academicJournals

Vol. 9(2), pp. 20-25, 30 January, 2014
DOI: 10.5897/IJPS2013.4083
ISSN 1992 - 1950 © 2014 Academic Journals
http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Vectorial reduced differential transform (VRDT) method for the solution of inhomogeneous Cauchy-Riemann system

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Accepted 27 January, 2014

It is well known that initial-value problem for the Cauchy-Riemann system is ill-posed and the problem with such Hadamard instability cannot be solved unless the initial data are analytic. In this paper, we present the vectorial reduced differential transform (VRDT) method to solve initial-value problem for the inhomogeneous Cauchy-Riemann system with analytic data. The VRDTM solution vector achieved is in the form of infinite series whose compact form is in agreement with the exact solution vector.

Key words: Inhomogeneous Cauchy-Riemann system, initial-value problem, vectorial reduced differential transform.

INTRODUCTION

This paper deals with the system of first-order linear equations:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}, \qquad x \in \mathbb{R}, y > 0, \tag{1}$$

for the desired vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ involving real-valued functions $u(x,y)$ and $v(x,y)$. Collectively, System (1) is elliptic while individually both the partial differential equations are hyperbolic for the ellipticity of the system we refer to Wendland (1979). If $f(x,y) \equiv 0 \equiv g(x,y)$, Equation (1) is the Cauchy-Riemann system and dependent variables u, v are analytic. Thinking of y as a time variable and of data for the vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ as being given on $y = 0$, we are mainly concerned with the inhomogeneous Cauchy-Riemann System (1) subject to the following initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \quad x \in \mathbb{R}, \tag{2}$$

where $\phi(x)$ and $\psi(x)$ are analytic.

It is well known that initial value problem for the Cauchy-Riemann system is ill-posed. The inherent instability of this system, for the first time was discussed by Hadamard (1923). Farmer and Howison (2006) illustrate the ill-posed nature of the system in various contexts.

The paper of Joseph and Saut (1990), which is the main source of motivation to our present work, associates the ill-posedness of Cauchy problem with the non-existence of solution to the initial-value problem for non-analytic data. They show that the problems which are Hadamard unstable cannot be solved unless the initial data are analytic. Reichel (1986) analyses several fast numerical methods based on solving initial-value problems for the Cauchy-Riemann system. She discusses the techniques for analytic continuation of

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Subject Classification: 35F40, 35J46.

conformal mappings and indicates the available methods for finding analytic continuations which use Taylor coefficients or their approximations for the analytic functions, see for example Baker and Graves-Morris (1981), Gustafson (1978), Niethammer (1977) and Henrici (1966). Reichel (1986) also shows the stability and accuracy of her schemes through numerous applications.

During the past couple of decades, researchers have been engrossed to constructing the approximate analytic solution for the partial differential equations. Zhou (1986) introduced the differential transform method. To solve Cauchy problem for various PDEs, the reduced differential transform method has recently been used by Keskin and Oturance (2009), Keskin (2010), Cenesiz et al. (2010), Taha (2011) and Hesam et al. (2012).

In this paper, we present vectorial reduced differential transform (VRDT) method to solve the initial-value problem for the inhomogeneous Cauchy-Riemann system. The method is applied in various situations of Cauchy-Riemann system with a variety of initial data. The VRDTM solutions are in the form of infinite series whose compact forms are in agreement with exact solutions.

VECTORIAL REDUCED DIFFERENTIAL TRANSFORM (VRDT)

Definition 1

Let $u(x,y)$ be an analytic function (obviously sufficiently smooth with respect to *x* and *y* in the domain of definition). The reduced differential transform $U_k(x)$ of $u(x,y)$ is defined as (Keskin and Oturance, 2009):

$$U_k(x) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}\,u(x,y)\right\}_{y=0}. \tag{3}$$

The inverse-RDT, $u(x,y)$ of $U_k(x)$ is defined as:

$$u(x,y) = \sum_{k=0}^{\infty} y^k U_k(x). \tag{4}$$

From (3) and (4) the following result is obtained:

$$u(x,y) = \sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}\,u(x,y)\right\}_{y=0}. \tag{5}$$

The basic RDTs are given in Table 1 and can be proved using Definitions (3) and (4), see for the details (Keskin, 2010).

