0}(t) = x_m(t^*) = d_m^*. \tag{15}$$

In order to carry out the iterations in every subinterval $[0,t_1), [t_1,t_2), [t_2,t_3), \ldots, [t_{j-1},t)$ of equal length h, we would need to know the values of the following (Odibat et al., 2010),

$$x_{1,0}^*(t) = x_1(t^*), \quad x_{2,0}^*(t) = x_2(t^*), \quad \ldots, \quad x_{m,0}^*(t) = x_m(t^*). \tag{16}$$

But, in general, we do not have these information at our clearance except at the initial point $t^* = t_0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{m,n}$ of the preceding subinterval,

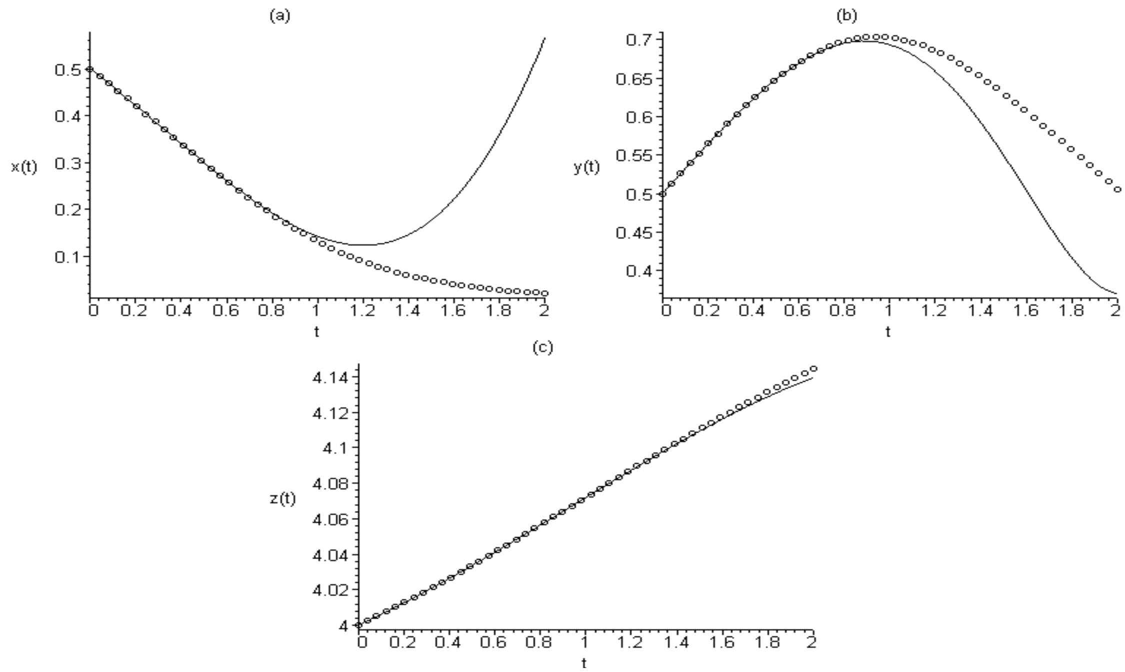

Figure 1. Local changes of x, y, z for 5-term DTM(line) and RK4 with time step $h = 0.001$(circle).

that is,

$$x_{1,0}^{*} \cong \varphi_{1,n}(t^{*}), x_{2,0}^{*} \cong \varphi_{2,n}(t^{*}), \ldots, x_{m,0}^{*} \cong \varphi_{m,n}(t^{*}). \tag{17}$$

RESULTS AND DISCUSSION

The model with Holling type II functional response

Taking the differential transformation of Equation (2) with respect to time *t* gives

$$X(k+1) = \left[X(k) + (b_1 - 1)\sum_{k_1=0}^{k} X(k_1)X(k-k1) - b_1 \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X(k_1)X(k2-k1)X(k-k2) - a_1 \sum_{k_1=0}^{k} X(k_1)Y(k-k1) - b_1 \sum_{k_1=1}^{k} (k-k_1+1)X(k_1)X(k-k1+1)\right] \Big/ \left((1+b_1X(0))(k+1)\right) \tag{18}$$

$$-a_2 \sum_{k_1=0}^{k} Y(k_1)Z(k-k1) + (a_1b_2 - b_1b_2d_1)\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X(k_1)Y(k2-k1)Y(k-k2)$$

$$Y(k+1) = \Bigg[-d_1 * Y(k) + (a_1 - b_1d_1)\sum_{k_1=0}^{k} X(k_1)Y(k-k1) - b_2d_1 \sum_{k_1=0}^{k} Y(k_1)Y(k-k1)$$

$$-a_2b_1 \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X(k_1)Y(k2-k1)Z(k-k2) - b_2 \sum_{k_1=1}^{k} (k-k_1+1)Y(k_1)Y(k-k1+1)$$

$$-b_1b_2 \sum_{k_2=1}^{k}\sum_{k_1=0}^{k_2} (k-k_2+1)X(k_1)Y(k2-k1)Y(k-k2+1)$$

$$-b_1 \sum_{k_1=1}^{k} (k-k_1+1)X(k_1)Y(k-k1+1)\Bigg] \Big/ \left((1+b_1X(0)+b_2Y(0)+b_1b_2X(0)Y(0))(k+1)\right) \tag{19}$$

$$Z(k+1) = \Bigg[-d_2Z(k) + (a_2 - d_2)\sum_{k_1=0}^{k} Y(k_1)Z(k-k1) -$$

$$-b_2 \sum_{k_1=1}^{k} (k-k_1+1)Y(k_1)Z(k-k1+1)\Bigg] \Big/ \left((1+b_2Y(0))(k+1)\right) \tag{20}$$

where $X(k), Y(k)$ and $Z(k)$ are the differential transformations of the corresponding functions $x(t), y(t)$ and $z(t)$, respectively, and the initial conditions are given by $X(0) = 0.5, Y(0) = 0.5$ and $Z(0) = 4$.

