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Full Length Research Paper

Solving system of nonlinear boundary value problems by modified variational iteration method

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In this paper, we use the modified variational iteration method (MVIM) for solving the system of nonlinear boundary value problems associated with obstacle problems. This modified variational method is an elegant coupling of variational iteration method and homotopy perturbation method. We give the examples of second-order, third-order and fourth-order system of nonlinear boundary value problems to illustrate the implementation and efficiency of the proposed MVIM. Results obtained in this paper may stimulate further research in this area.

Key words: Variational iteration method, homotopy perturbation method, system of nonlinear boundary value problems, correctional functional, Lagrange multiplier.

INTRODUCTION

It is well known a wide class of problems which arise in several branches of pure and applied sciences can be formulated as a system of boundary value problems. This system of boundary value problems can be obtained by using the penalty methods for solving the variational inequalities associated with the obstacle, unilateral, moving, free boundary value problems. To be more precise, we consider the following system of nonlinear boundary value problems:

$$Tu=\begin{cases} f\left(x,u\left(x\right)\right), & a\le x<c, \\ f\left(x,u\left(x\right)\right)+u\left(x\right)g\left(x\right)+r, & c\le x<d, \\ f\left(x,u\left(x\right)\right), & d\le x\le b, \end{cases} \tag{1}$$

with boundary conditions $u(a)=u(b)=\alpha_0,\ u'(a)=u'(b)=\alpha_1,\dots u^{(n-1)}(a)=u^{(n-1)}(b)=\alpha_{n-1}$, and continuity conditions of $u(x),u'(x),\dots,u^{(n-2)}(x)$ and $u^{(n-1)}(x)$ at internal points c and d of the interval $[a,b]$. Here T is differential operator of any order, $\alpha_i,\ i=0\dots n-1$, are constants and $g(x)$ is a linear continuous functions on $[a,b]$, whereas $f(x,u(x))=f(u)$ is nonlinear function. For the formulation, applications and numerical methods, see Al-Said et al. (2003, 1996, 1998), Gao et al. (2006), Geng et al. (2010), Kikuchi et al. (1988), Noor (1988, 2000, 2004, 2009) and Noor et al. (1993, 1994, 2003, 2010, 2010a, 2011, 2011a, 2011b, 2011c).

Several techniques have been developed for solving system of linear boundary value problems associated with obstacle problems. He (1999, 1999a, 2006, 2006a, 2007, 2008) developed the variational iteration and the homotopy perturbation methods for solving linear, nonlinear initial and boundary value problems. The origin of the variational iteration method can be traced back to Inokuti et al. (1978), but the real potential of the variational iteration method was explored by He (1999, 2007). The variational iteration method provides the solution in a rapid convergent series which may lead the solution in a closed form. Noor et al. (2010, 2011, 2011a, 2011b) have considered system of nonlinear higher order boundary value problems and applied the modified variation of parameters method for finding the approximate solution of the problems. Mohyud-Din et al. (2009) have developed the modified variational iteration method (MVIM) for solving system of nonlinear boundary value problems. The proposed technique is an elegant coupling of the

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variational iteration method and the homotopy perturbation method. The basic motivation of this paper is to apply the MVIM for solving the system of higher-order nonlinear boundary value problems associated with obstacle, unilateral and contact problems. In the present study we implement this technique for solving systems of nonlinear second, third and fourth-orders boundary value problems associated with obstacle, unilateral and contact problems.

MVIM

To convey the basic idea of the MVIM for differential equations, we consider the general differential equation of the form.

$$Lu(x)+Nu(x)=g(x), \quad (2)$$

Where L is linear operator, N is a nonlinear operator and g is a forcing term.

Following the technique of the variational iteration, we can construct a correction functional as follows

$$u_{k+1}(x)=u_k(x)+\int_0^x \lambda(\xi)\left(Lu_k(\xi)+N\tilde{u}_k(\xi)-g(\xi)\right)d\xi, \quad k=0,1,2,.. \quad (3)$$

where λ is a Lagrange multiplier, which can be identified via the optimally condition method. Here $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n=0,$ and is called as a correct functional.

We recall the homotopy perturbation method for solving the nonlinear Equation (2). Using the technique and idea of He (2008), one can construct a homotopy $H(v(x),p):\Omega \text{ x } [0,1]\rightarrow \mathrm{R},$ which satisfies:

$$H(v(x),p)=(1-p)[L(v(x))-L(u_0(x))]+p[L(v(x))+N(v(x)-g(x)]=0,$$

Where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of the solution of the Equation (2). Clearly, we have

$$H(v(x),0)=L(v(x))-L(u_0(x))=0,$$
$$H(v(x),1)=L(v(x))+N(v(x))-g(x)=0.$$

Changing the process of *p* from zero to unity is just that change of $H(v(x),\ p)$ from $L(v(x))-L(u_0(x))=0$ to $L(v(x))+N(v(x))-g(x)$. This is called homotopy and $L(v(x))-L(u_0(x))$ and $L(v(x))+N(v(x))-g(x)$ are called homotopic.

Now, we apply the homotopy perturbation method in the following form

$$\sum_{n=0}^{\infty}p^{(n)}u_n=u_0(x)+p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty}p^{(n)}L(u_n)+\sum_{n=0}^{\infty}p^{(n)}N(\tilde{u}_n)\right)d\xi-\int_0^x\lambda(\xi)g(\xi)d\xi \quad (4)$$

which is the coupling of variational iteration method and the homotopy perturbation method and is called the MVIM. The comparison of like powers of p provides solutions of various orders. According to the homotopy perturbation method, the solution of equation (4) can be written in the form of a power series in *p:*

$$v(x)=v_0(x)+pv_1(x)+p^2v_2(x)+...$$

Setting *p*=1, results in the approximate solution of Equation (2).

$$u(x)=\lim_{p\to 1}v(x)=v_0(x)+v_1(x)+v_2(x)+...$$

For the convergence of the homotopy perturbation technique (Biazar et al., 2009).

