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Effects of repeatedly used preconditioner on computational accuracy for nonlinear interval system of equations

Stephen Ehidiamhen Uwamusi

Department of Mathematics, Faculty of Physical Sciences, University of Benin, Benin City, Nigeria.
E-mail: uwamusi2000@yahoo.com.

We discuss Hansen-Sengupta operator in the context of circular interval arithmetic for the algebraic inclusion of zeros of interval nonlinear systems of equations. It was demonstrated by showing the effects of applying repeatedly preconditioners of inverses of the midpoint interval matrices on the well known Trapezoidal Newton method at each iteration cycle wherein, the work of Shokri (2008) was our major tool of investigation. It was shown that the Trapezoidal interval Newton method with inverse midpoint interval matrix as preconditioner is not a H-continuous map and that Baire category failed to hold in the sense of Aguelov et al. (2007). This was more so since it produced from our numerical example, not only overestimated results but, also results that are not finitely bounded which we compare with results computed previously given in Uwamusi.

Key words: Interval nonlinear systems of equations, Hansen–Sengupta operator, Trapezoidal Newton method, circular interval arithmetic, H-continuous map.

INTRODUCTION

We are interested in the solution of nonlinear interval system of equations

$$F(x) = 0 \tag{1}$$

where, $F : ID \subset IR^n \to IR^n$, and $[x] = \{[\underline{x}_1, \bar{x}_1] x ... x [\underline{x}_n, \bar{x}_n]\} \subset IR^n$ is a parallelepiped parallel to the axes often called a box for each $\underline{x}_i \le x \le \bar{x}_i$.

We assume that F is a smooth homeomorphism mapping with $F \in C^1(ID) \subseteq IR^n$.

We are interested in bounding the solution of (1) or establish their absence by a first order slope interval method. This means that given a box

$$[X] = [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2] \times ... \times [\underline{x}_n, \bar{x}_n] \subset IR^n \quad \text{evaluating } F_i$$

of the function F with desired interval based methods produce intervals $[F_i]$ of the function F which are guaranteed to validate and enclose the zeros of F even in the presence of nonlinearities and round off errors.

A method of reducing the width of co-multiplication for two intervals was proposed (Rump, 1999), This method is known to handle efficiently interval based evaluation of functions of several variables as well as finding enclosures of solution set of linear interval systems.

Our motive behind the paper is to extend Rump's interval operations wherein inverse isotonicity operation as enunciated in Uwamusi (2007, 2009) is applicable in the contexts of modified (Hansen and Sengupta, 1981) and Preconditioned Trapezoidal Newton method. It was expected that these methods are able to either guarantee that the system has no solution or to yield sharp bounds of the results computed.

We paid special emphasis on the efficiency of the method to guarantee convergence to the correct solutions in the results obtained.

By repeatedly solving the linear interval system

$$J(x)(\hat{x} - x) \ni -f(x), x \in X, x \subseteq D_i, \tag{2}$$

from

$$f(\hat{x}) \in f(x) + J(x)(\hat{x} - x), \qquad (3)$$

one obtains an equivalent system (1) where $J(x)$ is denoted by $A(x)$ in the form:

$$A(x)(\hat{x} - x) \ni -b(x). \qquad (4)$$

Method (4) does not only take into consideration the problem of dependencies but also has a simpler structure and its hull is also straight forward. Thus

$$x^{(k+1)} = N([x^{(k)}]) \cap (x^{(k)}), (k = 0,1,...,), \qquad (5)$$

where $N([x^{(k)}])$ is the interval Newton method.
Now suppose instead of solving method (4), we consider

$$f(\hat{x}) = f(x^{(k)}) + \int_o^1 J(x^{(k)} + t(\hat{x} - x^{(k)}))(\hat{x} - x^{(k)})dt \qquad (6)$$

where $x, \hat{x} \in ID$ and $t \in [0,1]$.

We can estimate $J(x^{(k)} + t(\hat{x} - x^{(k)}))$ in the interval [0,1] at the point t = 0. Following (Shokri, 2008), we estimate (6) by the trapezoidal rule in the form

$$0 = f(x^{(k)}) + \frac{1}{2}[J(x^{(k)}) + J(\hat{x})](\hat{x} - x^{(k)}), \qquad (7)$$

where from we obtain an iterative formula

$$x^{(k+1)} = x^{(k)} - 2[J(x^{(k)}) + J(\hat{x}^{(k+1)})]^{-1} f(x^{(k)} \cap x^{(k)}) \qquad (8)$$

and,

$$\hat{x}^{(k+1)} = x^{(k)} - J(x^{(k)})f(x^{(k)}), \qquad (9)$$

is the Newton step length which is a predictor.