For the above iterative system, we used a 5-term DTM, 5 term MsDTM with time step $h = 0.1, h = 0.01$ and RK4 with $h = 0.001$. Figures 1 and 2 show results obtanied from 5-term DTM and 5-term MsDTM with $h = 0.1$, respectively. It is not difficult to see that DTM is valid for small t. But, the MsDTM and RK4 solutions show good synchronization at time performed.

Table 1 shows the numerical outputs for 5-term MsDTM and RK4 for time span $[0,100]$. Figure 3 visually

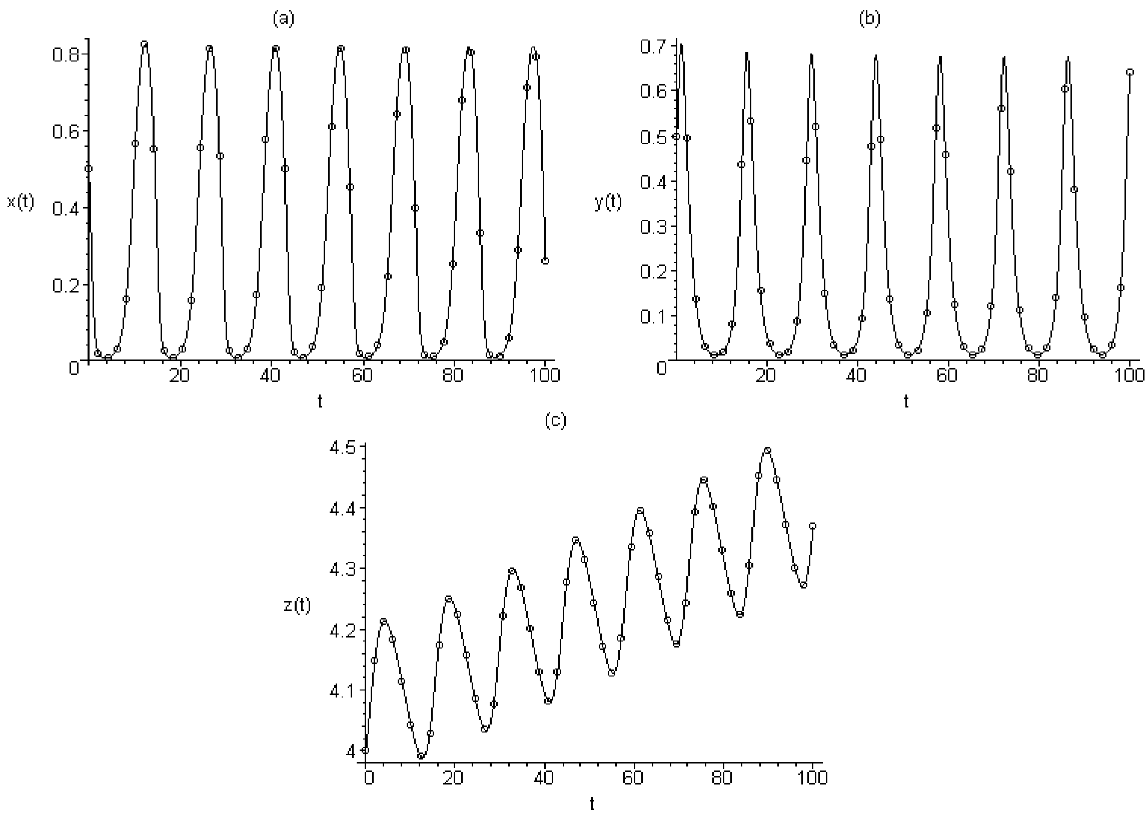

Figure 2. Local changes of x, y, z for 5-term multi-step DTM with $\Delta t = 0.1$(line) and RK4 with $h = 0.001$(circle).

Table 1. Numerical comparisons between the 5-term MsDTM and RK4 solutions.

t	$\Delta = \lvert MsDTM_{0.1} - RK4_{0.001} \rvert$			$\Delta = \lvert MsDTM_{0.01} - RK4_{0.001} \rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
5	1.430e-09	8.420e-09	1.300e-08	1.800e-10	2.000e-10	8.000e-09
10	7.100e-09	3.370e-09	1.200e-08	1.710e-08	2.500e-10	7.800e-08
15	2.800e-09	7.090e-08	3.700e-08	2.330e-08	1.960e-08	6.500e-08
20	1.033e-08	4.340e-09	1.700e-08	1.910e-09	3.200e-09	7.800e-08
25	7.790e-08	1.669e-08	1.800e-08	9.700e-09	5.600e-10	1.120e-07
30	2.017e-07	6.200e-09	1.000e-07	1.130e-08	2.030e-08	1.380e-07
35	3.772e-08	1.295e-08	2.300e-08	4.130e-09	1.820e-09	1.290e-07
40	9.090e-08	5.035e-08	4.100e-08	1.340e-08	1.144e-08	1.560e-07
45	1.385e-08	3.491e-07	1.550e-07	1.640e-09	1.860e-08	1.980e-07
50	1.124e-07	8.930e-09	3.500e-08	9.030e-09	4.060e-09	1.700e-07
55	3.630e-08	1.487e-07	6.900e-08	5.400e-09	2.086e-08	2.110e-07
60	3.110e-09	3.001e-07	1.440e-07	7.300e-10	1.960e-08	2.550e-07
65	3.263e-07	3.340e-09	3.700e-08	6.040e-08	7.800e-10	2.250e-07
70	3.990e-07	4.833e-07	2.030e-07	1.430e-08	1.170e-08	2.660e-07
75	1.423e-08	3.312e-07	1.700e-07	3.780e-09	6.040e-08	3.120e-07
80	7.736e-07	1.770e-08	3.400e-08	9.600e-08	1.037e-08	3.250e-07
85	1.340e-06	1.274e-06	4.220e-07	3.283e-07	2.508e-07	4.600e-07
90	4.594e-08	2.693e-07	1.350e-07	1.893e-08	4.923e-08	3.970e-07
95	1.128e-06	5.591e-08	2.700e-08	3.797e-07	2.472e-08	3.740e-07
100	1.737e-06	7.645e-07	6.260e-07	6.938e-07	1.327e-07	6.260e-07