NUMERICAL APPLICATIONS

In this section, we consider some examples of the system of boundary value problems associated with obstacle and unilateral problems to illustrate the implementation and efficiency of the MVIM. These examples are mainly due to Noor et al. (2010, 2011, 2011b).

Example 1

Consider following system of second-order nonlinear boundary value problems relevant to system by Noor et al. (2010). (1):

$$u''=\begin{cases}\frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & -1\le x<-\frac{1}{2},\\ \frac{u^3}{3!}+\frac{u^2}{2!}2u, & -\frac{1}{2}\le x<\frac{1}{2},\\ \frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & \frac{1}{2}\le x\le 1,\end{cases} \quad (5)$$

With boundary conditions $u(-1)=u(1)=0.$

We will use MVIM for solving a system of second-order nonlinear boundary value problems (5) and obtain

$$u_{k+1}(x)=\begin{cases}u_k(x)+\int_0^x\lambda(\xi)\left(u_k''-\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}+\tilde{u}_k+1\right)\right)ds, & -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x\lambda(\xi)\left(u_k''-\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}+2\tilde{u}_k\right)\right)ds, & -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x\lambda(\xi)\left(u_k''-\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}+\tilde{u}_k+1\right)\right)ds, & \frac{1}{2}\le x\le 1.\end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(\xi)=(\xi-x)$. Consequently,

$$u_{k+1}(x)=\begin{cases} u_k(x)+\int_0^x(\xi-x)\left(u_k''-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x(\xi-x)\left(u_k''-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+2u_k\right)\right)ds, & -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x(\xi-x)\left(u_k''-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & \frac{1}{2}\le x\le 1.\end{cases}$$

Applying MVIM, we consider three cases in the given domain.

Case 1: $-1\le x<-\frac{1}{2}$. In this case, we implement MVIM as follows,

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= c_2x+c_1\\ &+p\int_0^x(\xi-x)\left(\left(u_0''+pu_1''+p^2u_2''+\cdots\right)\right)ds,\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-p\int_0^x(\xi-x)\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have;

$$\begin{aligned} p^{(0)}&:u_0(x)=c_1+c_2x,\\ p^{(1)}&:u_1(x)=c_1+\frac{1}{2}x^2+\frac{1}{6}c_2x^3+\frac{1}{24}c_2^2x^4+\frac{1}{120}c_2^3x^5,\\ p^{(2)}&:u_2(x)=\frac{1}{2}(c_1-1)x^2+\frac{1}{6}(c_1c_2-c_2)x^3+\frac{1}{24}\left(1+c_1c_2^2-c_2^2\right)x^4+\frac{1}{30}\left(c_2-\frac{1}{4}c_2^3\right)x^5\\ &+\frac{11}{720}c_2^2x^6+\frac{1}{315}c_2^3x^7+\frac{1}{1920}c_2^4x^8+\frac{1}{17280}c_2^5x^9,\end{aligned}$$

Case 11: $-\frac{1}{2}\le x<\frac{1}{2}$. In this case, we have following approximations;

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= c_4x+c_3\\ &+p\int_0^x(\xi-x)\left(\left(u_0''+pu_1''+p^2u_2''+\cdots\right)\right)ds,\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-2p\int_0^x(\xi-x)\left(\left(u_0+pu_1+p^2u_2+\cdots\right)\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^{(0)}&:u_0(x)=c_3+c_4x,\\ p^{(1)}&:u_1(x)=c_3+\frac{1}{3}c_4x^3+\frac{1}{24}c_4^2x^4+\frac{1}{120}c_4^3x^5,\\ p^{(2)}&:u_2(x)=c_3x^2+\frac{1}{6}\left(c_3c_4-2c_4\right)x^3+\frac{1}{24}\left(c_3c_4^2-c_4^2\right)x^4+\frac{1}{30}\left(c_4-\frac{1}{4}c_4^3\right)x^5\\ &+\frac{1}{72}c_4^2x^6+\frac{3}{560}c_4^3x^7+\frac{1}{1920}c_4^4x^8+\frac{1}{17280}c_4^5x^9,\end{aligned}$$

Case III: $\frac{1}{2}\le x\le 1$. In this case, we implement MVIM as follows;

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= c_6x+c_5\\ &+p\int_0^x(\xi-x)\left(\left(u_0''+pu_1''+p^2u_2''+\cdots\right)\right)ds,\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x(\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-p\int_0^x(\xi-x)\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^{(0)}&:u_0(x)=c_1+c_2x,\\ p^{(1)}&:u_1(x)=c_1+\frac{1}{2}x^2+\frac{1}{6}c_2x^3+\frac{1}{24}c_2^2x^4+\frac{1}{120}c_2^3x^5,\\ p^{(2)}&:u_2(x)=\frac{1}{2}(c_1-1)x^2+\frac{1}{6}(c_1c_2-c_2)x^3+\frac{1}{24}\left(1+c_1c_2^2-c_2^2\right)x^4+\frac{1}{30}\left(c_2-\frac{1}{4}c_2^3\right)x^5\\ &+\frac{11}{720}c_2^2x^6+\frac{1}{315}c_2^3x^7+\frac{1}{1920}c_2^4x^8+\frac{1}{17280}c_2^5x^9,\\ &\vdots\end{aligned}$$

By using the MVIM, we have the following formula for getting series solution in the whole domain from the above cases;

$$u(x)=\begin{cases}\sum_{k=0}^{\infty}u_k(x), & \text{for } -1\le x\le -\frac{1}{2},\\ \sum_{k=0}^{\infty}u_k(x), & \text{for } -\frac{1}{2}\le x\le \frac{1}{2},\\ \sum_{k=0}^{\infty}u_k(x), & \text{for } \frac{1}{2}\le x\le 1.\end{cases}$$