Definition 1

An iteration is said to be numerically stable if it produces a sequence $\{x^{(k)}\}$ of approximations of the solution x^* such that for large k the relative error $\dfrac{\|x^{(k)} - x^*\|}{\|x^*\|}$ is of order $\eta(1 + cond(F,d))$ and η is the relative machine precision and d is a data vector (Wozniakowski, 1977).

Definition 2

An iteration is said to be well behaved if a slightly perturbed $x^{(k)}$ is an almost exact solution of a slightly perturbed problem (Wozniaskowski, 1977). This implies that $F(x^{(k)} + \delta x^{(k)}, d + \delta d_k) = 0(\eta^2)$ where $\dfrac{\|\delta x^{(k)}\|}{\|x^{(k)}\|}$ and $\dfrac{\|\delta d_k\|}{\|d\|}$ are of order η.

Definition 3

An interval function $F : X \to \overline{IR}^n$ is called S-continuous if its graph is a closed subset of $X \times \overline{R}^n$

Definition 4

An interval function $F : X \to \overline{IR}^n$ is Hausdorff continuous (H-continuous) if it is an S- continuous function that is minimal with respect to inclusion, that is if $\psi : D \to \overline{IR}^n$ is an S-continuous function then $\psi \subseteq F \Rightarrow \psi = F$ (Aguelov, 2007). Symbolically, we denote by $H(X)$, the set of H-continuous functions on X.

NOTATION

We denote $x \in IR^n$ and $A \in IR^{nxn}$ to signify the set of interval vectors and respectively interval matrices.
An interval vector x is said to be thick if there exist $x_1 \in X$ and $x_2 \in X$ with $x_1 \neq x_2$ such that the width $w(x) > 0$. An interval vector is said to be thin if for all $x_1 \in X$ and $x_2 \in X$ with $x_1 = x_2$, then the width $w(x) = x_2 - x_1$.
We hereby introduce interval operation due to Rump (1999) as follows: Let $(a,r) = [x \in R \| x - a | < r]$ where a is the centre and r is the radius. The basic interval operations $(+,-,o,/)$ such that for intervals $a = (a_1, r_1)$ and $b = (b_1, r_2) \in IR$ and $\circ \in \{+,-,o,/\}$, there follows $\{xoy | x \in a, y \in b\} \subseteq aob$. With these we have;

$$< a_1, r_1 > \pm < b_1, r_2 >=< a_1 \pm b_1, r_1 + r_2 >, \qquad (10)$$

$$< a_1, r_1 > \cdot < b_1, r_2 >=< a_1 b_1, |a_1| r_2 + |b_1| r_1 + r_1 r_2, \qquad (11)$$

$$< a_1, r_1 > / < b_1, r_2 >=< a_1, r_2 > . < b_1, r_2 >^{-1} \quad \text{where}$$
$$0 \notin < b_1, r_2 >. \qquad (12)$$

Inclusion isotonicity for intervals is implied by $< a_1, r_1 > \subseteq < b_1, r_2 >$ if and only if $|b_1 - a_1| \le r_2 - r_1 - < a_1, r_1 >=< -a_1, r_1 >$

These operations hold for commutativity and associativity but fail woefully for distributivity except for its subdistributivity, that is, $(a \pm b)c = ac \pm bc$ for $a, b, c \in IR$. A disk inversion due to (Carstensen and Petkovic, 1994) in the form of complex plane is adopted for our purpose as follows

$$[a_{ij}]^{-1} = \{a, r\}^{-1} =$$

$$\left\{ \frac{1}{a(1 - \frac{r^2}{|a|^2})}, \frac{r}{|a|^2 - r^2} \right\}, \qquad (13)$$

$$\left\{ \frac{1}{a}, \frac{r}{|a|(|a| - r)} \right\}, |a| > r \qquad (14)$$

$$\left\{ \frac{-a}{r^2 - |a|^2}, \frac{r}{r^2 - |a|^2} \right\}, |a| < r \qquad (15)$$

The emphasis is computing rigorous bounds on the solution of such systems with computable overestimation factor that is supposedly small. Such bounds enclose truncation, rounding and often modeling errors. The use of different specific interval machinery as inner enclosures to check the validity of the quality of bounds obtained dictates our interest in this paper.

