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The first integral method and its application for finding the exact solutions of nonlinear fractional partial differential equations (PDES) in the mathematical physics

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In this article we apply the first integral method to construct the exact solutions of some nonlinear fractional partial differential equations (PDES) in the sense of modified Riemann–Liouville derivatives, namely the nonlinear fractional Zoomeron equation and the nonlinear fractional Klein- Gordon-Zakharov system of equations. Based on a nonlinear fractional complex transformation, these two nonlinear fractional equations can be turned into nonlinear ordinary differential equations (ODE) of integer order. This method has more advantages: it is direct and concise.

Key words: First integral method, exact solutions, nonlinear fractional Zoomeron equation, nonlinear fractional Klein-Gordon-Zakharov system of equations.

INTRODUCTION

Fractional differential equations have been the focus of many studies due to there frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics (Miller and Ross, 1993; Kilbas et al., 2006; Podlubny, 1999). Recently, a large amount of literature has been provided to construct the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Several powerful methods have been proposed to obtain approximate and exact solutions of fractional differential equations, such as the Adomian decomposition method (El-Sayed et al., 2009; Safari et al., 2009), the variational iteration method (Inc, 2008; Wu and Lee, 2010; Fouladi et al., 2010), the homotopy analysis method (Song and Zhang, 2009;

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Abbasbanby and Shirzadi, 2010; Barania et al., 2010; Rashidi et al., 2009), the homotopy perturbation method (Ganji et al., 2010; Gepreel, 2011; Gupta and Singh, 2011), the Lagrange characteristic method (Jumarie, 2006a), the fractional sub-equation method (Zhang and Zhang, 2001), the local fractional variation iteration method (Yang and Baleanu, 2013; Liu et al., 2013; Wang et al., 2014; Zhao et al., 2014; Baleanu et al., 2014; He, 2012; Yang et al., 2013; Yang et al., 2013) and so on.

Jumarie (2006b) proposed a modified Riemann-Liouville derivative. With this kind of fractional derivative and some useful formulas, we can convert fractional differential equations into integer-order differential equations by using variable transformations. The first integral method (Feng and Roger, 2007; Feng, 2008; Raslan, 2008; Lu et al., 2010; Taghizadeh et al., 2011) can be used to construct the exact solutions for some time fractional differential equations.

The objective of this paper is to investigate the applicability and effectiveness of the first integral method on fractional nonlinear partial differential equations, namely the nonlinear fractional Zoomeron equation and the nonlinear fractional Klein- Gordon- Zakharov system of equations.

The modified Riemann-Liouville derivative and first integral method

Here, we first give some definitions and properties of the modified Riemann-Liouville derivative which are used further in this paper. Assume that $f:R\rightarrow R,\ x\rightarrow f(y)$ denote a continuous (but not necessarily differentiable) function. The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x(x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x(x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & 0\le\alpha<1,\\ \left[f^{(n)}(x)\right]^{(\alpha-n)}, & n\le\alpha<n+1,\ n\ge1\end{cases}\tag{1}$$

Some properties of the fractional modified Riemann-Liouville derivative were summarized and three useful formulas of them are:

$$D_x^\alpha x^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha},\ \gamma>0,\tag{2}$$

$$D_x^\alpha\left[f(x)g(x)\right]=f(x)D_x^\alpha g(x)+g(x)D_x^\alpha f(x),\tag{3}$$

$$D_x^\alpha\left[f(g(x))\right]=f_g'(g(x))D_x^\alpha g(x)=D_g^\alpha f(g(x))[g_x'(x)]^\alpha\tag{4}$$

which are consequences of the equation $d^\alpha x(t)=\Gamma(1+\alpha)dx(t)$.

Next, let us consider the following time fractional differential equation with independent variables $x=(x_1,x_2,...,x_m,t)$ and a dependent variable u:

$$F(u,D_t^\alpha u,u_{x_1},u_{x_2},u_{x_3},D_t^{2\alpha}u,u_{x_1x_1},u_{x_2x_2},...)=o.\tag{5}$$

Using the fractional complex transformation

$$u(x_1,x_2,x_3,...,x_m,t)=u(\xi),\quad \xi=x_1+k_1x_2+...+k_{m-1}x_m+\frac{ct^\alpha}{\Gamma(1+\alpha)}\tag{6}$$

where k_i,c are constants to be determined later; the fractional Equation (5) is reduced to the nonlinear ODE of integer orders:

$$H(u,u',u'',...)=0,\tag{7}$$

where $u'=\dfrac{du}{d\xi},u''=\dfrac{d^2u}{d\xi^2},.....$

We assume that Equation (7) has a solution in the form

$$u(\xi)=X(\xi),\tag{8}$$

and introduce a new independent variable $Y(\xi)=X'(\xi)$ which leads to a new system of equations

$$\begin{aligned}X'(\xi)&=Y(\xi),\\ Y'(\xi)&=G(X(\xi),Y(\xi)).\end{aligned}\tag{9}$$

Now, let us recall the first integral method. By using the division theorem for two variables in the complex domain C[X,Y] which is based on the Hilbert-Nullstellensatz Theorem (Bourbak, 1972), we can obtain one first integral to Equation (9) which can reduce Equation (7) to a first- order integrable ordinary differential equation. An exact solution to Equation (5) is obtained by solving this equation directly.

