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Full Length Research Paper

A passivity criterion for fixed-point state-space digital filters

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This paper proposes a new criterion for passivity of fixed-point state-space digital filters with saturation arithmetic and external interference. The criterion guarantees not only exponentially stability, but also passivity from the external interference to the output vector. The criterion takes the form of linear matrix inequality (LMI) and, hence, is computationally tractable. An illustrative example is given to demonstrate the effectiveness of the proposed criterion.

Key words: Passivity approach, exponential stability, digital filters, finite wordlength effects, linear matrix inequality (LMI).

INTRODUCTION

When a linear digital filter is implemented using fixed-point arithmetic on a digital computer or on special-purpose digital hardware, nonlinearities due to finite wordlength, namely, quantization and overflow are unavoidable. If the total number of quantization steps is large or, in other words, the internal wordlength is sufficiently long, then the effects of these nonlinearities can be regarded as decoupled or noninteracting and can be investigated separately. The stability properties of digital filters employing saturation overflow arithmetic have attracted the attention of several researchers (Ebert et al., 1969; Sandberg, 1979; Singh, 1990; Kar and Singh, 1998, 2003, 2005; Singh, 2007, 2008). However, most existing criteria for the stability of digital filters are only available under specific conditions, while in unfavorable environments with parameter uncertainty or external interference, these criteria will be of little use.

The passivity theory (Willems, 1972; Byrnes et al., 1991) is an effective tool to analyze the stability of a nonlinear system. It may deal with nonlinear systems using only the general characteristics of the input-output dynamics and offers elegant solutions for the proof of absolute stability. The passivity framework is a promising approach to the stability analysis of digital filters because it can lead to general conclusions on stability using only input-output characteristics.

A natural question arises: can we obtain a passivity criterion for digital filters? This paper gives an answer for this question. So far, to the best of the authors' knowledge, the passivity criterion of digital filters with saturation arithmetic and external interference has never been studied in the literature. This situation motivates our present investigation. In this paper, we propose a new passivity criterion for digital filters with saturation arithmetic and external interference.

This criterion is a new contribution to the topic of stability analysis for digital filters. The criterion guarantees that the digital filter is exponentially stable and passive from the external interference to the output vector. This criterion can be represented by a linear matrix inequality (LMI), which can be checked readily by using some standard numerical packages (Boyd et al., 1994; Gahinet et al., 1995).

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PROBLEM FORMULATION

The digital filter under consideration is described by:

$$x(r+1) = f(y(r)) + Gw(r), \tag{1}$$

$$y(r) = Ax(r) + Fw(r), \tag{2}$$

Where $f(y(r)) = [f_1(y_1(r))\ f_2(y_2(r))\ \cdots\ f_n(y_n(r))]^T$, $x(r) = [x_1(r)\ x_2(r)\ \cdots\ x_n(r)]^T \in R^n$ is a state vector, $w(r) = [w_1(r)\ w_2(r)\ \cdots\ w_m(r)]^T \in R^m$ is an external interference, $A \in R^{n\times n}$ is the coefficient matrix, and $G \in R^{n\times m}$, $F \in R^{n\times m}$ are known constant matrices.

Now we define $y(r) = [y_1(r)\ y_2(r)\ \cdots\ y_n(r)]^T \in R^n$ as an output vector. The following saturation nonlinearities:

$$f_i(y_i(r)) = \begin{cases} 1, & if\ \ y_i(r) > 1 \\ y_i(r), & if\ -1 \le y_i(r) \le 1 \\ -1, & if\ \ y_i(r) < -1 \end{cases} \tag{3}$$

are under consideration for $i = 1, 2, \cdots, n$. Note that the saturation nonlinearities are confined to the sector (0, 1), that is,

$$f_i(0) = 0, \quad -1 \le \frac{f_i(y_i(r))}{y_i(r)} \le 1, \quad i = 1, 2, \cdots, n. \tag{4}$$

The objective of this study is to obtain a new LMI criterion in the passivity framework. Specifically, find a proper LMI criterion such that the digital filter 1 to 2 with $w(r) = 0$ is exponentially stable ($\| x(r) \| \le \varepsilon \eta^r$, where ε and η are constants satisfying $\varepsilon \ge 1$ and $0 < \eta < 1$) and

$$\sum_{k=0}^{r} w^T(k) y(k) + \beta \ge \sum_{k=0}^{r} \Omega(x(k)), \qquad \forall r \ge 0, \tag{5}$$

Where β is a nonnegative constant and $\Omega(x(k))$ is a positive semi-definite storage function.

LMI BASED PASSIVITY CRITERION

The new criterion is given in the following theorem.