Consequently, we have the following series solution

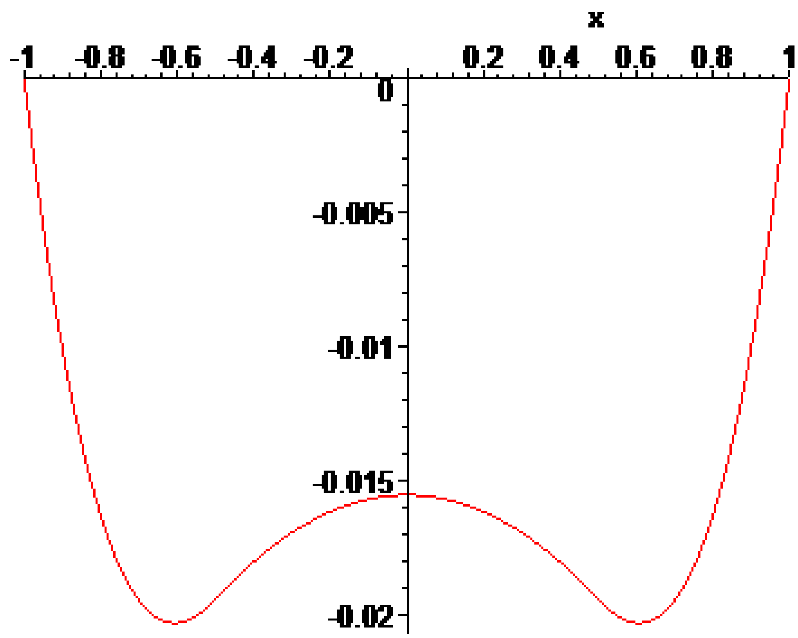

Figure 1. Graphical representation of series solution of system of second-order nonlinear boundary value problem (5) by modified variational iteration method.

$$u(x)=\begin{cases} c_1+c_2x+\frac{1}{2}c_1x^2+\frac{1}{6}c_1c_2x^3+\frac{1}{24}\left(1+c_1c_2^{\,2}\right)x^4+\frac{1}{30}c_2x^5+\frac{11}{720}c_2^{\,2}x^6+\frac{1}{315}c_2^{\,3}x^7 \\ +\frac{1}{1920}c_2^{\,4}x^8+\frac{1}{17280}c_2^{\,5}x^9, & -1\le x<-\frac{1}{2}, \\ c_3+c_4x+c_3x^2+\frac{1}{6}c_3c_4x^3+\frac{1}{24}c_3c_4^{\,2}x^4+\frac{1}{30}c_4x^5+\frac{1}{72}c_4^{\,2}x^6+\frac{3}{560}c_4^{\,3}x^7 \\ +\frac{1}{1920}c_4^{\,4}x^8+\frac{1}{17280}c_4^{\,5}x^9, & -\frac{1}{2}\le x<\frac{1}{2}, \\ c_5+c_6x+\frac{1}{2}c_5x^2+\frac{1}{6}c_5c_6x^3+\frac{1}{24}\left(1+c_5c_6^{\,2}\right)x^4+\frac{1}{30}c_6x^5+\frac{11}{720}c_6^{\,2}x^6+\frac{1}{315}c_6^{\,3}x^7 \\ +\frac{1}{1920}c_6^{\,4}x^8+\frac{1}{17280}c_6^{\,5}x^9, & \frac{1}{2}\le x\le 1. \end{cases} \quad (6)$$

By using boundary conditions and continuity conditions at $x=-\frac{1}{2}$ and $x=\frac{1}{2}$, we have a system of nonlinear equations. By using Newton's method for solving system of nonlinear equations, we have the following values of unknown constants:

$$c_1=-.0045417349,\ \ c_2=.0337710150,\ \ c_3=-.0155380140,$$
$$c_4=-.0000000001,\ \ c_5=-.0045417350,\ c_6=-.0337710150. \quad (7)$$

By using values of unknowns from (7) into (6), we have following series solution of system of second-order nonlinear boundary value problems (5) (Figure 1)

$$u(x)=\begin{cases} -.0045417349+.0337710150x-.002270867450x^2-2.556316623\times10^{-5}\,x^3 \\ +.04166645085\,x^4+.0011257005x^5+1.7424022\times10^{-5}\,x^6+1.2227052\times10^{-7}x^7 \\ +6.774468474\times10^{-10}\,x^8+2.542007517\times10^{-12}x^9, & -1\le x<-\frac{1}{2}, \\ -.0155380140-1.0\times10^{-10}\,x-.01553801x^2+2.58966\times10^{-13}x^3-6.474172\times10^{-24}\,x^4 \\ -3.3333333\times10^{-12}x^5+1.3888888\times10^{-22}x^6-5.3571428\times10^{-33}x^7+5.2083333\times10^{-44}\,x^8 \\ -5.787037037\times10^{-55}x^9, & -\frac{1}{2}\le x<\frac{1}{2}, \\ -.0045417350-.0077211134\,x-.03377101\,x^2+2.556316\times10^{-5}x^3+.041666450\,x^4 \\ -.00112570050\,x^5+1.7424022\times10^{-5}x^6\ -1.2227052\times10^{-7}x^7+6.77446847\times10^{-10}\,x^8 \\ -2.54200751\times10^{-12}x^9, & \frac{1}{2}\le x\le 1. \end{cases}$$

Example 2

Consider the following system of third-order nonlinear boundary value problems by Noor et al. (2011b):

$$u'''=\begin{cases} \frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & -1\le x<-\frac{1}{2}, \\ \frac{u^3}{3!}+\frac{u^2}{2!}-3u+2, & -\frac{1}{2}\le x<\frac{1}{2}, \\ \frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & \frac{1}{2}\le x\le 1, \end{cases} \quad (8)$$

with boundary conditions $u(-1)=u(1)=u'(-1)=0.$

By using MVIM, we have the following correction functional of (8):

$$u_{k+1}(x)=\begin{cases} u_k(x)+\int_0^x \lambda(\xi)\left(u_k''' -\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}+\tilde{u}_k+1\right)\right)ds, & -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x \lambda(\xi)\left(u_k''' -\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}-3\tilde{u}_k+2\right)\right)ds, & -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x \lambda(\xi)\left(u_k''' -\left(\frac{\tilde{u}_k^3}{3!}+\frac{\tilde{u}_k^2}{2!}+\tilde{u}_k+1\right)\right)ds, & \frac{1}{2}\le x\le 1.\end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(\xi)=\frac{1}{2}(\xi-x)^2$. In this case, we have;

$$u_{k+1}(x)=\begin{cases} u_k(x)+\int_0^x \frac{1}{2}(\xi-x)^2\left(u_k''' -\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x \frac{1}{2}(\xi-x)^2\left(u_k''' -\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}-3u_k+2\right)\right)ds, & -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x \frac{1}{2}(\xi-x)^2\left(u_k''' -\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & \frac{1}{2}\le x\le 1.\end{cases}$$

Hence, we have following three cases in the given domain.