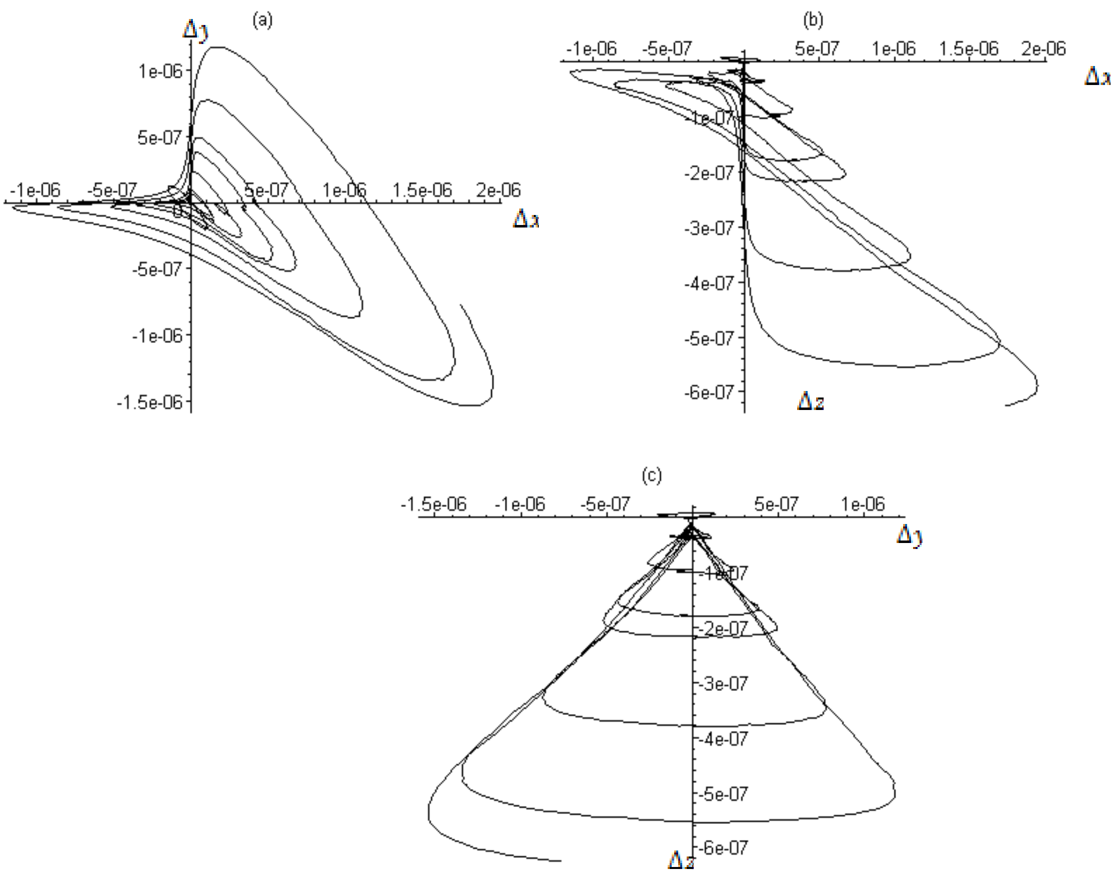

Figure 3. Differences between the 5-term MsDTM with $\Delta t = 0.1$ and RK4 with $h = 0.001$.

displays plots of various differences between 5-term MsDTM with $\Delta t = 0.1$ and RK4 on time step $h = 0.001$ for each variable. They indicates that the result of MsDTM are in agreement with those obtained by the RK4 method (Figure 4).

The model with Beddington-DeAngelis functional response

Taking the differential transformation of Equation (2) with respect to time *t* gives

$$X(k+1) = \Bigg[(r_0 K_0 a_2 - a_1 K_1) \sum_{k_1=0}^{k} X(k_1) Y(k-k1) + (r_0 K_0 a_3 - r_0) \sum_{k_1=0}^{k} X(k_1) X(k-k1)$$

$$+ (a_1 - r_0 a_2) \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1) X(k2-k1) X(k-k2) - r_0 a_3 \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} X(k_1) X(k2-k1) X(k-k2)$$

$$- a_2 K_1 \sum_{k_1=1}^{k} (k-k_1+1) Y(k_1) Y(k-k_1+1) - (a_3 K_1 - 1) \sum_{k_1=1}^{k} (k-k_1+1) X(k_1) X(k-k_1+1)$$

$$+ a_2 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k-k_2+1) X(k_1) Y(k2-k1) X(k-k2+1)$$