Theorem 1

If we assume that there exist symmetric positive definite matrices P, S, a positive diagonal matrix M, and a positive scalar δ such that:

$$\begin{bmatrix} \delta A^T A + S - P & MA & -\frac{1}{2} A^T \\ A^T M & P - \delta I - 2M & PG \\ -\frac{1}{2} A & G^T P & G^T PG - F \end{bmatrix} < 0, \tag{6}$$

then the digital filter (1 to 2) is passive from the external interference $w(r)$ to the output vector $y(r)$.

Proof

Consider the following Lyapunov function: $V(x(r)) = x^T(r) P x(r)$. Along the trajectory of (1), we have:

$$\begin{aligned} \Delta V(x(r)) &= V(x(r+1)) - V(x(r)) \\ &= [f(Ax(r)) + Gw(r)]^T P [f(Ax(r)) + Gw(r)] - x^T(r) P x(r) \\ &= f^T(Ax(r)) P f(Ax(r)) + f^T(Ax(r)) PGw(r) + w^T(r) G^T P f(Ax(r)) \\ &\quad + w^T(r) G^T PGw(r) - x^T(r) P x(r) + 2 f^T(Ax(r)) M [Ax(r) - f(Ax(r))] \\ &\quad - 2 f^T(y(r)) M [y(r) - f(y(r))]. \end{aligned}$$

Adding and subtracting $w^T(r)[Ax(r) + Fw(r)]$, we obtain:

$$\begin{aligned} \Delta V(x(r)) &= f^T(Ax(r)) P f(Ax(r)) + f^T(Ax(r)) PGw(r) + w^T(r) G^T P f(Ax(r)) - \frac{1}{2} w^T(r) A x(r) \\ &\quad - \frac{1}{2} x^T(r) A^T w(r) - w^T(r) F w(r) + w^T(r) G^T PGw(r) - x^T(r) P x(r) \\ &\quad + 2 f^T(Ax(r)) M [Ax(r) - f(Ax(r))] + w^T(r)[Ax(r) + Fw(r)] \\ &\quad - 2 f^T(y(r)) M [y(r) - f(y(r))]. \end{aligned}$$

From (4), it is clear that:

$$f^T(Ax(r)) f(Ax(r)) = \| f(Ax(r)) \|^2 \le \| Ax(r) \|^2 = (Ax(r))^T Ax(r). \tag{7}$$

Then, for a positive scalar δ, we have:

$$\delta [x^T(r) A^T A x(r) - f^T(Ax(r)) f(Ax(r))] \ge 0. \tag{8}$$

Using (8), a new bound for $\Delta V(x(r))$ can be obtained as:

$$\Delta V(x(r)) \le f^T(Ax(r))Pf(Ax(r)) + f^T(Ax(r))PGw(r) + w^T(r)G^T Pf(Ax(r)) - \frac{1}{2}w^T(r)Ax(r)$$

$$-\frac{1}{2}x^T(r)A^T w(r) - w^T(r)Fw(r) + w^T(r)G^T PGw(r) - x^T(r)Px(r)$$

$$+2f^T(Ax(r))M[Ax(r) - f(Ax(r))] + w^T(r)[Ax(r) + Fw(r)]$$

$$-2f^T(y(r))M[y(r) - f(y(r))] + \delta[x^T(r)A^T Ax(r) - f^T(Ax(r))f(Ax(r))]$$

$$= \begin{bmatrix} x(r) \\ f(Ax(r)) \\ w(r) \end{bmatrix}^T \begin{bmatrix} \delta A^T A + S - P & MA & -\frac{1}{2}A^T \\ A^T M & P - \delta I - 2M & PG \\ -\frac{1}{2}A & G^T P & G^T PG - F \end{bmatrix} \begin{bmatrix} x(r) \\ f(Ax(r)) \\ w(r) \end{bmatrix}$$

$$-x^T(r)Sx(r) + w^T(r)[Ax(r) + Fw(r)] + \Phi(r), \tag{9}$$

Where $\Phi(r) = -2f^T(y(r))M[y(r) - f(y(r))]$.

Note that $\Phi(r)$ is nonpositive in view of (3). If the LMI (6) is satisfied, we have:

$$\begin{aligned}\Delta V(x(r)) &< -x^T(r)Sx(r) + w^T(r)[Ax(r) + Fw(r)] \\ &= -x^T(r)Sx(r) + w^T(r)y(r).\end{aligned} \tag{10}$$

Summation both sides of (10) from 0 to r gives:

$$V(x(r)) - V(x(0)) < -\sum_{k=0}^{r} x^T(k)Sx(k) + \sum_{k=0}^{r} w^T(k)y(k).$$

Let $\beta = V(x(0))$. Since $V(x(r)) \ge 0$,

$$\begin{aligned}\sum_{k=0}^{r} w^T(k)y(k) + \beta &> \sum_{k=0}^{r} x^T(k)Sx(k) + V(x(r)) \\ &\ge \sum_{k=0}^{r} x^T(k)Sx(k).\end{aligned} \tag{11}$$

The relation (11) satisfies (5). Therefore, the digital filter (1 to 2) is rendered to be passive from the external interference $w(r)$ to the output vector $y(r)$. This completes the proof.