Division theorem

Suppose that $P(x,y)$ and $Q(x,y)$ are polynomials in C[X,Y] and $P(x,y)$ is irreducible in C[X,Y]. IF $Q(x,y)$ vanishes at all zero points of $P(x,y)$, then there exists a polynomial $H(x,y)$ in C[X,Y] such that

$$Q(x,y)=P(x,y)H(x,y). \qquad (10)$$

Applications

Here, we present two examples to illustrate the applicability of the first integral method to solve nonlinear fractional partial differential equations.

Example 1. The nonlinear fractional Zoomeron equation

This equation is well- known (Alquran and Al-Khaled, 2012; Abazar, 2011) and can be written in the form:

$$D_{tt}^{2\alpha}[\frac{u_{xy}}{u}]-[\frac{u_{xy}}{u}]_{xx}+2D_t^{\alpha}[u^2]_x=0,\ 0<\alpha\le 1. \qquad (11)$$

For our purpose, we introduce the following transformations

$$u(x,y,t)=u(\xi),\ \xi=\ell x+cy-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}, \qquad (12)$$

where ℓ, c, ω are constants. Substituting Equation (12) into (11), we have the ODE:

$$\ell c\,\omega^2\left(\frac{u''}{u}\right)''-c\ell^3\left(\frac{u''}{u}\right)''-2\ell\omega(u^2)''=0 \qquad (13)$$

Integrating Equation (13) twice with respect to ξ, we get

$$\ell c(\omega^2-\ell^2)u''-2\ell\omega u^3-ru=0 \qquad (14)$$

where r is a non zero constant of integration, while the second constant of integration is vanishing.
On using Equations (8) and (9), then Equation (14) is equivalent to the two dimensional autonomous system:

$$X'(\xi)=Y(\xi) \qquad (15)$$

$$Y'(\xi)=\frac{rX(\xi)+2\ell\omega X^3(\xi)}{\ell c(\omega^2-\ell^2)}, \qquad (16)$$

where $X(\xi)=u(\xi)$.
According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Equations (15) and (16) and $Q(X,Y)$ is an irreducible polynomial in the complex domain C[X.Y] such that

$$Q(X,Y)=\sum_{i=0}^{m}a_i(X)Y^i(\xi)=0 \qquad (17)$$

where $a_i(X),\ (i=1,2,...m)$ are polynomials in X and $a_m(X)\neq 0$. Due to the Division theorem, there exists a polynomial $[g(X)+h(X)Y(\xi)]$ in the complex domain C[X,Y] such that

$$\frac{dQ}{d\xi}=\frac{\partial Q}{\partial X}\frac{dX}{d\xi}+\frac{\partial Q}{\partial Y}\frac{dY}{d\xi}=[g(X)+h(X)Y(\xi)]\sum_{i=0}^{m}a_i(X)Y^i(\xi) \qquad (18)$$

Let us now consider two cases.

Case 1

If m = 1.

Substituting Equations (15) and (16) into Equation (18) and equating the coefficients of $Y^i(\xi),(i=0,1,2)$ on both sides of Equation (18), we have respectively

$$g(X)a_0(X)=a_1(X)[\frac{rX(\xi)+2\omega\ell X^3(\xi)}{\ell c(\omega^2-\ell^2)}], \qquad (19)$$

$$\frac{da_o(X)}{dX}=g(X)a_1(X)+h(X)a_o(X), \qquad (20)$$

$$\frac{da_1(X)}{dX}=h(X)a_1(X). \qquad (21)$$

Since $a_i(X),\ (i=0,1)$ are polynomials, then from Equation (21) we deduce that $a_1(X)$ is a constant and $h(X)=0$. For simplicity we take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$ we conclude that $\deg(g(X))=1$. Suppose that $g(X)=A_1X+B_o$, then we find

$$a_0(X)=\frac{1}{2}A_1X^2+B_oX+A_o, \qquad (22)$$

where A_1, B_o, A_o are constants to be determined. Substituting $a_0(X), a_1(X), g(X)$ into Equation (19) we get

$$[A_1X(\xi)+B_0][\frac{1}{2}A_1X^2(\xi)+B_0X(\xi)+A_0]-[\frac{rX(\xi)+2\omega\ell X^3(\xi)}{\ell c(\omega^2-\ell^2)}]=0 \qquad (23)$$

Setting the coefficients of powers of $X(\xi)$ to zero, we obtain the following system of algebraic equations:

$$A_0B_0=0,\ A_1A_0+B_0^2=\frac{r}{\ell c(\omega^2-\ell^2)},\ \frac{3}{2}A_1B_0=0,\ A_1^2=\frac{4\omega}{c(\omega^2-\ell^2)} \quad (24)$$

On solving these algebraic equations, we have the results

$$A_1=\pm 2\sqrt{\frac{\omega}{c(\omega^2-\ell^2)}},\ B_0=0,\ A_0=\frac{\pm r}{2\ell\sqrt{\omega c(\omega^2-\ell^2)}} \quad (25)$$