Case I: $-1\le x<-\frac{1}{2}$. In this case, we implement MVIM as follows;

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= \frac{1}{2}c_3x^2+c_2x+c_1\\ &+p\int_0^x \frac{1}{2}(\xi-x)^2\left(\left(u_0'''+pu_1'''+p^2u_2'''+\cdots\right)\right)ds,\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have

$$p^{(0)}: u_0(x)=c_1+c_2x+\frac{1}{2}c_3x^2,$$

$$p^{(1)}: u_1(x)=c_1+\frac{1}{2}c_3x^2-\frac{1}{6}x^3-\frac{1}{24}c_2x^4-\frac{1}{120}c_2^2x^5-\frac{1}{720}c_2^3x^6,$$

$$\begin{aligned} p^{(2)}: u_2(x)=&\frac{1}{6}(c_3-c_1)x^3-\frac{1}{24}(c_1c_2+1)x^4-\frac{1}{120}\left(c_2-c_3+c_1c_2^2\right)x^5-\frac{1}{720}\left(c_2^2-1+3c_2c_3\right)x^6\\ &-\left(\frac{1}{5040}c_2^3-\frac{1}{1008}c_2+\frac{1}{840}c_3c_2^2\right)x^7+\frac{1}{2520}c_2^2x^8+\frac{1}{181440}c_2^3x^9+\frac{1}{129600}c_2^4x^{10}\\ &+\frac{1}{1425600}c_2^5x^{11},\end{aligned}$$

Case II: $-\frac{1}{2}\le x<\frac{1}{2}$. In this case, we have following approximations;

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= \frac{1}{2}c_6x^2+c_5x+c_4\\ &+p\int_0^x \frac{1}{2}(\xi-x)^2\left(\left(u_0'''+pu_1'''+p^2u_2'''+\cdots\right)\right)ds,\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-p\int_0^x \frac{1}{2}(\xi-x)^2\left(-3\left(u_0+pu_1+p^2u_2+\cdots\right)+2\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have;

$$p^{(0)}: u_0(x)=c_4+c_5x+\frac{1}{2}c_6x^2,$$

$$p^{(1)}: u_1(x)=c_4+\frac{1}{2}c_6x^2-\frac{1}{3}x^3+\frac{1}{8}c_5x^4-\frac{1}{120}c_5^2x^5-\frac{1}{720}c_5^3x^6,$$

$$\begin{aligned} p^{(2)}: u_2(x)=&\frac{1}{6}(3c_4+c_6)x^3-\frac{1}{24}(c_4c_5+2)x^4-\frac{1}{120}\left(c_4c_5^2-c_5-c_6\right)x^5-\frac{1}{720}\left(c_5^2+6+3c_5c_6\right)x^6\\ &-\frac{1}{5040}\left(c_5^3-17c_5+6c_6c_5^2\right)x^7+\frac{1}{20160}c_5^2x^8-\frac{1}{8640}c_5^3x^9+\frac{1}{129600}c_5^4x^{10}\\ &+\frac{1}{1425600}c_5^5x^{11},\end{aligned}$$

Case III: $\frac{1}{2}\le x\le 1$. In this case, we implement MVIM as follows

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= \frac{1}{2}c_9x^2+c_8x+c_7\\ &+p\int_0^x (\xi-x)\left(\left(u_0'''+pu_1'''+p^2u_2'''+\cdots\right)\right)ds,\\ &-p\int_0^x (\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds\\ &-p\int_0^x (\xi-x)\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds\\ &-p\int_0^x (\xi-x)\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds.\end{aligned}$$

Comparing the coefficients of like powers of p, we have;

$$p^{(0)}: u_0(x) = c_7 + c_8 x + \frac{1}{2} c_9 x^2,$$

$$p^{(1)}: u_1(x) = c_7 + \frac{1}{2} c_9 x^2 - \frac{1}{6} x^3 - \frac{1}{24} c_8 x^4 - \frac{1}{120} c_8^2 x^5 - \frac{1}{720} c_8^3 x^6,$$

$$p^{(2)}: u_2(x) = \frac{1}{6}(c_9 - c_8) x^3 - \frac{1}{24}(c_7 c_8 + 1) x^4 - \frac{1}{120}\left(c_8 - c_9 - c_7 c_8^2\right) x^5 - \frac{1}{720}\left(c_8^2 - 1 + 3 c_8 c_9\right) x^6$$
$$- \left(\frac{1}{5040} c_8^3 - \frac{1}{1008} c_8 + \frac{1}{840} c_9 c_8^2\right) x^7 + \frac{1}{2520} c_8^2 x^8 + \frac{1}{181440} c_8^3 x^9 + \frac{1}{129600} c_8^4 x^{10}$$
$$+ \frac{1}{1425600} c_8^5 x^{11},$$