$$+ a_3 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k-k_2+1) X(k_1) X(k2-k1) X(k-k2+1) \Bigg] \Big/ \Big((k+1)(K_1 + (a_3 K_1 - 1) X(0) \qquad (21)$$

$$+ a_2 K_1 Y(0) - a_2 X(0) Y(0) - a_3 X(0)^2) \Big)$$

$$Y(k+1) = \Bigg[(a_4 a_6 - a_3 a_6 - a_5 a_3) \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} X(k_1) Y(k2-k1) Z(k-k2)$$

$$+ (a_4 a_7 - a_3 a_7) \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} X(k_1) Y(k2-k1) Y(k-k2)$$

$$- (a_5 a_2 + a_2 a_6) \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Z(k_1) Y(k2-k1) Y(k-k2) - a_2 a_7 \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1) Y(k2-k1) Y(k-k2)$$

$$+ (a_4 a_8 - a_3 a_8) \sum_{k_1=0}^{k} X(k_1) Y(k-k1) - (a_5 + a_6) \sum_{k_1=0}^{k} Z(k_1) Y(k-k1)$$

$$- (a_2 a_8 + a_7) \sum_{k_1=0}^{k} Y(k_1) Y(k-k1) - a_8 Y(k) - a_6 \sum_{k_1=1}^{k} (k-k_1+1) Z(k_1) Y(k-k1+1)$$

$$- a_3 a_8 \sum_{k_1=1}^{k} (k-k_1+1) X(k_1) Y(k-k1+1) - (a_2 a_8 + a_7) \sum_{k_1=1}^{k} (k-k_1+1) Y(k_1) Y(k-k1+1)$$

$$- a_2 a_6 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k-k2+1) Y(k_1) Z(k2-k1) Y(k-k2+1)$$

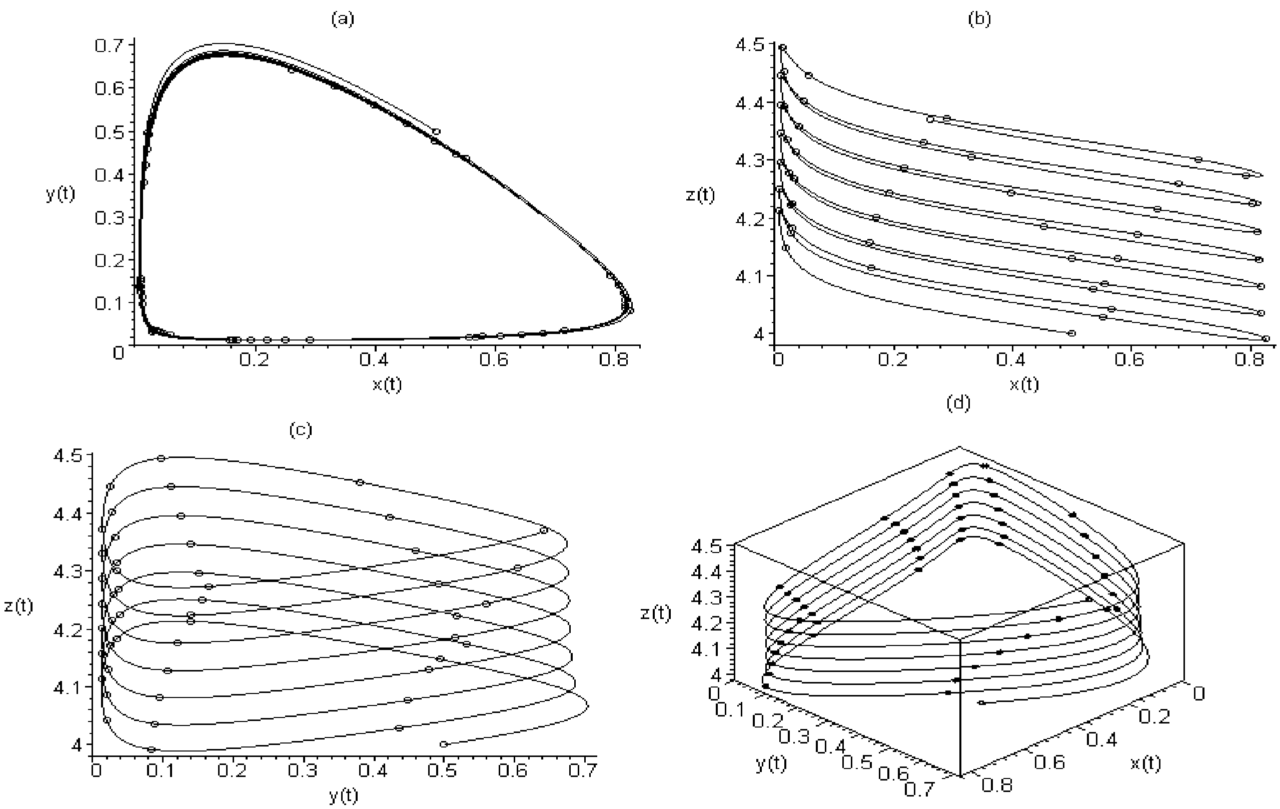

Figure 4. Phase portraits for 5-term multi-step DTM with $\Delta t = 0.1$(line) and RK4 with $h = 0.001$(circle), with time span $[0,100]$.