From Equations (17), (22) and (25) we conclude that

$$Y(\xi)=-a_0(X)=\frac{\pm r}{2\ell\sqrt{\omega c(\omega^2-\ell^2)}}\pm\sqrt{\frac{\omega}{c(\omega^2-\ell^2)}}X^2(\xi) \quad (26)$$

and consequently we obtain the equation

$$X'(\xi)=\frac{\pm r}{2\ell\sqrt{\omega c(\omega^2-\ell^2)}}\pm\sqrt{\frac{\omega}{c(\omega^2-\ell^2)}}X^2(\xi) \quad (27)$$

Equation (27) is just the well-known Riccati equation. With reference to the article (Ma and Fuchssteiner, 1996) the authors proved that the Riccati equation $V'=\alpha_0+\alpha_1 V+\alpha_2 V^2$, where $\alpha_0,\alpha_1,\alpha_2$ are constants such that $\alpha_2\neq 0$ has the following solutions:

(i) If $\Delta=\alpha_1^2-4\alpha_0\alpha_2>0$, then

$$V(\xi)=\begin{cases}\frac{-1}{2\alpha_2}[\alpha_1+\sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{2}\xi-\frac{\varepsilon\ln\xi_0}{2})] & if\quad \xi_0>0\\ \frac{-1}{2\alpha_2}[\alpha_1+\sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{2}\xi-\frac{\varepsilon\ln(-\xi_0)}{2})] & if\quad \xi_0<0\\ \frac{-1}{2\alpha_2}[\alpha_1+\varepsilon\sqrt{\Delta}] & if\quad \xi_0=0\end{cases} \quad (28)$$

(ii) If $\Delta=\alpha_1^2-4\alpha_0\alpha_2<0$, then

$$V(\xi)=\begin{cases}\frac{-1}{2\alpha_2}[\alpha_1-\sqrt{-\Delta}\tan(\frac{\sqrt{-\Delta}}{2}\xi+\xi_0)],\\ \frac{-1}{2\alpha_2}[\alpha_1+\sqrt{-\Delta}\cot(\frac{\sqrt{-\Delta}}{2}\xi+\xi_0)].\end{cases} \quad (29)$$

(iii) If $\Delta=\alpha_1^2-4\alpha_0\alpha_2=0$, then

$$V(\xi)=\frac{-\alpha_1}{2\alpha_2}-\frac{1}{\alpha_2\xi+\xi_0} \quad (30)$$

where ξ_0 is an arbitrary constant and $\varepsilon=\pm 1$.

With the aid of Equations (28) - (30) the solutions of Equation (11) have the forms:

(i) If $\frac{r}{\ell c(\omega^2-\ell^2)}<0,\ \frac{r}{\ell\omega}<0$, we get

$$\begin{aligned}u_1(\xi)&=\pm\sqrt{\frac{-r}{2\ell\omega}}\tanh[\sqrt{\frac{-r}{2\ell c(\omega^2-\ell^2)}}\ \xi-\frac{\varepsilon\ln\xi_0}{2}] \quad if\quad \xi_0>0\\ u_2(\xi)&=\pm\sqrt{\frac{-r}{2\ell\omega}}\coth[\sqrt{\frac{-r}{2\ell c(\omega^2-\ell^2)}}\ \xi-\frac{\varepsilon\ln(-\xi_0)}{2}] \quad if\quad \xi_0<0\\ u_3(\xi)&=\pm\frac{\varepsilon}{2}\sqrt{\frac{-r}{2\ell\omega}} \quad if\quad \xi_0=0\end{aligned} \quad (31)$$

(ii) If $\frac{r}{\ell c(\omega^2-\ell^2)}>0,\ \frac{r}{\ell\omega}>0$, we get

$$\begin{aligned}u_4(\xi)&=\pm\sqrt{\frac{r}{2\ell\omega}}\tan[\sqrt{\frac{r}{2\ell c(\omega^2-\ell^2)}}\ \xi+\xi_0]\\ u_5(\xi)&=\pm\sqrt{\frac{r}{2\ell\omega}}\cot[\sqrt{\frac{r}{2\ell c(\omega^2-\ell^2)}}\ \xi+\xi_0]\end{aligned} \quad (32)$$

(iii) If $\frac{\omega}{c(\omega^2-\ell^2)}>0,\ r=0$, we get

$$u_6(\xi)=\frac{\pm 1}{\sqrt{\frac{r}{c(\omega^2-\ell^2)}}\ \xi\pm\xi_0}, \quad (33)$$

where ξ is given by Equation (12).

As a result, we find the periodic and solitary solutions of Equation (11) are new and when $\alpha=1$ they are also different from the solutions found in (Alquran and Al-Khaled, 2012; Abazar, 2011).

Case 2

If m = 2.