Hence, we have the following series solution

$$u(x) = \begin{cases} c_1 + c_2 x + \frac{1}{2} c_3 x^2 + \frac{1}{6}(c_3 - c_1 - 1) x^3 - \frac{1}{24}(c_1 c_2 + 1 + c_2) x^4 - \frac{1}{120}\left(c_2 - c_3 + c_2^2 + c_1 c_2^2\right) x^5 \\ - \frac{1}{720}\left(c_2^2 + c_2^3 - 1 + 3 c_2 c_3\right) x^6 - \left(\frac{1}{5040} c_2^3 - \frac{1}{1008} c_2 + \frac{1}{840} c_3 c_2^2\right) x^7 + \frac{1}{2520} c_2^2 x^8 \\ + \frac{1}{181440} c_2^3 x^9 + \frac{1}{129600} c_2^4 x^{10} + \frac{1}{1425600} c_2^5 x^{11}, & -1 \le x < -\frac{1}{2}, \\ c_4 + c_5 x + \frac{1}{2} c_6 x^2 + \frac{1}{6}(3 c_4 + c_6 - 2) x^3 - \frac{1}{24}(c_4 c_5 + 2 - 3 c_5) x^4 - \frac{1}{120}\left(c_4 c_5^2 - c_5 + c_5^2 - c_6\right) x^5 \\ - \frac{1}{720}\left(c_5^2 + c_5^3 + 6 + 3 c_5 c_6\right) x^6 - \frac{1}{5040}\left(c_5^3 - 17 c_5 + 6 c_6 c_5^2\right) x^7 + \frac{1}{20160} c_5^2 x^8 \\ - \frac{1}{8640} c_5^3 x^9 + \frac{1}{129600} c_5^4 x^{10} + \frac{1}{1425600} c_5^5 x^{11}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ c_7 + c_8 x + \frac{1}{2} c_9 x^2 + \frac{1}{6}(c_9 - c_7 - 1) x^3 - \frac{1}{24}(c_7 c_8 + 1 + c_8) x^4 - \frac{1}{120}\left(c_8 - c_9 + c_8^2 + c_7 c_8^2\right) x^5 \\ - \frac{1}{720}\left(c_8^2 + c_8^3 - 1 + 3 c_8 c_9\right) x^6 - \left(\frac{1}{5040} c_8^3 - \frac{1}{1008} c_8 + \frac{1}{840} c_9 c_8^2\right) x^7 + \frac{1}{2520} c_8^2 x^8 \\ + \frac{1}{181440} c_8^3 x^9 + \frac{1}{129600} c_8^4 x^{10} + \frac{1}{1425600} c_8^5 x^{11}, & \frac{1}{2} \le x \le 1. \end{cases} \quad (9)$$

By using boundary conditions and continuity conditions at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, we have a system of nonlinear equations. By using Newton's method for system of nonlinear equations, we have the following values of unknown constants:

$$c_1 = .2186789500,\ c_2 = .3090217400,\ c_3 = -.1759445100,\ c_4 = .2133531700,$$
$$c_5 = .2758655200,\ c_6 = -.3142669500,\ c_7 = .2105692200,\ c_8 = .2911758100,$$
$$c_9 = -.368184170 \quad (10)$$

From (90 and (10), we have following series solution of system of second-order nonlinear boundary value problems (8) (Figure 2)

$$u(x) = \begin{cases} .2186789500 + .3090217400\,x - .087972255\,x^2 - .232437243\,x^3 - .0573582620\,x^4 \\ - .00207878574\,x^5 + .00144181631\,x^6 + 3.20716105 \times 10^{-4}\,x^7 + 3.78946173 \times 10^{-5}\,x^8 \\ + 1.789067592 \times 10^{-6}\,x^9 + 7.03640992 \times 10^{-8}\,x^{10} + 1.976730582 \times 10^{-9}\,x^{11} \\ + 2.671067731 \times 10^{-16}\,x^{12} + 1.586431737 \times 10^{-19}\,x^{13}, & -1 \le x < -\frac{1}{2}, \\ .3831795500 - 5.1 \times 10^{9}\,x - .4581052550\,x^2 - 1.533333333 \times 10^{-9}\,x^3 + .07361132750\,x^4 \\ + 1.878815358 \times 10^{-10}\,x^5 + .00103976601\,x^6 + 5.22182952 \times 10^{-12}\,x^7 - .0014880952\,x^8 \\ - 2.670304233 \times 10^{-13}\,x^9 + 6.450892857 \times 10^{-23}\,x^{10} + 1.960680465 \times 10^{-31}\,x^{11} \\ + 5.084476461 \times 10^{41}\,x^{12} - 1.994679227 \times 10^{50}\,x^{13}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ .3841038600 - .0077211134\,x - .4335136550\,x^2 - .03575997167\,x^3 + .09379713209\,x^4 \\ + .001698941726\,x^5 - .002592978883\,x^6 - 3.70547476 \times 10^{-5}\,x^7 + 2.495676630 \times 10^{-5}\,x^8 \\ - 1.2801513 \times 10^{-7}\,x^9 + 3.61426098 \times 10^{-10}\,x^{10} - 3.3441216 \times 10^{-13}\,x^{11} + 2.6710699 \times 10^{-16}\,x^{12} \\ - 1.586433380 \times 10^{-19}\,x^{13}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Example 3

Consider following system of fourth-order nonlinear boundary value problems:

$$u^{(iv)} = \begin{cases} \frac{u^3}{3!} + \frac{u^2}{2!} + u + 1, & -1 \le x < -\frac{1}{2}, \\ \frac{u^3}{3!} + \frac{u^2}{2!} - 3u + 2, & -\frac{1}{2} \le x < \frac{1}{2}, \\ \frac{u^3}{3!} + \frac{u^2}{2!} + u + 1, & \frac{1}{2} \le x \le 1, \end{cases} \quad (11)$$

with boundary conditions $u(-1) = u(1) = u\left(-\frac{1}{2}\right) = u\left(\frac{1}{2}\right) = 0.$

By using MVIM, we have the following correction functional of (11):

$$u_{k+1}(x) = \begin{cases} u_k(x) + \int_0^x \lambda(\xi)\left(u_k^{(iv)} - \left(\frac{\tilde{u}_k^3}{3!} + \frac{\tilde{u}_k^2}{2!} + \tilde{u}_k + 1\right)\right) d\xi, & -1 \le x < -\frac{1}{2}, \\ u_k(x) + \int_0^x \lambda(\xi)\left(u_k^{(iv)} - \left(\frac{\tilde{u}_k^3}{3!} + \frac{\tilde{u}_k^2}{2!} - 3\tilde{u}_k + 2\right)\right) d\xi, & -\frac{1}{2} \le x < \frac{1}{2}, \\ u_k(x) + \int_0^x \lambda(\xi)\left(u_k^{(iv)} - \left(\frac{\tilde{u}_k^3}{3!} + \frac{\tilde{u}_k^2}{2!} + \tilde{u}_k + 1\right)\right) d\xi, & \frac{1}{2} \le x \le 1. \end{cases}$$