The definition of reduced differential transform can be extended to the vectors of analytic functions which are given as follow:

Definition 2

Let $[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T$ denotes a column vector with elements as analytic functions (obviously sufficiently smooth with respect to x, y in the domain of definition). The vectorial reduced differential transform (VRDT) of the vector $[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T$, given by the vector $[U_k(x)\quad V_k(x)\quad W_k(x)\quad \cdots]^T$, is defined as:

$$[U_k(x)\quad V_k(x)\quad W_k(x)\quad \cdots]^T = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T\right\}_{y=0} \tag{6}$$

The inverse-VRDT vector, $[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T$ of vector $[U_k(x)\quad V_k(x)\quad W_k(x)\quad \cdots]^T$ is further defined by:

$$[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T = \left[\sum_{k=0}^{\infty} y^k U_k(x)\quad \sum_{k=0}^{\infty} y^k V_k(x)\quad \sum_{k=0}^{\infty} y^k W_k(x)\quad \cdots\right]^T. \tag{7}$$

The following result is immediately obtained from (6) and (7).

$$[u(x,y)\quad v(x,y)\quad w(x,y)\quad \cdots]^T = \left[\sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}u(x,y)\right\}_{y=0}\quad \sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}v(x,y)\right\}_{y=0}\quad \sum_{k=0}^{\infty}\frac{y^k}{k!}\left\{\frac{\partial^k}{\partial y^k}w(x,y)\right\}_{y=0}\quad \cdots\right]^T \tag{8}$$

VRDT METHOD TO SOLVE INITIAL VALUE PROBLEM FOR THE INHOMOGENEOUS CAUCHY-RIEMANN SYSTEM

A proper posing of initial-value problem for the Cauchy-Riemann system is very important since the existence of a unique solution which is also guaranteed by the Cauchy-Kowalevsky theorem (Walter, 1985), may not continuously depend upon initial data (Hadamard, 1923). Joseph and Saut (1990) show that the problems with Hadamard instability cannot be solved unless the initial data are analytic.

To solve the initial-value Problem (2) with analytic data functions, for the inhomogeneous Cauchy-Riemann System (1), the VRDT method proceeds as follows. The VRDT on Problem (1) and (2) yields the following recurrence relations:

$$(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k + \begin{bmatrix}F(x)\\G(x)\end{bmatrix}_k, \tag{9}$$

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0 = \begin{bmatrix}\phi(x)\\\psi(x)\end{bmatrix}, \tag{10}$$

where the vectors $(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1}$, $\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k$ and $\begin{bmatrix}F(x)\\G(x)\end{bmatrix}_k$ represent the VRDTs for the vectors $\frac{\partial}{\partial y}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}$, $\frac{\partial}{\partial x}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}$ and $\begin{bmatrix}f(x,y)\\g(x,y)\end{bmatrix}$, respectively.

Table 1. Basic reduced differential transforms.

S/N	Function	Reduced Differential Transform
1.	$u(x,y)$	$U_k(x) = \frac{1}{k!}\left\{\frac{\partial^k}{\partial y^k}\,u(x,y)\right\}_{y=0}$
2.	$u(x,y) \pm v(x,y)$	$U_k(x) \pm V_k(x)$
3.	$\alpha\, u(x,y)$	$\alpha\, U_k\,(x)$, α being a constant
4.	$u(x,y)\; v(x,y)$	$\sum_{r=0}^{k} U_r(x)\, V_{k-r}(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x)$
5.	$x^m y^n\; u(x,y)$	$x^m U_{k-n}(x)$
6.	$x^m y^n$	$x^m\,\delta(k-n), \quad \delta(k) = \begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}$
7.	$\frac{\partial^n}{\partial x^n}\; u(x,y)$	$\frac{\partial^n}{\partial x^n} U_k(x)$
8.	$\frac{\partial^r}{\partial y^r}\; u(x,y)$	$(k+1)(k+2)\ldots(k+r)U_{\,k+r}(x)$