In this case, Equations (17) and (18) respectively reduce to

$$Q[X(\xi),Y(\xi)]=a_o(X)+a_1(X)Y(\xi)+a_2(X)Y^2(\xi)=0 \quad (34)$$

and

$$\frac{da_0(X)}{dX}+\frac{da_1(X)}{dX}Y(\xi)+\frac{da_2(X)}{dX}Y^2(\xi)]Y(\xi)+[a_1X+2a_2(X)Y(\xi)][\frac{rX(\xi)+2\omega\ell X^3(\xi)}{\ell c(\omega^2-\ell^2)}]$$
$$=[g(X)+h(X)Y(\xi)][a_0X+a_1(X)Y(\xi)+a_2(X)Y^2(\xi)] \quad (35)$$

Equating the coefficient of powers of $Y(\xi)$ on both sides of Equation (35) we have

$$\frac{da_2(X)}{dX}=h(X)a_2(X) \quad (36)$$

$$\frac{da_1(X)}{dX}=g(X)a_2(X)+h(X)a_1(X) \quad (37)$$

$$\frac{da_0(X)}{dX}+2a_2(X)[\frac{rX(\xi)+2\omega\ell X^3(\xi)}{\ell c(\omega^2-\ell^2)}]=g(X)a_1(X)+h(X)a_0(X) \quad (38)$$

$$g(X)a_0(X)=a_1(X)[\frac{rX(\xi)+2\omega\ell X^3(\xi)}{\ell c(\omega^2-\ell^2)}] \quad (39)$$

Since $a_i(X),\ (i=0,1,2)$ are polynomials, then from Equation (36) we deduce that $a_2(X)$ is constant and $h(X)=0$. For simplicity we take $a_2(X)=1$. Balancing the degrees of $g(X)$ and $a_1(X)$ we conclude that deg $(g(X))=1$, and hence we get

$$g(X)=A_1X+B_o, \quad (40)$$

and $a_1(X)=\frac{1}{2}A_1X^2+B_oX+A_o$ (41)

where A_1,B_o,A_o are constants to be determined, such that $A_1\neq 0$. Now, Equation (38) becomes

$$\frac{da_0(X)}{dX}=[\frac{-2r}{\ell c(\omega^2-\ell^2)}+A_1A_0+B_0^2]X(\xi)+\frac{3}{2}A_1B_0X^2(\xi)$$
$$+[\frac{-4\omega}{c(\omega^2-\ell^2)}+\frac{1}{2}A_1^2]X^3(\xi)+A_0B_0 \quad (42)$$

and

$$a_0(X)=[\frac{-r}{\ell c(\omega^2-\ell^2)}+\frac{1}{2}A_1A_0+\frac{1}{2}B_0^2]X^2(\xi)+\frac{1}{2}A_1B_0X^3(\xi)$$
$$+[\frac{-\omega}{c(\omega^2-\ell^2)}+\frac{1}{8}A_1^2]X^4(\xi)+A_0B_0X(\xi)+d \quad (43)$$

where *d* is the constant of integration.

Substituting Equations (40), (41) and (43) into Equation (39) and equating the coefficients of powers of $X(\xi)$ we get

$$A_1=\frac{\pm 4\omega}{\sqrt{\omega c(\omega^2-\ell^2)}},\ B_0=0,\ A_0=\frac{\pm r}{\ell\sqrt{\omega c(\omega^2-\ell^2)}},\ d=\frac{r^2}{4\omega c\ell^2(\omega^2-\ell^2)}$$

Consequently, we deduce that

$$a_0(X)=\frac{r}{\ell c(\omega^2-\ell^2)}X^2(\xi)+\frac{\omega}{c(\omega^2-\ell^2)}X^4(\xi)+\frac{r^2}{4\omega\ell^2c(\omega^2-\ell^2)} \quad (44)$$

and

$$a_1(X)=\pm\frac{2\omega}{\sqrt{\omega c(\omega^2-\ell^2)}}X^2(\xi)\pm\frac{r}{\ell\sqrt{\omega c(\omega^2-\ell^2)}} \quad (45)$$

Substituting Equations (44), (45) into Equation (34) we deduce after some reduction that

$$Y(\xi)=-\frac{1}{2}a_1(X) \quad (46)$$

and hence

$$X'(\xi)=\pm\frac{r}{2\ell\sqrt{\omega c(\omega^2-\ell^2)}}\pm\frac{\omega}{\sqrt{\omega c(\omega^2-\ell^2)}}X^2(\xi) \quad (47)$$

which has the same form Equation (27) and gives the same solutions (31) to (33). This shows that the two cases m=1 and m=2 give the same solutions. Comparing our results with the results ((Alquran and Al-Khaled, 2012; Abazar, 2011), it can be seen that our solutions are new.

Example 2. The nonlinear fractional Klein-Gordon-Zakharov equations

These equations are well- known (Thornhill and Haar, 1978; Dendy, 1990; Ebadi et al., 2010; Shang et al., 2008) and can be written in the following system:

$$D_{tt}^{2\alpha}u-u_{xx}+u+\beta_1uv=0,$$

$$D_{tt}^{2\alpha}v-v_{xx}-\beta_2(|u|)_{xx}=0,\ 0<\alpha\leq 1, \quad (48)$$

with $u(x,t)$ is a complex function and $v(x,t)$ is a real function, where β_1,β_2 are nonzero real parameters. This system describes the interaction of the Langmuir wave and the ion acoustic in a high frequency plasma. Using the wave variable

$$u(x,t)=\phi(x,t)\exp\left[i(kx+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}+\xi_1)\right], \qquad (49)$$

where $\phi(x,t)$ is a real-valued function, k,ω are real constants to be determined and ξ_1 is an arbitrary constant. Then the system (48) is carried to the following PDE system:

$$\begin{aligned} &D_{tt}^{2\alpha}\phi-\phi_{xx}+(k^2-\omega^2+1)\phi+\beta_1 v\,\phi=0\\ &\omega D_t^{\alpha}\phi-k\,\phi_x=0 \qquad (50)\\ &D_{tt}^{2\alpha}v-v_{xx}-\beta_2(\phi^2)_{xx}=0 \end{aligned}$$