Corollary 1: Without the external interference, the digital filter (1 to 2) is exponentially stable.

Proof

Note that $V(x(r))$ satisfies the following Rayleigh inequality (Strang, 1986):

$$\lambda_{\min}(P)\|x(r)\|^2 \le V(x(r)) \le \lambda_{\max}(P)\|x(r)\|^2, \tag{12}$$

Where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the maximum and minimum Eigen values of the matrix.

When $w(r) = 0$, we have:

$$\Delta V(x(r)) < -x^T(r)Sx(r) \le -\lambda_{\min}(S)\|x(r)\|^2 \tag{13}$$

From (10). According to Theorem 1 of (Lee, 2002), (12) and (13) guarantee the exponential stability. This completes the proof.

Remark 1

Various efficient convex optimization algorithms can be used to check whether the LMI (6) is feasible. In this paper, in order to solve the LMI, we utilize MATLAB LMI Control Toolbox (Gahinet et al., 1995), which implements state-of- the-art interior-point algorithms.

NUMERICAL EXAMPLE

Consider a second-order system (1 to 2) with:

$$A = \begin{bmatrix} 0.25 & 0.5 \\ -0.5 & 0.8 \end{bmatrix},\quad G = \begin{bmatrix} 0.13 & 0.1 \\ -0.1 & 0.01 \end{bmatrix},\quad F = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \tag{14}$$

and the external interference $w(r)$ is given by:

$$w(r) = \begin{cases} [0.1n_1(r)\ \ 0.15n_2(r)]^T, & 0 \le r \le 50, \\ [0\ \ 0]^T, & otherwise, \end{cases} \tag{15}$$