THE METHOD

Central to our discussion, we review the following definition.

Definition 4

We say that a sequence of interval matrices $[A^k]$ converges if the lower and upper bounds converge,

or equivalently, if the midpoints and radii converge (Neumaier, 1990). This means that

$$\lim_{k \to \infty} A^k = \left[\lim_{k \to \infty} \underline{A}^{(k)}, \lim_{k \to \infty} \overline{A}^{(k)} \right],$$

$$mid\left(\lim_{k \to \infty} A^{(k)} \right) = \lim_{k \to \infty} mid(A^{(k)})$$

$$rad\left(\lim_{k \to \infty} A^{(k)} \right) = \lim_{k \to \infty} rad(A^{(k)})$$

Thus the sequence
$A^{(o)} \supseteq A^{(1)} \supseteq ..... \supseteq A^{(k)} \supseteq A^{(k+1)} \supseteq$ of nested interval matrices converges to the limit $\lim_{k \to \infty} A^{(k)} = \bigcap_{k \ge o} A^{(k)}$

Theorem 1

Let $F : D \subset IR^n \to IR^n$ be continuously differentiable and assume that $(IGA(f'([x]^o))$ exists for some interval vector $[x]^o \subseteq D$ (Alefeld, 1984). Assume that $f'([x]^o)$ exists

(a) if $N([x]) \subseteq [x]$ \qquad (16)

for some $[x] \subseteq [x]^o$ then f has a zero x* in [x] which is unique even in [x]°.
  Assuming further that

$$\rho(Q) < 1 \quad \text{where } Q = \left| I - IGA(f'([\bar{x}]^o f'([x]^o)) \right| \qquad (17)$$

(b) If f has a zero x* in [x]° then the sequence $\{[x]^k\}_{k=o}^\infty$ defined by

$$x^{(k+1)} = N([x^{(k)}]) \cap [x^{(k)}], (k = 0,1,....), \qquad (18)$$

is well behaved, $x* \in [x]^k$ and $\lim_{k \to \infty} it [x]^k = x*$

In particular, $\{[x]^{(k)}\}_{k=0}^\infty$ is monotonically decreasing and x* is unique in [x]°
  Moreover \hfill if
$$df'([x]_{ij}) \le \alpha \|d[x]\|_\infty, \alpha \ge 0, \quad 1 \le i, j \le n \quad for \quad \forall [x] \subseteq [x]^o$$
, then $\left\| d[x]^{(k+1)} \right\| \le \eta d[x]^{(k)} \|_\infty^2, \eta \ge 0,$

(c) $N[x]^{ko} \cap [x]^{ko} = 0$ for some $k \ge 0$ if and only if

$f(x) \neq 0 \quad \forall x \in [x^0]$.

Proof is given in Alefeld (1984, 2008) for example.

Theorem 2

Let $A \in R^{nxn}$ be an H-matrix with positive diagonal elements and let $R^n \to R^n$ be continuous, diagonal and isotone (Neumaier, 1990). Then the function $f : R^n \to R^n$ defined by $\overline{F}(\overline{x}) = A\overline{x} + Q(x)$ has a unique zero $x^* \in R^n$. Moreover the inequality

$$|x^* - \overline{x}| \leq \omega, \tag{19}$$

holds for every nonnegative vector $w \in R^n$ satisfying

$$< A > w \geq |f(\overline{x})| \tag{20}$$

Proof is given in Neumaier (1990).