Setting

$$v(x,t)=v(\xi),\quad \phi(x,t)=\phi(\xi),\quad \xi=\omega x+\frac{kt^{\alpha}}{\Gamma(1+\alpha)}+\xi_2 \qquad (51)$$

where ξ_2 is an arbitrary constant, then we get

$$v(\xi)=\frac{\beta_2\omega^2\phi^2(\xi)}{(k^2-\omega^2)}+C, \qquad (52)$$

and

$$\phi''(\xi)+\ell_1\phi(\xi)+\ell_2\phi^3(\xi)=0 \qquad (53)$$

where $\ell_1=\frac{(k^2-\omega^2+\beta_1 C+1)}{(k^2-\omega^2)}$, $\ell_2=\frac{\omega^2\beta_2\beta_1}{(k^2-\omega^2)^2}$, C is an integration constant, and $k\neq\pm\omega$.

On using Equations (8) and (9), we deduce that Equation (53) is equivalent to the two dimensional autonomous system:

$$X'(\xi)=Y(\xi) \qquad (54)$$

$$Y'(\xi)=-\ell_1 X(\xi)-\ell_2 X^3(\xi) \qquad (55)$$

where $X(\xi)=\phi(\xi)$.

Now we consider the two cases.

Case 1

If m = 1.

Substituting Equation (54) and (55) into Equation (18) and equating the coefficients of $Y^i(\xi),(i=0,1,2)$ on both sides of Equation (18), we have respectively.

$$g(X)a_0(X)=a_1(X)[-\ell_1 X(\xi)-\ell_2 X^3(\xi)], \qquad (56)$$

$$\frac{da_o(X)}{dX}=g(X)a_1(X)+h(X)a_o(X), \qquad (57)$$

$$\frac{da_1(X)}{dX}=h(X)a_1(X). \qquad (58)$$

Since $a_i(X),\ (i=0,1)$ are polynomials we deduce that $h(X)=0$, $a_1(X)=1$ and deg $(g(X))=1$. Then we obtain

$$g(X)=A_1X+B_o, \qquad (59)$$

$$a_0(X)=\frac{1}{2}A_1X^2+B_oX+A_o \qquad (60)$$

Substituting $a_0(X),a_1(X),g(X)$ into Equation (56) we get

$$[A_1X(\xi)+B_0][\frac{1}{2}A_1^2X(\xi)+B_0X(\xi)+A_0]+\ell_1X(\xi)+\ell_2X^3(\xi)=0 \qquad (61)$$

Setting the coefficients of powers of $X(\xi)$ to zero, we obtain

$$A_0B_0=0,\ A_1A_0+B_0^2=-\ell_1,\ \frac{3}{2}A_1B_0=0,\ A_1^2=-2\ell_2$$

On solving these algebraic equations, we have the results

$$A_1=\pm\sqrt{-2\ell_2},\ B_0=0, A_0=\pm\frac{\ell_1}{\sqrt{-2\ell_2}},\ \ell_2<0.$$

Now, we deduce that

$$X'(\xi)=\pm\frac{\ell_1}{\sqrt{-2\ell_2}}\mp\frac{1}{2}\sqrt{-2\ell_2}X^2(\xi) \qquad (62)$$

which represents the well- known Riccati equation. With the help of Equations (28) to (30) the solutions of the system (48) can be written in the forms:

(i) If $\ell_1<0$ and $\ell_2<0$ we get the hyperbolic and rational solutions

$$\begin{cases} u_1(\xi)=\pm\sqrt{-\ell_1/\ell_2}\ \tanh(\sqrt{\frac{\ell_1}{2}}\ \xi-\frac{\varepsilon\ln\xi_0}{2})\exp(i\,\xi),\ if\quad \xi_0>0 \\ v_1(\xi)=\frac{-\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}\ \tanh^2(\sqrt{\frac{\ell_1}{2}}\ \xi-\frac{\varepsilon\ln\xi_0}{2})+C\ ,\ if\quad \xi_0>0 \end{cases} \tag{63}$$

$$\begin{cases} u_2(\xi)=\pm\sqrt{-\ell_1/\ell_2}\ \coth(\sqrt{\frac{\ell_1}{2}}\ \xi-\frac{\varepsilon\ln(-\xi_0)}{2})\exp(i\,\xi),\ if\quad \xi_0<0 \\ v_2(\xi)=\frac{-\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}\ \coth^2(\sqrt{\frac{\ell_1}{2}}\ \xi-\frac{\varepsilon\ln(-\xi_0)}{2})+C\ ,\ if\quad \xi_0<0 \end{cases} \tag{64}$$

$$\begin{cases} u_3(\xi)=\pm\sqrt{-\ell_1/\ell_2}\ \exp(i\,\xi), \\ V_3(\xi)=\frac{-\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}+C\ , \end{cases} \tag{65}$$