Making the correction functional stationary, the Lagrange

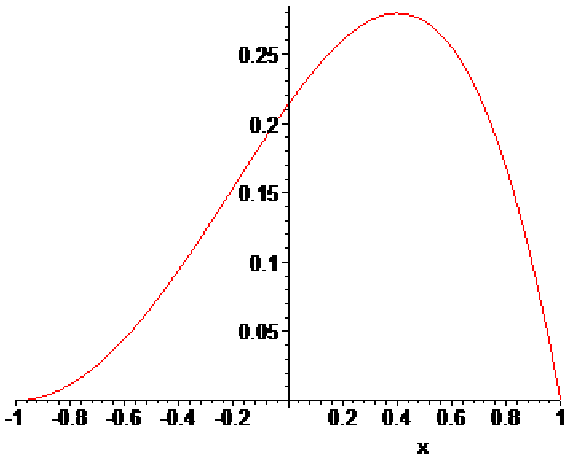

Figure 2. Graphical representation of series solution of system of third-order nonlinear boundary value problem (8) by modified variational iteration method.

multipliers is identified as $\lambda(\xi)=\frac{(\xi-x)^3}{3!}.$ Consequently, we have,

$$u_{k+1}(x)=\begin{cases} u_k(x)+\int_0^x \frac{(\xi-x)^3}{3!}\left(u_k^{(iv)}-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & -1\le x<-\frac{1}{2}, \\ u_k(x)+\int_0^x \frac{(\xi-x)^3}{3!}\left(u_k^{(iv)}-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}-3u_k+2\right)\right)ds, & -\frac{1}{2}\le x<\frac{1}{2}, \\ u_k(x)+\int_0^x \frac{(\xi-x)^3}{3!}\left(u_k^{(iv)}-\left(\frac{u_k^3}{3!}+\frac{u_k^2}{2!}+u_k+1\right)\right)ds, & \frac{1}{2}\le x\le 1. \end{cases}$$

Applying MVIM, we have following three cases in the given domain.

Case I: $-1\le x<-\frac{1}{2}$. In this case, we implement MVIM as follows

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= c_4\frac{x^3}{3!}+c_3\frac{x^2}{2!}+c_2x+c_1 \\ &+p\int_0^x \frac{(\xi-x)^3}{3!}\left(\left(u_0^{(iv)}+pu_1^{(iv)}+p^2u_2^{(iv)}+\cdots\right)\right)ds, \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds. \end{aligned}$$

Comparing the coefficients of like powers of p, we have

$$p^{(0)}:u_0(x)=c_4\frac{x^3}{3!}+c_3\frac{x^2}{2!}+c_2x+c_1,$$

$$p^{(1)}:u_1(x)=c_1+\frac{1}{2}c_3x^2+\frac{1}{6}x^3c_4+\frac{1}{24}x^4+\frac{1}{120}x^5c_2+\frac{1}{720}x^6c_2^2+\frac{1}{5040}x^7c_2^3,$$

$$\begin{aligned} p^{(2)}:u_2(x)=&\left(\frac{1}{24}c_1-\frac{1}{24}c_3\right)x^4+\left(\frac{1}{120}c_2c_1-\frac{1}{120}c_4\right)x^5+\left(\frac{1}{720}c_2^2c_1+\frac{1}{720}c_3-\frac{1}{720}\right)x^6 \\ &+\frac{1}{1680}\left(\frac{1}{3}c_4+c_3c_2-\frac{1}{3}c_2\right)x^7+\left(\frac{1}{40320}+\frac{1}{6720}c_2^2c_3+\frac{1}{10080}c_2c_4-\frac{1}{40320}c_2^2\right)x^8 \\ &+\left(\frac{1}{36288}c_2c_4-\frac{1}{362880}c_2^2+\frac{1}{60480}\right)c_2x^9+\frac{11}{1814400}c_2^2x^{10}+\frac{29}{39916800}c_2^3x^{11} \\ &+\frac{1}{13305600}c_2^4x^{12}+\frac{1}{172972800}c_2^5x^{13}, \end{aligned}$$

Case II: $-\frac{1}{2}\le x<\frac{1}{2}$. In this case, we have following approximations

$$\begin{aligned} u_0+pu_1+p^2u_2+\cdots &= c_8\frac{x^3}{3!}+c_7\frac{x^2}{2!}+c_6x+c_5 \\ &+p\int_0^x \frac{(\xi-x)^3}{3!}\left(\left(u_0^{(iv)}+pu_1^{(iv)}+p^2u_2^{(iv)}+\cdots\right)\right)ds, \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds \\ &-p\int_0^x \frac{(\xi-x)^3}{3!}\left(-3\left(u_0+pu_1+p^2u_2+\cdots\right)+2\right)ds. \end{aligned}$$

Comparing the coefficients of like powers of p, we have

$$p^{(0)}:u_0(x)=c_8\frac{x^3}{3!}+c_7\frac{x^2}{2!}+c_6x+c_5,$$

$$p^{(1)}:u_1(x)=c_5+\frac{1}{2}c_7x^2+\frac{1}{6}x^3c_8+\frac{1}{12}x^4-\frac{1}{40}x^5c_6+\frac{1}{720}x^6c_6^2+\frac{1}{5040}x^7c_6^3,$$

$$\begin{aligned} p^{(2)}:u_2(x)=&-\left(\frac{1}{8}c_5+\frac{1}{24}c_7\right)x^4+\frac{1}{120}(c_6c_5-c_8)x^5+\left(\frac{1}{720}c_6^2c_5-\frac{1}{240}c_7-\frac{1}{360}\right)x^6 \\ &+\frac{1}{1680}(c_6+c_6c_7-c_8)x^7+\left(-\frac{1}{6720}-\frac{1}{40320}c_6^2+\frac{1}{6720}c_6^2c_7+\frac{1}{10080}c_8c_6\right)x^8 \\ &+\left(\frac{19}{362880}-\frac{1}{362880}c_6^2+\frac{1}{36288}c_6c_8\right)c_6x^9+\frac{1}{403200}c_6^2x^{10}-\frac{59}{39916800}c_6^3x^{11} \\ &+\frac{1}{13305600}c_6^4x^{12}+\frac{1}{172972800}c_6^5x^{13}, \end{aligned}$$