The (Hansen and Sengupta (1981) and Uwamusi (2009) method is defined by

$$H(x, \overline{x}) = x^{(k+1)} = \Gamma(R\overline{A}, RD \cdot rad(A)abs(x^{(k)} + Rb_D, x^{(k)})) \tag{21}$$

where abs(x) denotes interval extension of the absolute value function which should not be confused with $|x|$. It is understood that in the limit

$$x^\infty = \Gamma(R\overline{A}, a, x^\infty), \tag{22}$$

where it is set that:

$$a = RD.rad(A)abs(x^\infty) + Rb_D, \tag{23}$$

and

$$x_i^\infty \subseteq \frac{\left(a_i - \sum_{j \neq i} \left((R\overline{A})_{ij} \right) \right)}{(R\overline{A})_{ii}} \tag{24}$$

and

$$rad(x_i^\infty) \leq \frac{1}{(R\overline{A})_{ii}} (Rad(a_i) + \sum_{j \neq i} |R\overline{A}|_{ij} rad(x_k^\infty). \tag{25}$$

We expect the interval matrix A to be regular. Sufficient conditions for verifying regularity of interval matrices have been discussed (Rohn, 2011a, 2011b, 2010, 2005). The numerator in Equation (24) contains zero. By the definition of optimal preconditioner in Kearfott (1990) and

due to analysis given in Kearfott and Xing (1993), we are able to conclude that $\inf\left((R\overline{A})_{ii} \right) = 1$.

The width w of the solution vector is then guided by

$$w(x^{(k+1)} - x^{(k)}) = w\left[RF(x^{(k)}) + \sum_{\substack{j=2 \\ j \neq i}} [R_{ij}A_{ij}](x_k - X_k) \right]$$

The preconditioned system (21) has an M- matrix centered about the identity matrix I with right hand vector in the form

$$M[x] = r, \tag{26}$$

where $M \in IR^{nxn}$, and $r \in IR^n$.

That means $\overline{M}_{ij} = M_{\underline{ij}} > 0$ for $i \neq k$ and that

$$\overline{M}_{ii} + M_{\underline{ii}} = 2.$$

The solution set of method (21) is bounded by the inequalities:

$$\underline{M}|x| \leq D \, mid \, (r) + rad \, (r)$$

$$\overline{M}|x| \geq D \, mid \, (r) + rad \, (r)$$

where D is the diagonal matrix defined by

$$D_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \quad and \quad x_i \geq 0 \\ -1 & if \quad i = j \quad and \quad x_i \leq 0 \end{cases}$$

(Shi and Tian, 1999) obtained inequalities for method (21) in the form:

$$M_{\underline{ii}}|x_i| + \sum_{j \neq i} M_{\underline{ij}}|x_j| = |mid \, (r_i) + rad \, (r_i)|, \tag{27}$$

and

$$M_{\underline{jj}}|x_j| + \sum_{k \neq j} M_{\underline{jk}}|x_k| = |mid(r_j) + rad(r_j)| \quad \forall j \neq i, \tag{28}$$

$$\overline{M}_{ii} D_{ii} x_i + \sum_{j \neq i} \overline{M}_{ij}|x_j| \geq -\overline{r}_i, \tag{29}$$

$$M_{\underline{ii}} D_{ii} x_i + \sum_{j \neq i} M_{\underline{ij}}|x_j| \leq -r_{\underline{i}}, \tag{30}$$

Table 1. Results for preconditioned trapezoidal Interval Newton method of Equation 8.

Iteration	Results in midpoint-radius vector	$\left\|F(X^{(k)})\right\|_{\infty}$
1	[0.33169161,0.012680616] [0.5362032,0.01241961] [0.794301886,0.006144126]	2.0118213×10^{-2}
2	[0.337703974,0.000558556] [0.585171538,0.0060343] [0.801821044,0.00038168]	1.153693×10^{-3}
3	[0.338826244,0.000003618] [0.586294836 0.0000038] [0.799870224,0.000001674]	1.0659344×10^{-2}
4	[0.341908146,0.000000038] [0.591894964,0.000000096] [0.804670818,0.000000808	1.1499132×10^{-2}
5	?	?

where from the following are valid:

$$M_{\underline{ii}}|x_i| + \sum_{j \neq i} M_{\underline{ij}}|x_j| \leq \frac{(D_{ii}-1)r_i}{2} + \frac{(D_{ii}+1)r_i}{2},$$

$$\bar{M}_{ii}|x_i| + \sum_{j \neq i} M_{ij}|x_j| \geq \frac{(D_{ii}-1)\bar{r}_i}{2} + \frac{(D_{ii}+1)\bar{r}_i}{2}.$$

Setting $x' = (|x_1|,|x_2|,....,|x_{i-1}|,x_i,|x_{i+1}|,...,|x_n|)^T$ we can infer from the analysis presented in (Rohn, 1993) that $M(x'-|x|)+|x| \leq Mr$. Thus

$$x' = x_i = b_i + ((I-A)x)_i \leq b' radb)_i + (A''|x|)_i = (r+A''|x|)_i$$

where A'' is a matrix of perturbation bound. With the aforementioned exposition, we infer that Hansen-Sengupta method converges for any starting point for the interval nonlinear systems of equations.