$$-a_3 a_6 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k - k2 + 1) X(k_1) Z(k2 - k1) Y(k - k2 + 1)$$

$$-a_3 a_7 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k - k2 + 1) X(k_1) Y(k2 - k1) Y(k - k2 + 1)$$

$$-a_2 a_7 \sum_{k_2=1}^{k} \sum_{k_1=0}^{k_2} (k - k2 + 1) Y(k_1) Y(k2 - k1) Y(k - k2 + 1) \Bigg] \Big/ \Big((k+1)(a_2 a_6 Y(0) Z(0)$$

$$+a_2 a_7 Y(0)^2 + (a_2 a_8 + a_7) Y(0) + a_3 a_6 X(0) Z(0) + a_3 a_7 X(0) Y(0) + a_3 a_8 X(0) + a_6 Z(0) + a_8)\Big) \quad (22)$$

$$Z(k+1) = \Bigg[(a_9 - a_7 a_{10}) \sum_{k_1=0}^{k} Y(k_1) Z(k - k1) - a_{10} a_8 Z(k) - a_6 a_{10} \sum_{k_1=0}^{k} Z(k_1) Z(k - k1)$$

$$-a_6 \sum_{k_1=0}^{k} (k - k_1 + 1) Z(k_1) Z(k - k1 + 1)$$

$$-a_7 \sum_{k_1=0}^{k} (k - k_1 + 1) Y(k_1) Z(k - k1 + 1) \Bigg] \Big/ \Big((k+1)(a_8 + a_7 Y(0) + a_6 Z(0))\Big) \quad (23)$$

where $X(k), Y(k)$ and $Z(k)$ are the differential transformations of the corresponding functions $x(t), y(t)$ and $z(t)$, respectively, and the initial conditions are given by $X(0) = 0.1, Y(0) = 0.1$ and $Z(0) = 0.1$.

For the above iterative system, we used a 5-term DTM, 5 term MsDTM with time step $\Delta t = 0.1, \Delta t = 0.01$ and RK4 with $h = 0.001$. Figures 5 and 6 show results obtanied from 5-term DTM and 5-term MsDTM with $h = 0.1$, respectively. It is not difficult to see that DTM is valid for small t. But, the MsDTM and RK4 solutions show good synchronization at time performed.

Table 2 indicate the numerical outputs for 5-term MsDTM and RK4 for time span $[0,100]$. Figure 7 visually displays plots of various differences between 5-term MsDTM with $\Delta t = 0.1$ and RK4 on time step $h = 0.001$ for each variable. They indicates that the result of MsDTM are excellent in agreement with those obtained by the RK4 method. Observation shows that the accuracy between both time steps used are considered very precise. We could also see that a smaller time step ($h = 0.01$) exhibits a small maximum error. At the same time, increasing the number of iteration will also help

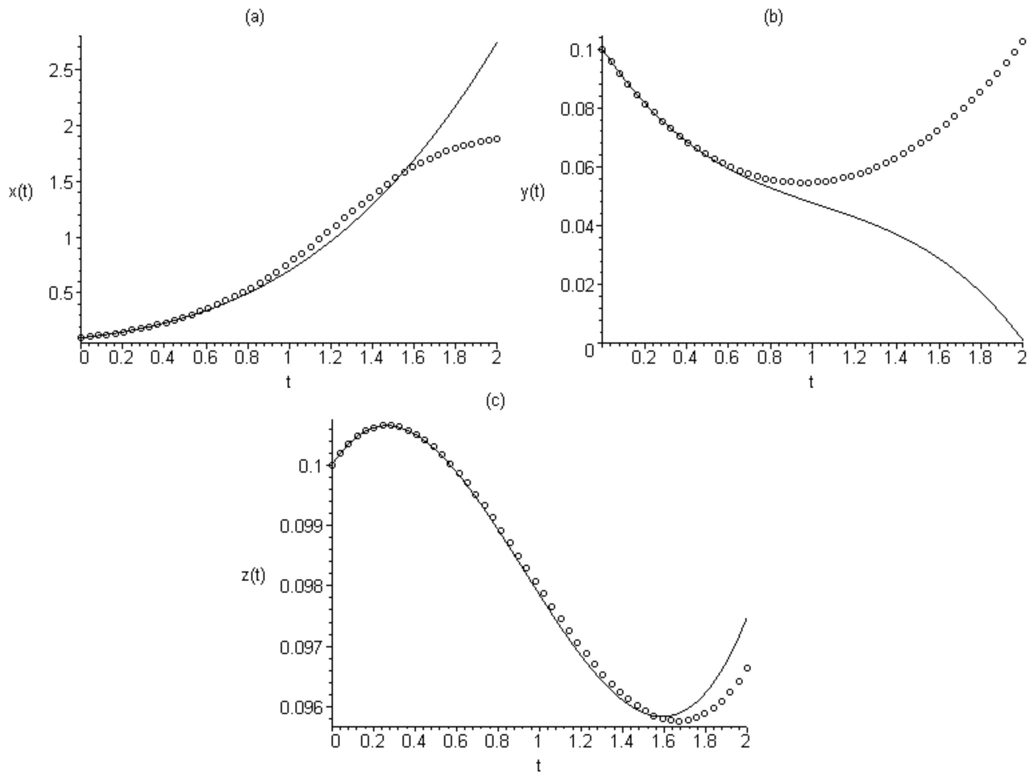

Figure 5. Local changes of x, y, z for 5-term DTM(line) and RK4 with $h = 0.001$ (circle).