Case III: $\frac{1}{2}\le x\le 1$. In this case, we implement MVIM as follows

$$u_0+pu_1+p^2u_2+\cdots=c_{12}\frac{x^3}{3!}+c_{11}\frac{x^2}{2!}+c_{10}x+c_9$$
$$+p\int_0^x\frac{(\xi-x)^3}{3!}\left(\left(u_0^{(iv)}+pu_1^{(iv)}+p^2u_2^{(iv)}+\cdots\right)\right)ds,$$
$$-p\int_0^x\frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^3}{3!}ds$$
$$-p\int_0^x\frac{(\xi-x)^3}{3!}\frac{\left(u_0+pu_1+p^2u_2+\cdots\right)^2}{2!}ds$$
$$-p\int_0^x\frac{(\xi-x)^3}{3!}\left(\left(u_0+pu_1+p^2u_2+\cdots\right)+1\right)ds.$$

Comparing the coefficients of like powers of p, we have

$$p^{(0)}:u_0(x)=c_{12}\frac{x^3}{3!}+c_{11}\frac{x^2}{2!}+c_{10}x+c_9,$$
$$p^{(1)}:u_1(x)=c_9+\frac{1}{2}c_{11}x^2+\frac{1}{6}x^3c_{12}+\frac{1}{24}x^4+\frac{1}{120}x^5c_{10}+\frac{1}{720}x^6c_{10}^{\ 2}+\frac{1}{5040}x^7c_{10}^{\ 3},$$
$$p^{(2)}:u_2(x)=\left(\frac{1}{24}c_9-\frac{1}{24}c_{11}\right)x^4+\left(\frac{1}{120}c_{10}c_9-\frac{1}{120}c_{12}\right)x^5+\left(\frac{1}{720}c_{10}^{\ 2}c_9+\frac{1}{720}c_{11}-\frac{1}{720}\right)x^6$$
$$+\frac{1}{1680}\left(\frac{1}{3}c_{12}+c_{11}c_{10}-\frac{1}{3}c_{10}\right)x^7+\left(\frac{1}{40320}+\frac{1}{6720}c_{10}^{\ 2}c_{11}+\frac{1}{10080}c_{10}c_{12}-\frac{1}{40320}c_{10}^{\ 2}\right)x^8$$
$$+\left(\frac{1}{36288}c_{10}c_{12}-\frac{1}{362880}c_{10}^{\ 2}+\frac{1}{60480}\right)c_{10}x^9+\frac{11}{1814400}c_{10}^{\ 2}x^{10}+\frac{29}{39916800}c_{10}^{\ 3}x^{11}$$
$$+\frac{1}{13305600}c_{10}^{\ 4}x^{12}+\frac{1}{172972800}c_{10}^{\ 5}x^{13},$$

Hence, we have the following series solution

$$u(x)=\begin{cases}c_1+c_2x+\frac{1}{2}c_3x^2+\frac{1}{6}x^3c_4+(1+c_1-c_3)\frac{1}{24}x^4+(c_1c_2-c_4+c_2)\frac{1}{120}x^5\\ +\left(c_2^{\ 2}+c_3-1+c_2^{\ 2}c_1\right)\frac{1}{720}x^6+\left(\frac{1}{5040}c_4+\frac{1}{1680}c_3c_2+\frac{1}{5040}c_2^{\ 3}-\frac{1}{5040}c_2\right)x^7\\ +\left(\frac{1}{40320}+\frac{1}{6720}c_2^{\ 2}c_3+\frac{1}{10080}c_2c_4-\frac{1}{40320}c_2^{\ 2}\right)x^8\\ +\left(\frac{1}{36288}c_2c_4-\frac{1}{362880}c_2^{\ 2}+\frac{1}{60480}\right)c_2x^9+\frac{11}{1814400}c_2^{\ 2}x^{10}+\frac{29}{39916800}c_2^{\ 3}x^{11}\\ +\frac{1}{13305600}c_2^{\ 4}x^{12}+\frac{1}{172972800}c_2^{\ 5}x^{13}, & -1\le x<-\frac{1}{2},\\ c_5+c_6x+\frac{1}{2}c_7x^2+\frac{1}{6}x^3c_5x^4-\left(\frac{1}{8}c_5+\frac{1}{24}c_7-\frac{1}{12}\right)x^4+\frac{1}{120}(c_6c_5-c_8-3c_6)x^5\\ +\left(\frac{1}{720}c_6^{\ 2}+\frac{1}{720}c_6^{\ 2}c_5-\frac{1}{240}c_7-\frac{1}{360}\right)x^6+\frac{1}{1680}\left(c_6+c_6c_7-c_8+\frac{1}{3}c_6^{\ 3}\right)x^7\\ -\left(\frac{1}{6720}+\frac{1}{40320}c_6^{\ 2}-\frac{1}{6720}c_6^{\ 2}c_7-\frac{1}{10080}c_8c_6\right)x^8+\frac{c_6}{36288}\left(\frac{19}{10}-\frac{1}{10}c_6^{\ 2}+c_6c_8\right)x^9\\ +\frac{1}{403200}c_6^{\ 2}x^{10}-\frac{59}{39916800}c_6^{\ 3}x^{11}+\frac{1}{13305600}c_6^{\ 4}x^{12}+\frac{1}{172972800}c_6^{\ 5}x^{13}, & -\frac{1}{2}\le x<\frac{1}{2},\\ c_9+c_{10}x+\frac{1}{2}c_{11}x^2+\frac{1}{6}x^3c_{12}+(1+c_9-c_{11})\frac{1}{24}x^4+(c_9c_{10}-c_{12}+c_{10})\frac{1}{120}x^5\\ +\left(c_{10}^{\ 2}+c_{11}-1+c_{10}^{\ 2}c_9\right)\frac{1}{720}x^6+\left(\frac{1}{5040}c_{12}+\frac{1}{1680}c_{11}c_{10}+\frac{1}{5040}c_{10}^{\ 3}-\frac{1}{5040}c_{10}\right)x^7\\ +\left(\frac{1}{40320}+\frac{1}{6720}c_{10}^{\ 2}c_{11}+\frac{1}{10080}c_{10}c_{12}-\frac{1}{40320}c_{10}^{\ 2}\right)x^8\\ +\left(\frac{1}{36288}c_{10}c_{12}-\frac{1}{362880}c_{10}^{\ 2}+\frac{1}{60480}\right)c_{10}x^9+\frac{11}{1814400}c_{10}^{\ 2}x^{10}+\frac{29}{39916800}c_{10}^{\ 3}x^{11}\\ +\frac{1}{13305600}c_{10}^{\ 4}x^{12}+\frac{1}{172972800}c_{10}^{\ 5}x^{13}, & \frac{1}{2}\le x\le 1.\end{cases}$$