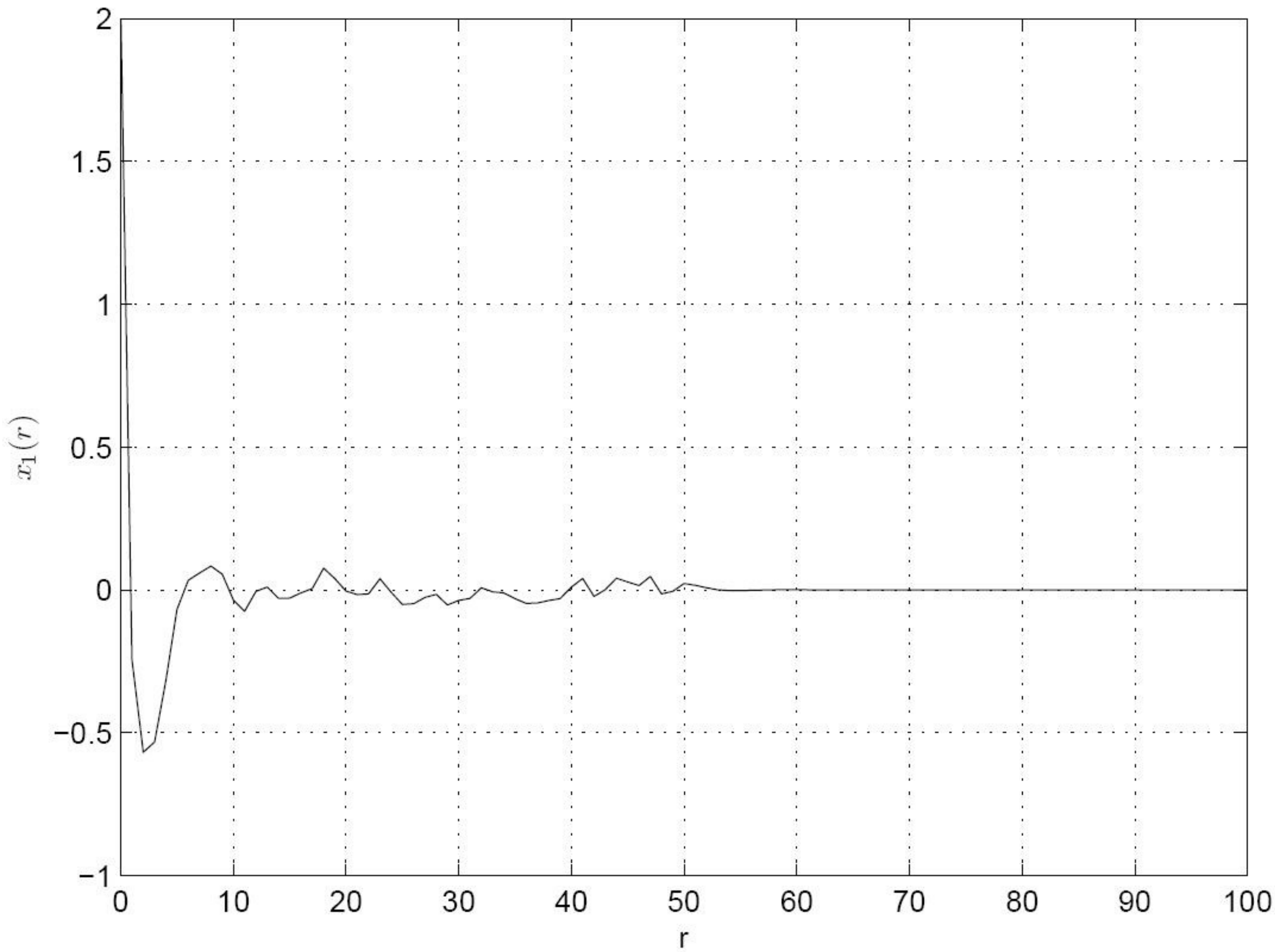

Figure 1. Response of the state $x_1(r)$.

Where $n_1(r)$ and $n_2(r)$ are white Gaussian random sequences with mean 0 and variance 1.

Solving the LMI (6) by the convex optimization technique of MATLAB software gives:

$$P = \begin{bmatrix} 2.2021 & 2.2262 \\ 2.2262 & 9.5741 \end{bmatrix}, \quad S = \begin{bmatrix} 0.2814 & 0.3603 \\ 0.3603 & 1.0371 \end{bmatrix},$$
$$M = \begin{bmatrix} 5.1844 & 0 \\ 0 & 9.0774 \end{bmatrix}, \quad \delta = 0.2720.$$

It can be easily verified that each of the criteria in previous works (Ebert et al., 1969; Sandberg, 1979; Singh, 1990; Kar and Singh, 1998, 2003, 2005; Singh, 2007, 2008) fails in the example given by (1 to 2) with parameters (14) and(15). On the other hand, it turns out that the criterion (6) verifies the passivity when $w(r) \neq 0$ and the exponential stability when $w(r) = 0$ in this example. Figures 1 and 2 show state trajectories when the initial states are given by $x(0) = [2 \ -1.5]^T$. These figures show the state vector $x(r)$ of the digital filter is bounded on the interval where the external interference $w(r)$ exists. In addition, it is shown that the state vector $x(t)$ converges to zero after the external interference $w(r)$ disappears.

CONCLUSION

In this paper, a new LMI criterion for passivity of fixed-point digital filters with saturation arithmetic and

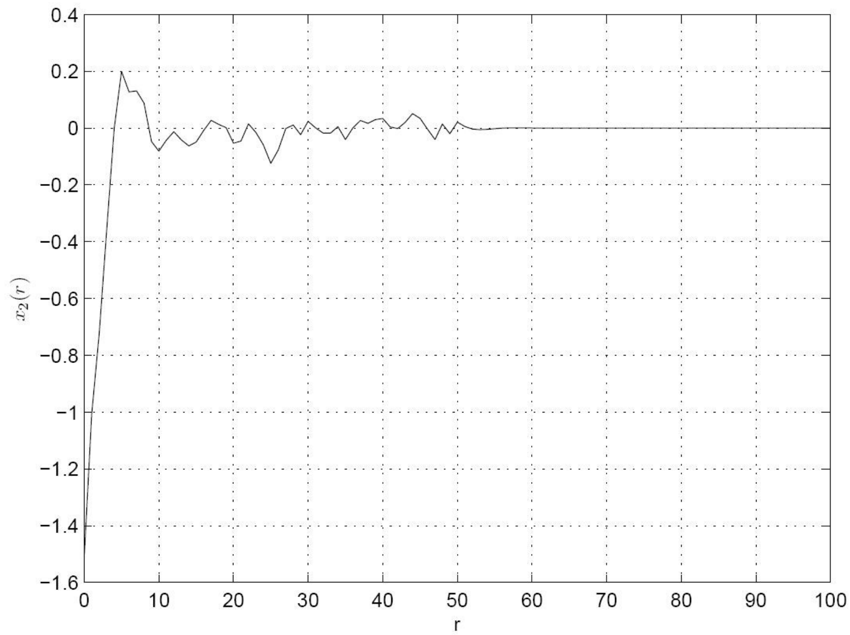

Figure 2. Response of the state $x_2(r)$.

external disturbance has been presented. The proposed criterion guaranteed that the digital filter is exponentially stable and passive from the external interference to the output vector. A numerical example was given to show the effectiveness of the proposed criterion.

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