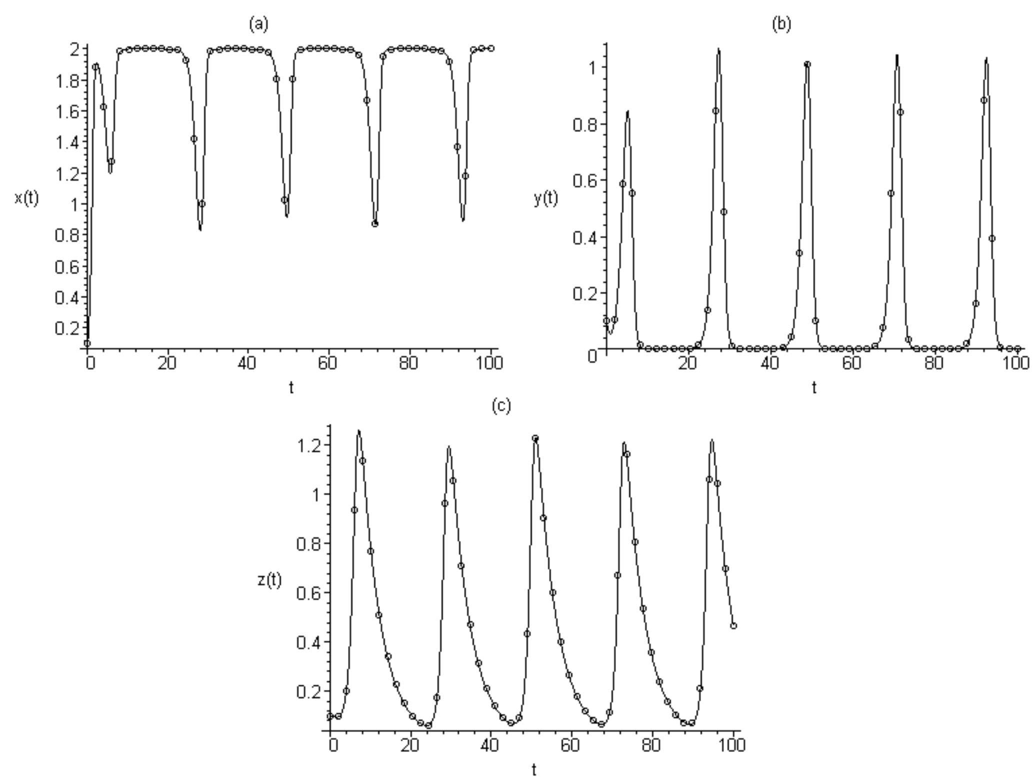

Figure 6. Local changes of x, y, z for 5-term multi-step DTM with $\Delta t = 0.1$(line) and RK4 with $h = 0.001$(circle).

Table 2. Numerical comparisons between the 5-term MsDTM and RK4 solutions.

t	$\Delta = \lvert MsDTM_{0.1} - RK4_{0.001} \rvert$			$\Delta = \lvert MsDTM_{0.01} - RK4_{0.001} \rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
5	4.575e-05	4.365e-05	4.064e-05	6.400e-08	4.510e-08	2.380e-08
10	8.300e-07	9.382e-07	1.175e-05	2.000e-09	9.681e-10	1.690e-08
15	8.200e-08	1.401e-07	3.328e-06	5.000e-09	1.411e-10	5.700e-09
20	1.603e-06	3.058e-06	1.700e-09	4.000e-09	2.733e-09	2.700e-09
25	2.630e-04	4.531e-04	7.097e-05	2.520e-07	4.419e-07	6.838e-08
30	3.338e-04	1.737e-04	1.476e-04	2.960e-07	1.554e-07	1.700e-07
35	3.100e-08	3.146e-08	1.423e-04	1.000e-09	4.060e-11	1.354e-07
40	3.780e-07	7.143e-07	5.203e-05	1.000e-09	9.938e-10	4.660e-08
45	5.136e-05	9.879e-05	4.018e-06	6.400e-08	1.237e-07	6.510e-09
50	1.041e-03	1.191e-03	1.064e-03	1.493e-06	1.605e-06	1.548e-06
55	4.900e-08	2.795e-08	2.360e-04	1.000e-09	3.641e-10	2.696e-07
60	1.990e-07	3.599e-07	8.646e-05	8.000e-09	1.919e-10	1.039e-07
65	1.205e-05	2.342e-05	2.647e-05	7.000e-09	1.550e-08	2.945e-08
70	1.508e-03	1.380e-03	6.004e-04	1.258e-06	1.173e-06	4.366e-07
75	9.798e-06	1.007e-05	5.491e-04	1.100e-08	8.488e-09	4.575e-07
80	5.500e-08	1.039e-07	2.076e-04	5.000e-09	6.800e-12	1.493e-07
85	2.552e-06	4.899e-06	7.477e-05	3.000e-09	3.173e-09	6.590e-08
90	4.427e-04	7.841e-04	8.323e-05	2.690e-07	4.722e-07	4.056e-08
95	8.836e-04	4.263e-04	3.951e-04	5.230e-07	2.259e-07	3.690e-07
100	2.700e-08	1.361e-08	3.553e-04	2.000e-09	2.893e-10	3.623e-07