EXAMPLE

We illustrate with the following problem discussed in (Uwamusi, 2009):

$$F(x) = \begin{cases} x_2^2 - 3x_1^2 \\ x_1^2 + x_1 x_3 + x_3^2 - 3x_2^2 \\ x_2^2 + x_2 + 1 - 3x_3^2 \end{cases}$$

$$x^{(o)} = (\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4}), \varepsilon = 0.01$$

Using a result due to (Schafer, 2007) which is a version of Miranda's theorem (Miranda, 1940), we are able to show that the so called Trapezoidal Newton method has no rational map with a fixed point solution to the given problem. Further insight into this regard can be found in Rohn (1993, 2005). Again Trapezoidal Newton method is also not a H-continuous map since Baire category (Aguelov et al., 2005; Anguelov, 2007; Angueov et al., 2006) failed to hold as we found in the given problem. This means that the graph completion operator is not an inclusion isotone for this type of method.

Conclusion

The paper reported a defect which is common with multiple applications of a preconditioner in the interval based Hansen-Sengupta method for finding solution to interval nonlinear system of equations. In particular, we studied this effect on Trapezoidal interval Newton method. The inherent problems encountered arose as a result, where there are many paths near some points, it was discovered from our investigation that the Trapezoidal Newton algorithm based on multiple usage of preconditioner may produce not only overly an overestimated results but also results that are not finitely bounded as shown in Table 1.

On the other hand, computed values from Uwamusi (2009) as shown in Table 2 wherein we incorporated Hansen-Sengupta method of Equation 21 via inclusion isotonicity of inverse disk in the sense of Carstensen and Petkovic (1994) produced quite satisfactory results in the sense that monotonic and inclusion isotonicity property of

Table 2. Results for Hansen-Sengupta method with error bounds.

Iteration	Result Mid (X_k), Rad (X_k)	$\left\|F(\bar{X_k})\right\|_\infty$
1	0.358937022, 1.7095978 × 10^{-2} 0.600208287, 1.6381989 × 10^{-2} 0.808328540, 8.809388 × 10^{-3}	2.6257369 × 10^{-2}
2	0.336461223, 3.350353 × 10^{-3} 0.585636548, 2.417562 × 10^{-3} 0.801816087, 8.112458 × 10^{-3}	3.351703 × 10^{-3}
3	0.337953381, 3.4978 × 10^{-5} 0.585355878, 3.4033 × 10^{-5} 0.801709854, 1.5718 × 10^{-5}	2.18688 × 10^{-4}
4	0.337917117, 3.6264 × 10^{-5} 0.585289640, 6.6708 × 10^{-5} 0.801634504, 4.15 × 10^{-7}	3.1 × 10^{-8}
5	0.337917117, 4 × 10^{-9} 0.585289640, 7.305 × 10^{-12} 0.801634504, 1.9341 × 10^{-12}	0
6	0.337917117,0 0.585289640,0 0.801634504,0	0

interval arithmetic with order preserving hold. Such a similar case was reported in Kearfortt and Xing (1993) which narrowed closely our findings. While the solutions obtained from using Interval Trapezoidal Newton method expanded suddenly in third iteration, that of Uwamusi (2009) method provided numerically good solution as sequence of iterates approach infinity.

Termination criterion for the iteration is $\dfrac{\left\|F(x^{(k)})\right\|}{\left\|F(x^{(0)})\right\|} < 10^{-12}$

or $\left\|s^{(k)}\right\| < 10^{-10}$ and $s^{(k)} = x^{(k+1)} - x^{(k)}$.

In the case of Table 1, we halt the iteration when there was no longer any noticeable improvement after four successive iteration as the solution obtained began to diverge from the true solution. Thus the way an iterative method is written or evaluated will greatly harm the quality of interval solutions. This suggests that it is strongly recommended in interval based iteration to reduce as much as possible correlations among intervals as were seen in our investigation in the Trapezoidal Newton method.

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