(12)

By using boundary conditions and continuity conditions at $x=-\frac{1}{2}$ and $x=\frac{1}{2}$, we have a system of nonlinear equations. By using Newton's method for system of nonlinear equations, we have the following values of unknown constants:

$$c_1=.3841038600,\ c_2=.0077211118,\ c_3=-.8670273200,\ c_4=.2145598100,$$
$$c_5=.3831795500,\ c_6=-.0000000051,\ c_7=-.9162105100,\ c_8=-.0000000092, \quad (13)$$
$$c_9=.3841038600,\ c_{10}=-.0077211134,\ c_{11}=-.8670273100,\ c_{12}=-.2145598300.$$

From (13) and (12), we have following analytic solution of system of forth-order nonlinear boundary value problem (11) (Figure 3),

$$u(x)=\begin{cases}.3841038600+.0077211118x-.4335136600x^2+.03575996833x^3+.09379713250x^4\\ -.001698941579x^5-.002592978897x^6+3.70547448\times10^{-5}x^7+2.49567662\times10^{-5}x^8\\ +1.280151057\times10^{-7}x^9+3.614259489\times10^{-9}x^{10}+3.344119612\times10^{-13}x^{11}\\ +2.671067731\times10^{-16}x^{12}+1.586431737\times10^{-19}x^{13}, & -1\le x<-\frac{1}{2},\\ .3831795500-5.1\times10^{-9}x-.4581052550x^2-1.533333333\times10^{-9}x^3+.07361132750x^4\\ +1.878815358\times10^{-10}x^5+.00103976601x^6+5.22182952\times10^{-12}x^7-.00014880952x^8\\ -2.670304233\times10^{-13}x^9+6.450892857\times10^{-23}x^{10}+1.960680465\times10^{-31}x^{11}\\ +5.084476461\times10^{41}x^{12}-1.994679227\times10^{-50}x^{13}, & -\frac{1}{2}\le x<\frac{1}{2},\\ .3841038600-.0077211134x-.4335136550x^2-.03575997167x^3+.09379713209x^4\\ +.001698941726x^5-.002592978883x^6-3.705474769\times10^{-5}x^7+2.495676630\times10^{-5}x^8\\ -1.2801513\times10^{7}x^9+3.61426098\times10^{-10}x^{10}-3.3441216\times10^{-13}x^{11}+2.6710699\times10^{-16}x^{12}\\ -1.586433380\times10^{-19}x^{13}, & \frac{1}{2}\le x\le 1.\end{cases}$$

Conclusion

In this paper, we have used the MVIM for solving system of second, third and fourth-orders obstacle boundary value problems. The proposed technique is an elegant coupling of the variational iteration method and the homotopy perturbation method. In the MVIM, main advantages of both the techniques have been united together to find the approximate solution of nonlinear obstacle boundary value problems in more efficient way. The results are also presented graphically which demonstrate the nature of obstacle. We analyzed that our proposed method is well suited for use in higher order obstacle boundary value problems as it provides best solution in less number of iterations and reduces the computational work. This is another advantage of this method. The ideas and techniques of this paper may stimulate further applications of the MVIM for solving

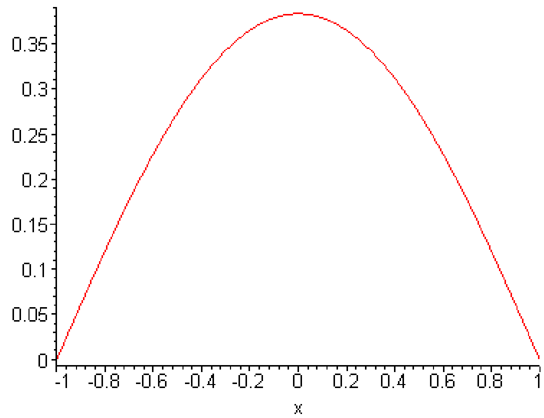

Figure 3. Graphical representation of series solution of system of forth-order nonlinear boundary value problem (11) by modified variational iteration method.

more complicated problems, which arise in several braches of pure and applied sciences such as transportation, network analysis, optimization, risk analysis and financial and equilibrium problems.

Future directions

We would like to mention that the system of nonlinear boundary value associated with obstacle and unilateral problems considered in this paper can be studied from different point of views such sensitivity analysis, dynamical and stability analysis. Several other analytical and numerical techniques including finite difference, finite element, splines and nonpolynomial spline can be used to solve such type of nonlinear system of boundary value problems. This is an interesting problem from both application point of view and numerical analysis to verify the implementation and efficiency of the proposed iterative methods. This is another direction of future research in the dynamic and fast growing field of mathematical and engineering sciences. The ideas and technique of the auxiliary principle technique may be extended for solving the mixed quasi variational-like inequalities and the equilibrium problems. We hope that this direction of research will yield some new and novel applications of these techniques.

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