(ii) If $\ell_1<0$ and $\ell_2<0$ we get

$$\begin{cases} u_4(\xi)=\pm\sqrt{-\ell_1/\ell_2}\ \tan(\sqrt{\frac{-\ell_1}{2}}\ \xi+\xi_0)\exp(i\,\xi) \\ v_4(\xi)=\frac{-\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}\ \tan^2(\sqrt{\frac{-\ell_1}{2}}\ \xi+\xi_0)+C, \end{cases} \tag{66}$$

$$\begin{cases} u_5(\xi)=\pm\sqrt{-\ell_1/\ell_2}\ \cot(\sqrt{\frac{-\ell_1}{2}}\ \xi+\xi_0)\exp(i\,\xi) \\ v_5(\xi)=\frac{-\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}\ \cot^2(\sqrt{\frac{-\ell_1}{2}}\ \xi+\xi_0)+C, \end{cases} \tag{67}$$

(iii) If $\ell_1=0$ and $\ell_2<0$ we get the rational solutions

$$\begin{cases} u_6(\xi)=\frac{\pm1}{[\sqrt{\frac{-\ell_2}{2}}\xi+\xi_0]}\exp(i\,\xi) \\ v_6(\xi)=\frac{\beta_2\omega}{(k^2-\omega^2)[\sqrt{\frac{-\ell_2}{2}}\xi+\xi_0]^2}+C \end{cases} \tag{68}$$

Case 2

If m = 2.

Substituting Equations (54) and (55) in to Equation (18) and equating the coefficients of powers of $Y\ (\xi)$ on both sides of Equation (18), we get

$$\frac{da_2(X)}{dX}=h(X)a_2(X) \tag{69}$$

$$\frac{da_1(X)}{dX}=h(X)a_1(X)+g(X)a_2(X) \tag{70}$$

$$\frac{da_0(X)}{dX}+2a_2(X)[-\ell_1X-\ell_2X^3]=g(X)a_1(X)+h(X)a_0(X) \tag{71}$$

$$g(X)a_0(X)=a_1(X)[-\ell_1X-\ell_2X^3]. \tag{72}$$

Since $a_i(X),\ (i=0,1,2)$ are polynomials, then from Equation (69) we deduce that $a_2(X)$ is a constant and $h(X)=0$. For simplicity we take $a_2(X)=1$. Balancing the degrees of $g(X)$ and $a_1(X)$ we conclude that $\deg(g(X))=1$, and hence we get

$$g(X)=A_1X+B_o, \tag{73}$$

$$a_1(X)=\frac{1}{2}A_1X^2+B_oX+A_o, \tag{74}$$

where A_1,B_o,A_o are constants to be determined, such that $A_1\neq0$. Now, Equation (71) reduces to

$$\frac{da_0(X)}{dX}=[2\ell_1+A_0A_1+B_0^2]X+\frac{3}{2}A_1B_0X^2+[2\ell_2+\frac{1}{2}A_1^2]X^3 \tag{75}$$

Integrating Equation (75) with respect to $X\ (\xi)$, we have

$$a_0(X)=[\ell_1+\frac{1}{2}A_0A_1+\frac{1}{2}B_0^2]X^2+\frac{1}{2}A_1B_0X^3+[\frac{\ell_2}{2}+\frac{1}{8}A_1^2]X^4+d \tag{76}$$

where d is the constant of integration.

Substituting Equations (73), (74) and (76) into Equation (72) and equating the coefficients of powers of $X\ (\xi)$ we get

$$A_0=\pm\ell_1\sqrt{\frac{-2}{\ell_2}},\ A_1=\pm2\sqrt{-2\ell_2},B_0=0,\ d=-\frac{\ell_1^2}{2\ell_2}$$

where $\ell_2<0$.

Consequently, we deduce that

$$a_0(X) = 3\ell_1 X^2(\xi) - \frac{1}{2}\ell_2 X^4(\xi) - \frac{\ell_1^2}{2\ell_2} \tag{77}$$

and

$$a_1(X) = \pm\sqrt{-2\ell_2}\, X^2(\xi) \pm \ell_1\sqrt{\frac{-2}{\ell_2}}. \tag{78}$$

Substituting Equations (77) and (78) into Equation (34) we deduce that

$$Y(\xi) = -\frac{1}{2}a_1(X) \pm \sqrt{-2\ell_1}X(\xi). \tag{79}$$

Hence we conclude that

$$X'(\xi) = \pm\frac{\ell_1}{2}\sqrt{\frac{-2}{\ell_2}} \pm \sqrt{-2\ell_1}X(\xi) \pm \frac{1}{2}\sqrt{-2\ell_2}X^2(\xi), \tag{80}$$

which represents the generalized Riccati equation. With the help of Equations (28) and (30) the solutions of the system (48) can be written in the forms:

(i) If $\ell_1 < 0$ and $\ell_2 < 0$ we get the hyperbolic and rational solutions

$$\begin{cases} u_7(\xi) = \sqrt{\ell_1/\ell_2}\,[1 \pm \sqrt{2}\tanh(\sqrt{-\ell_1}\,\xi - \frac{\varepsilon \ln \xi_0}{2})\exp(i\,\xi),\ if \quad \xi_0 > 0 \\ v_7(\xi) = \frac{\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}[1 \pm \tanh(\sqrt{-\ell_1}\,\xi - \frac{\varepsilon \ln \xi_0}{2})]^2 + C,\ if \quad \xi_0 > 0 \end{cases} \tag{81}$$

$$\begin{cases} u_8(\xi) = \sqrt{\ell_1/\ell_2}\,[1 \pm \sqrt{2}\coth(\sqrt{-\ell_1}\,\xi - \frac{\varepsilon \ln(-\xi_0)}{2})\exp(i\,\xi),\ if \quad \xi_0 < 0 \\ v_8(\xi) = \frac{\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}[1 \pm \coth(\sqrt{-\ell_1}\,\xi - \frac{\varepsilon \ln(-\xi_0)}{2})]^2 + C,\ if \quad \xi_0 < 0 \end{cases} \tag{82}$$

$$\begin{cases} u_9(\xi) = \sqrt{\ell_1/\ell_2}\,[1 \pm \varepsilon\sqrt{2}]\exp(i\,\xi), \\ v_9(\xi) = \frac{\beta_2\ell_1\omega^2}{\ell_2(k^2-\omega^2)}[1 \pm \varepsilon\sqrt{2})]^2 + C, \end{cases} \tag{83}$$

(ii) If $\ell_1 = 0$ and $\ell_2 < 0$

In this case we can show that $X(\xi)$ is complex. Since $X(\xi) = \phi(\xi)$, then $\phi(\xi)$ is complex. This contradicts (49) where $\phi(\xi)$ should be real. Thus this case is rejected.

(iii) If $\ell_1 = 0$ and $\ell_2 < 0$ we get the rational solutions

$$\begin{cases} u_{10}(\xi) = \frac{\pm 1}{[\sqrt{\frac{-\ell_2}{2}}\,\xi \pm \xi_0]}\exp(i\,\xi) \\ v_{10}(\xi) = \frac{\beta_2\ell_1\omega^2}{(k^2-\omega^2)[\sqrt{\frac{-\ell_2}{2}}\,\xi \pm \xi_0]^2} + C \end{cases} \tag{84}$$

Comparing our results with the results in (Thornhill and Haar, 1978; Dendy, 1990; Ebadi et al., 2010; Shang et al., 2008) it can be seen that our results are new.

PHYSICAL EXPLANATIONS OF SOME OF OBTAINED SOLUTIONS

Here, we will present some graphs of our results to visualize the underlying mechanism of the original equations. Using the mathematical software Maple 15, we plot some of these obtained solutions which are shown in Figures 1 and 2.

The nonlinear fractional Zoomeron

The obtained solutions incorporate three types of explicit solutions namely, the hyperbolic, trigonometric and rational function solutions. From these explicit results, it easy to say that the solutions (31) are kink, singular and rational solutions respectively. While the solutions (32) are periodic solution and Equation (33) are the rational solutions. For more convenience the graphical representations of $u_1(x,t)$ and $u_4(x,t)$ of Equation (11) are shown in Figure 1.

The nonlinear fractional Klein-Gordan-Zakharov system of Equations (48)

The obtained solutions (63) are kink solutions, the solutions (64) are the singular, the solution (65) are rational. While the solutions (66) and (67) are periodic and the solutions (68) are rational. The graphical solutions (63) and (66) are shown in Figure 2.

CONCLUSIONS

The first integral method is applied successfully for finding the exact solutions of the nonlinear fractional Zoomeron equation and the nonlinear fractional Klein-Gordon- Zakharov system of equations. The performance of this method is reliable and effective and gives more solutions. Thus, we deduce that the proposed method can be extended to solve many systems of other areas such that physics, biology and chemistry. With the aid of the maple, we have assured the correctness of the