Equations (9) and (10) straightaway produce all vectors $\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_k$ iteratively. The inverse-VRDTs of the set of vectors $\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0$, $\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_1$, $\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_2$, …, $\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_n$ then give the VRDTM solution vector $\begin{bmatrix}\tilde{u}(x,y)\\ \tilde{v}(x,y)\end{bmatrix}$ in the form:

$$\begin{bmatrix}\tilde{u}(x,y)\\ \tilde{v}(x,y)\end{bmatrix}_n = \begin{bmatrix}\sum_{k=0}^{n} y^k U_k(x)\\ \sum_{k=0}^{n} y^k V_k(x)\end{bmatrix}, \qquad (11)$$

where n is the order of approximation for the solution. The exact solution vector $\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix}$ of Problems (1) and (2) then becomes:

$$\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix} = \begin{bmatrix}\lim_{n\to\infty} \tilde{u}_n(x,y)\\ \lim_{n\to\infty} \tilde{v}_n(x,y)\end{bmatrix}. \qquad (12)$$

IMPLEMENTATION OF VRDT METHOD

The method is implemented on a variety of initial-value problems for the homogeneous and inhomogeneous Cauchy-Riemann systems. The data of the model problems is given in the Table 2.

Model Problem-A

In this type of model problem, we consider the homogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix}, \quad x\in\mathbb{R}, y>0, \qquad (13)$$

subject to the initial condition

$$\begin{bmatrix}u(x,0)\\ v(x,0)\end{bmatrix} = \begin{bmatrix}0\\ \sinh x\end{bmatrix}, x\in\mathbb{R}, \qquad (14)$$

having exact solution vector given by:

$$\begin{bmatrix}u(x,y)\\ v(x,y)\end{bmatrix} = \begin{bmatrix}\sin y\cosh x\\ \cos y\sinh x\end{bmatrix}. \qquad (15)$$

The VRDT on (13) and (14) gives:

$$(k+1)\begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_{k+1} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}U_x(x)\\ V_x(x)\end{bmatrix}_k, \qquad (16)$$

$$\begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_0 = \begin{bmatrix}0\\ \sinh x\end{bmatrix}, \qquad (17)$$

and iterative relations (16) and (17) yield the following vectors:

$$\begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_0 = \begin{bmatrix}0\\ \sinh x\end{bmatrix}, \begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_1 = \begin{bmatrix}\cosh x\\ 0\end{bmatrix}, \begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_2 = \begin{bmatrix}0\\ \frac{-1}{2!}\sinh x\end{bmatrix}, \begin{bmatrix}U(x)\\ V(x)\end{bmatrix}_3 = \begin{bmatrix}\frac{-1}{3!}\cosh x\\ 0\end{bmatrix}, \cdots \qquad (18)$$

Table 2. Data of model problems at a glance.

Prob.	$f(x,y)$	$g(x,y)$	$\phi(x)$	$\psi(x)$
A	0	0	0	$\sinh x$
B	0	0	$\sin(x)$	$\cos(x)$
C	$(a^2-1)e^{-y}\sin(ax)$	0	$\sin(ax)$	$a\cos(ax)$
D	$(a+b)\sin(ax)\cos(by)$	$(a-b)\cos(ax)\sin(by)$	0	$\cos(ax)$
E	$4y^3\sin(4x)-2(1-x)\sin(4y)$	$4x(2-x)\cos(4y)+4y^4\cos(4x)$	0	0
F	$4y^3\sin(x)$	$4y^3+y^4\cos(x)$	0.001	0.001

Using inverse-VRDTs of (18), we obtain the VRDTM solution vector:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \left\{y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right\}\cosh x \\ \left\{1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right\}\sinh x \end{bmatrix}, \quad (19)$$

whose compact form takes the form:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \sin y \\ \sinh x \cos y \end{bmatrix},$$

which is the same as exact solution vector.