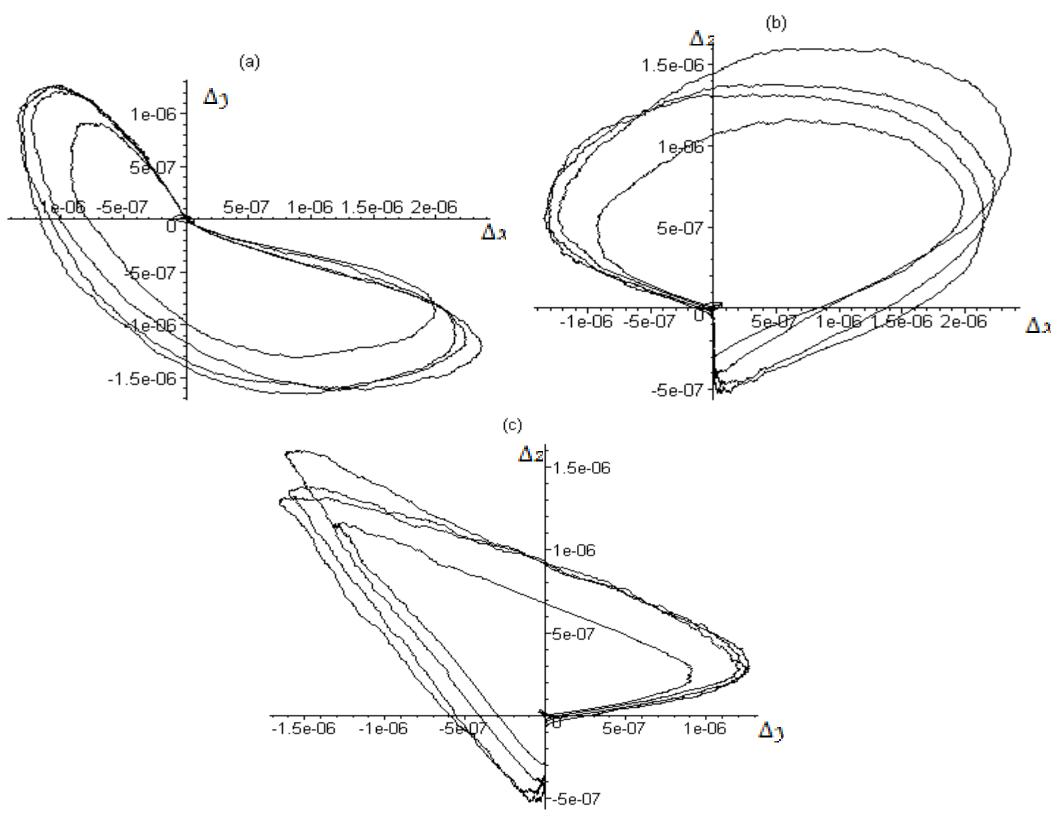

Figure 7. Differences between the 5-term MsDTM with $\Delta t = 0.1$ and RK4 with $h = 0.001$.

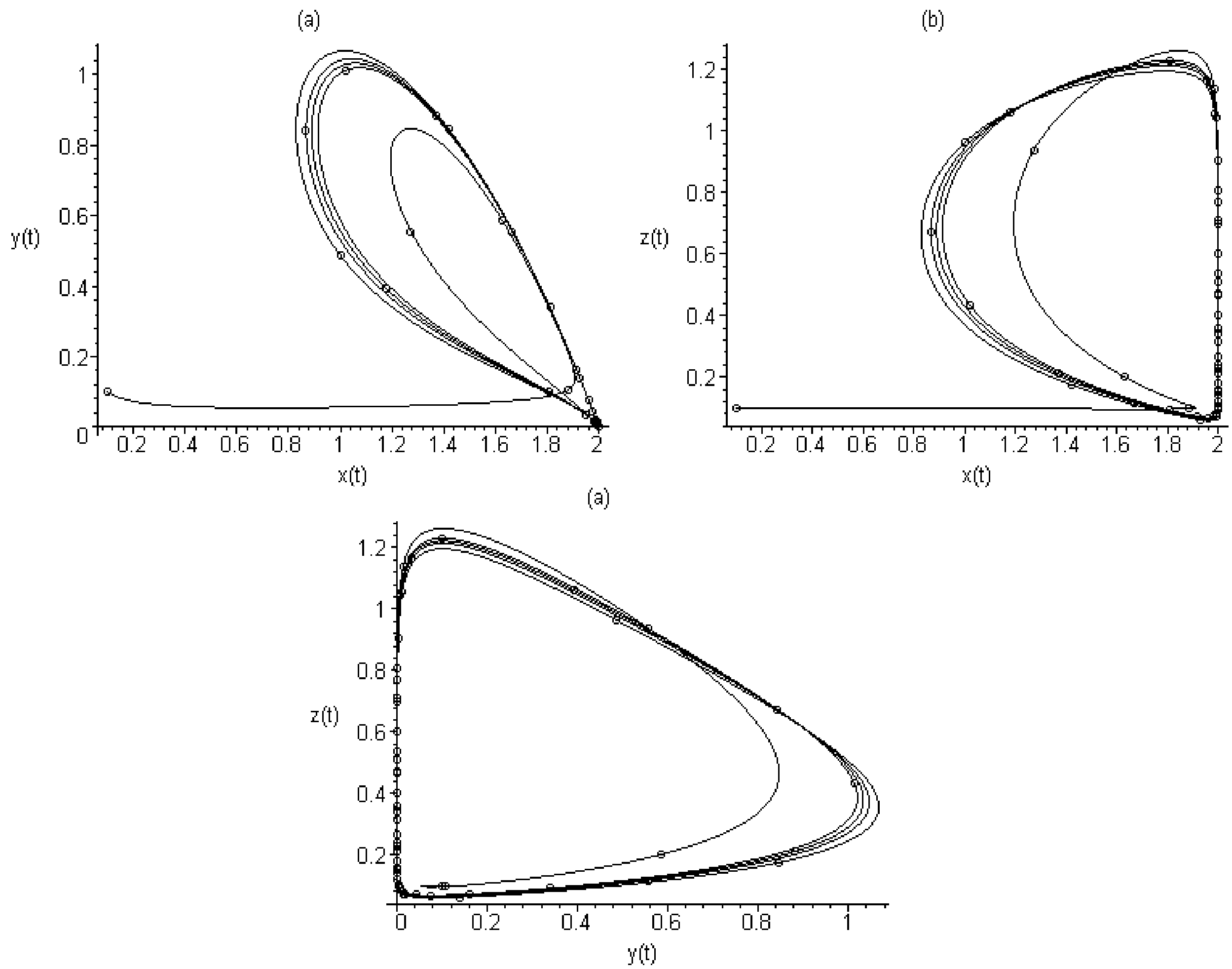

Figure 8. Phase portraits for 5-term multi-step DTM with $h = 0.1$(line) and RK4 with $h = 0.001$(circle), with time span $[0,100]$.