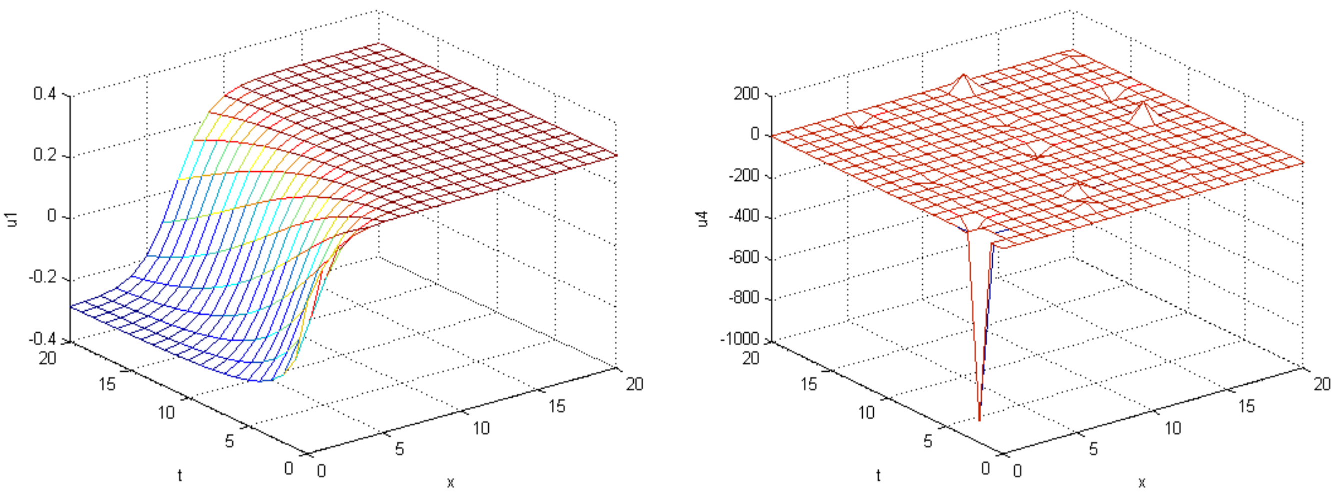

Figure 1. The plot of the solutions $u_1(\xi)$ and $u_4(\xi)$ of the nonlinear fractional Zoomeron equation.

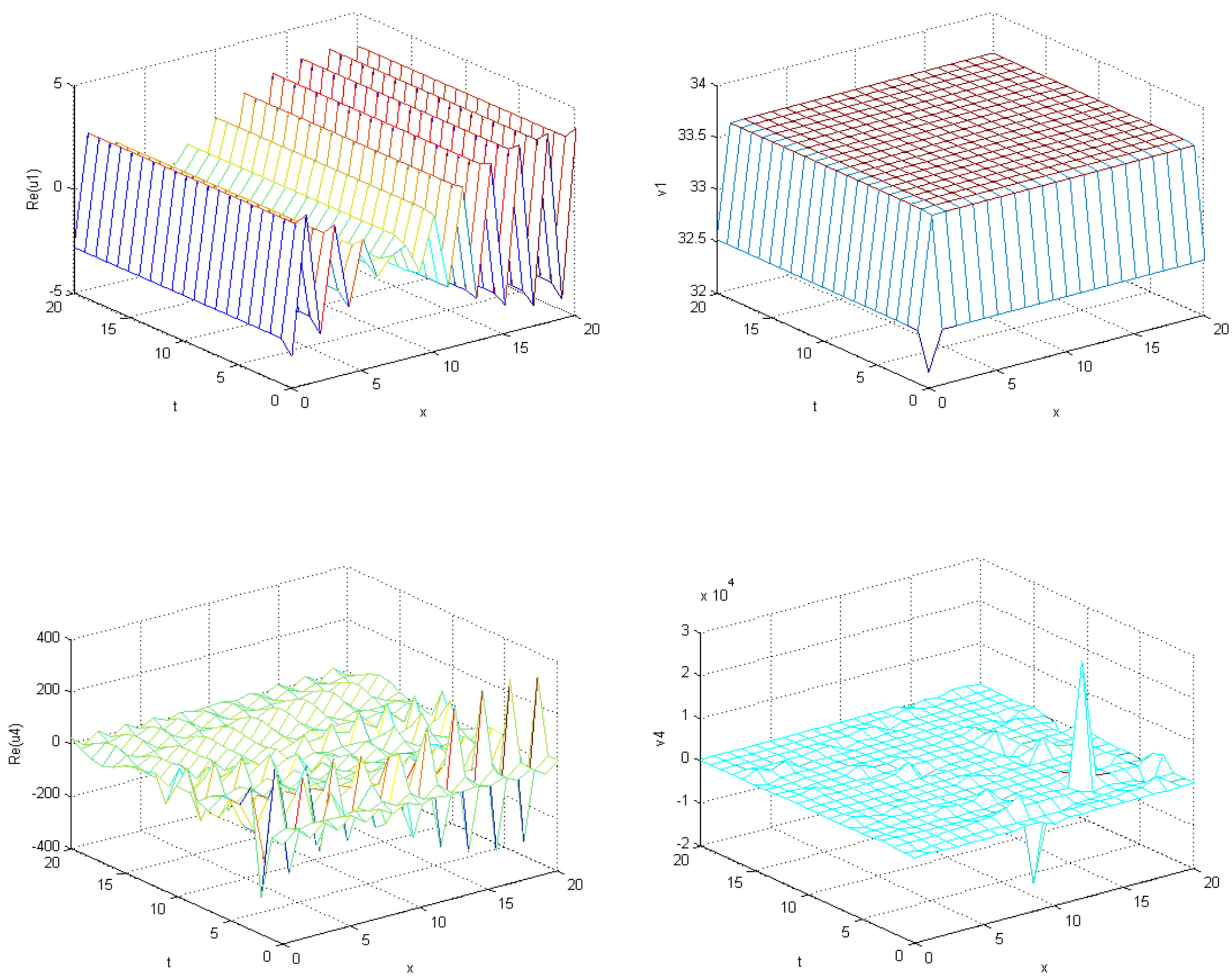

Figure 2. The plots of some of solutions of nonlinear fractional Klein-Gordon-Zakharov equations.

obtained solutions by putting them back to the original equations.

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Conflict of Interests

The author(s) have not declared any conflict of interests.

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