Model Problem-B

We consider the homogeneous Cauchy-Riemann system (13) subject to initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (20)$$

having the exact solution vector given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(x) \\ e^{-y}\cos(x) \end{bmatrix}. \quad (21)$$

Now we use the VRDT method to solve this problem. The VRDT on (13), (20) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k, \quad (22)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}. \quad (23)$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} -\sin(x) \\ -\cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2!}\sin(x) \\ \frac{1}{2!}\cos(x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{1}{3!}\sin(x) \\ -\frac{1}{3!}\cos(x) \end{bmatrix}, \dots .$$

Finally the inverse-VRDTs of the above vectors give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \left\{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\right\}\sin(x) \\ \left\{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\right\}\cos(x) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(x) \\ e^{-y}\cos(x) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-C

We consider the following inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} (a^2-1)e^{-y}\sin(ax) \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \quad (24)$$

where *a* is a given parameter, subject to initial condition

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (25)$$

having the exact solution vector given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(ax) \\ a\,e^{-y}\cos(ax) \end{bmatrix}. \quad (26)$$

It is interesting to note that the problem takes the form of model problem-B if $a = 1$. The VRDT on (24) and (25) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} (a^2-1)\sin(ax)\left\{\delta(k) - \delta(k-1) + \frac{1}{2!}\delta(k-2) - \dots\right\} \\ 0 \end{bmatrix}, \quad (27)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}. \quad (28)$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} \sin(ax) \\ a\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} -\sin(ax) \\ -a\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2!}\sin(ax) \\ \frac{a}{2!}\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{1}{3!}\sin(ax) \\ -\frac{a}{3!}\cos(ax) \end{bmatrix}, \dots .$$

Finally the inverse-VRDTs of the above vectors give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\}\sin(ax) \\ a\{1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \dots\}\cos(ax) \end{bmatrix} = \begin{bmatrix} e^{-y}\sin(ax) \\ a\, e^{-y}\cos(ax) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-D

For the inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} (a+b)\sin(ax)\cos(by) \\ (a-b)\cos(ax)\sin(by) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \quad (29)$$

where a, b are given parameters, we prescribe the following initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \quad x \in \mathbb{R}, \quad (30)$$

whose exact solution vector is given by:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sin(ax)\sin(by) \\ \cos(ax)\cos(by) \end{bmatrix}. \quad (31)$$

Now we use VRDT method to solve this initial-value problem. The VRDT on (29) and (30) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} (a+b)\sin(ax)\left\{\delta(k) - \frac{b^2}{2!}\delta(k-2) + \dots\right\} \\ (a-b)\cos(ax)\left\{b\delta(k-1) - \frac{b^3}{3!}\delta(k-3) + \dots\right\} \end{bmatrix}, \quad (32)$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}. \quad (33)$$

The above iterative relations then yield the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} b\sin(ax) \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} 0 \\ -\frac{b^2}{2!}\cos(ax) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} -\frac{b^3}{3!}\sin(ax) \\ 0 \end{bmatrix}, \dots \quad (34)$$

Finally the inverse-VRDTs of vectors in (34) give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} \{by - \frac{(by)^3}{3!} + \frac{(by)^5}{5!} - \dots\}\sin(ax) \\ \{1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \dots\}\cos(ax) \end{bmatrix} = \begin{bmatrix} \sin(ax)\sin(by) \\ \cos(ax)\cos(by) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-E

In this problem, VRDT method is applied on inhomogeneous Cauchy-Riemann system for the homogeneous initial data. We consider the following inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} 4y^3\sin(4x) - 2(1-x)\sin(4y) \\ 4x(2-x)\cos(4y) + 4y^4\cos(4x) \end{bmatrix}, \quad x \in \mathbb{R}, y > 0, \quad (35)$$

subject to initial condition:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x \in \mathbb{R}, \quad (36)$$

having exact solution vector given as:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\sin(4y) \end{bmatrix}. \quad (37)$$

Now we use the VRDT method to solve this problem. The VRDT on (35) and (36) gives:

$$(k+1)\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} U_x(x) \\ V_x(x) \end{bmatrix}_k + \begin{bmatrix} 4\sin(4x)\,\delta(k-3) - 2(1-x)\left\{4\delta(k-1) - \frac{4^3}{3!}\delta(k-3) + \cdots\right\} \\ 4x(2-x)\left\{\delta(k) - \frac{4^2}{2!}\delta(k-2) + \cdots\right\} + 4\cos(4x)\,\delta(k-4) \end{bmatrix},$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using above iterative relation we obtain the following vectors:

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 4x(2-x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_3 = \begin{bmatrix} 0 \\ -\frac{4^3}{3!}x(2-x) \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_4 = \begin{bmatrix} \sin(4x) \\ 0 \end{bmatrix}, \begin{bmatrix} U(x) \\ V(x) \end{bmatrix}_5 = \begin{bmatrix} 0 \\ \frac{4^5}{5!}x(2-x) \end{bmatrix}, \dots$$

The inverse-VRDTs of the above vectors finally give:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} U_k(x)\, y^k \\ \sum_{k=0}^{\infty} V_k(x)\, y^k \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\left\{4y - \frac{(4y)^3}{3!} + \frac{(4y)^5}{5!} - \cdots\right\} \end{bmatrix} = \begin{bmatrix} y^4\sin(4x) \\ x(2-x)\sin(4y) \end{bmatrix},$$

which is the exact solution vector.

Model Problem-F

In this problem, experimentation is made with initial data $\begin{bmatrix}0.001\\0.001\end{bmatrix}$ instead of homogeneous. We consider the inhomogeneous Cauchy-Riemann system:

$$\frac{\partial}{\partial y}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}+\begin{bmatrix}4y^3\sin(x)\\4y^3+y^4\cos(x)\end{bmatrix},\quad x\in\mathbb{R},y>0, \tag{38}$$

subject to initial condition:

$$\begin{bmatrix}u(x,0)\\v(x,0)\end{bmatrix}=\begin{bmatrix}0.001\\0.001\end{bmatrix},\quad x\in\mathbb{R}, \tag{39}$$

having exact solution vector given by:

$$\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}=\begin{bmatrix}y^4\sin(x)+0.001\\y^4+0.001\end{bmatrix}. \tag{40}$$

Now we use the VRDT method to solve this problem. The VRDT on (38) and (39) gives:

$$(k+1)\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_{k+1}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}U_x(x)\\V_x(x)\end{bmatrix}_k+\begin{bmatrix}4\sin(x)\,\delta(k-3)\\4\delta(k-3)+\cos(x)\,\delta(k-4)\end{bmatrix}, \tag{41}$$

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0=\begin{bmatrix}0.001\\0.001\end{bmatrix}. \tag{42}$$

Using iterative relations (41) and (42) we obtain the following vectors:

$$\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_0=\begin{bmatrix}0.001\\0.001\end{bmatrix},\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_1=\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_2=\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_3=\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_4=\begin{bmatrix}\sin(x)\\1\end{bmatrix},\begin{bmatrix}U(x)\\V(x)\end{bmatrix}_5=\begin{bmatrix}0\\0\end{bmatrix},\dots$$

The inverse-VRDTs of the above vectors finally give:

$$\begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}=\begin{bmatrix}\sum_{k=0}^{\infty}U_k(x)\,y^k\\\sum_{k=0}^{\infty}V_k(x)\,y^k\end{bmatrix}=\begin{bmatrix}y^4\sin(x)+0.001\\y^4+0.001\end{bmatrix},$$

which is the exact solution vector.

It is noticed that all the model problems yield the VRDTM solution vectors in the form of infinite series whose compact forms are in agreement with the exact solutions. The results reveal that the technique is computationally less expensive in terms of mathematical manipulations as compared to other ones (for example, Adomian method, homotopy perturbation method and differential transform method). The VRDT method is a reliable and quite powerful technique that neither requires discretization nor perturbation.

Conclusion

For solving an initial-value problem for the Cauchy-Riemann system with analytic data, vectorial reduced differential transform (VRDT) method has been presented. The technique has been tested on a variety of homogeneous and inhomogeneous Cauchy-Riemann systems with various types of initial data. The VRDTM solution vector achieved, in each case, is in the form of infinite series whose compact form is in agreement with the exact solution vector. The technique is quite reliable and powerful.

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