enhance the accuracy level. We do note, however, that the results displayed by the chaotic case is less accurate compared to the non-chaotic case. This is due to the fact that its chaotic state has sensitive dependence on initial conditions.

Figure 7 visually displays plots of various differences between MsDTM with time steps $\Delta t = 0.1$ and RK4 on time step $h = 0.001$ for each state variable.

In Figure 8, we have displayed typical phase portraits for young tilapia (prey) population $x(t)$, developed tilapia (predator)population, $y(t)$and tucunare fish (top-predator) population z(t) of the system (2) with initial values $X(0) = (0.1,0.1,0.1)$.

Period doubling bifurcation leads to chaos of system (2) with initial values X(0) = (0.1, 0.1, 0.1),(a) Quasi-periodic solutions for $K_0 = 2$, (b) phase portrait of 2T-period solution for $K_0 = 2.5$, (c) phase portrait of 4T-period solution for $K_0 = 2.80$ and (d) phase portrait of 8T-period solution for $K_0 = 2.85$ in Figure 9 .

Conclusion

In this work, we carefully apply the multi-step DTM, a reliable modification of the DTM, that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. The validity of the proposed method has been successful by applying it for the chaotic three-species food chain models. The method were used in a direct way without using linearization, perturbation or restrictive assumptions. It provides the solutions in terms of convergent series with easily computable components and the results have shown remarkable performance.

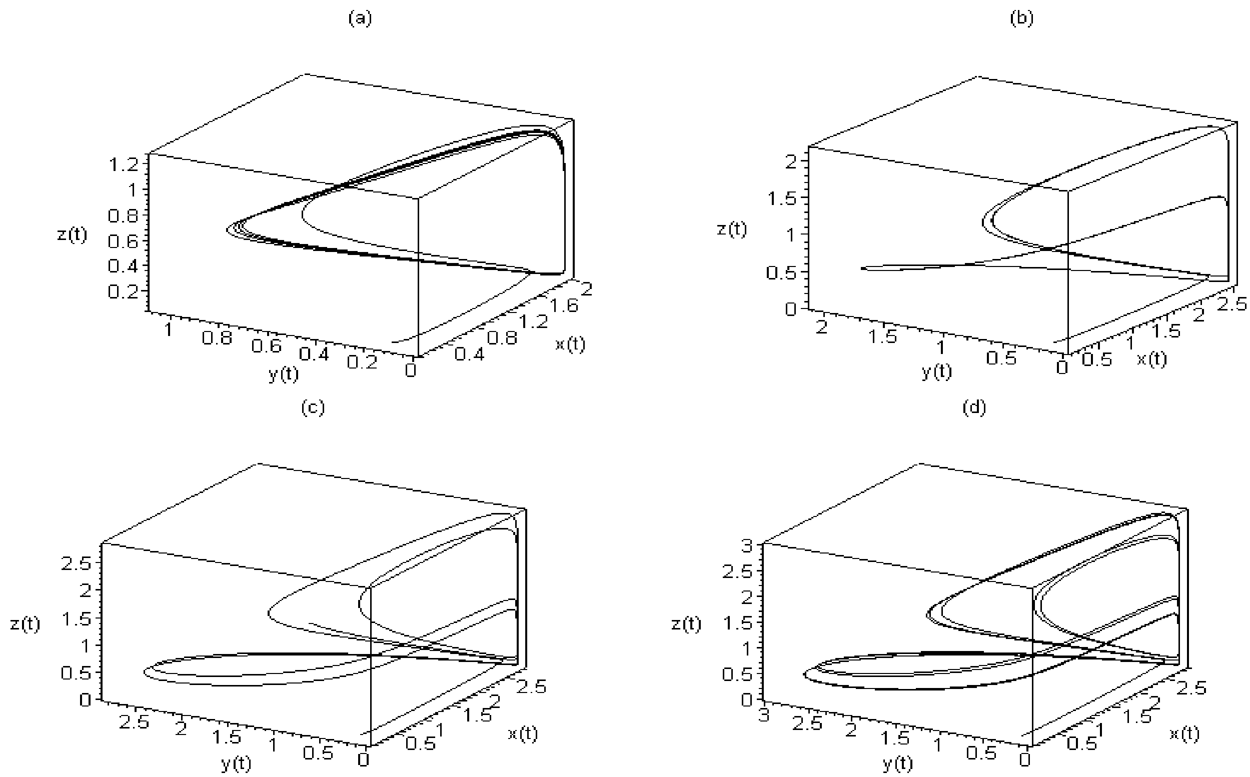

Figure 9. Attractors for 5-term multi-step DTM with $h = 0.1$ when (a) $K_0 = 2$ (b) $K_0 = 2.5$ (c) $K_0 = 2.80$ (d) $K_0 = 2.85$.

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