International Journal of the Physical Sciences Vol. 6(33), pp. 7512 - 7523, 9 December, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.335

Full Length Research Paper

Convergence of iterative methods for solving Kawahara equation

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Accepted 5 September, 2011

In this paper, a Kawahara equation is solved by using the Adomian's decomposition method, modified Adomian's decomposition method, variational iteration method, modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.

Key words: Kawahara equation, Adomian decomposition method (ADM), modified Adomian decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), homotopy perturbation method (HPM), modified homotopy perturbation method (MHPM), homotopy analysis method (HAM).

INTRODUCTION

Kawahara equation plays an important role in mathematical physics (Bongsoo, 2009; Hunter and Scheurle, 1988; Kawahara, 1972). In recent years, some works have been done in order to find the numerical solution of this equation, for example (Abbasbandy, 2010; Biazar et al., 2008; Tutalar, 2006; Assas, 2009; Yuan and Wu, 2008; Ass, 2009; Matinfar, 2008). In this work, we developed the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve the Kawahara equation as follows:

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0, \quad (1)$$

where α, β and γ are some arbitrary constants.
With the initial conditions:

$$u(x, 0) = f(x). \quad (2)$$

In order to obtain an approximate solution of Equation 1, let us integrate one time Equation 1 with respect to t using the initial conditions, we obtain:

$$u(x,t) = f(x) - \alpha \int_0^t F(u(x,t))dt - \beta \int_0^t D^3(u(x,t))dt - \gamma \int_0^t D^5(u(x,t))dt \quad (3)$$

where,

$$D^i(u(x,t)) = (\partial^i u(x,t))/\partial x^i, \; i = 3, 5,$$

$$F(u(x,t)) = u(x,t) \; \partial u(x,t)/\partial x.$$

In Equation 3, we assume f(x) is bounded for all x in J = [a, T]. The terms $D^i(u(x, t))$ (i = 3, 5) and F (u(x, t)) are Lipschitz continuous with $| D3(u) - D3(u*) | \le L2 | u - u* |$, $| F(u) - F(u*) | \le L1 | u - u* |$ and $| D5(u) - D5(u*) | \le L3 | u - u* |$.

ITERATIVE METHODS

Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation:

$$Lu + Ru + Nu = g_1, \quad (4)$$

where u(x, t) is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms and g_1 is the source term. Applying the inverse operator L^{-1} to both sides of Equation 4 and using the given conditions, we obtain:

$$u(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \quad (5)$$

where the function $f_1(x)$ represents the terms arising from integrating the source term g_1. The nonlinear operator $Nu = G_1(u)$ is decomposed as:

$$G_1(u) = \sum_{n=0}^{\infty} An, \quad (6)$$

where A_n, $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^m}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\right]_{\lambda=0.} \quad (7)$$

The first Adomian polynomials (Behriy, 2009; Wazwaz, 2001) are:

$$A_0 = G_1(U_0),$$

$$A_1 = u_1 G'_1(U_0),$$

$$A_2 = u_2 G'_1(U_0) + \frac{1}{2!} u_1^2 G_1''(u_0),$$

$$A_3 = u_3 G'_1(U_0) + u_1 u_2 G_1''(u_0), + \frac{1}{3!} u_1^3 G_1'''(u_0). \quad (8)$$

Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in Equation 4 as the following series:

$$u(x,t) = \sum_{n=0}^{\infty} u_i(x,t), \quad (9)$$

where the components u_0, u_1, . . .u_n, which can be determined recursively as:

$$u_0 = f(x)$$
$$u_1 = -\alpha \int_0^t A_0(x,t)dt - \beta \int_0^t B_0(x,t)dt - \gamma \int_0^t L_0(x,t)dt,$$
$$\vdots$$
$$u_{n+1} = -\alpha \int_0^t A_n(x,t)dt - \beta \int_0^t B_n(x,t)dt - \gamma \int_0^t L_n(x,t)dt, \qquad n \geq 0 \quad (10)$$

Substituting Equation 8 in 10 leads to the determination of the components of u.

Modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz (Fariborz and Sadigh, 2011a). The modified forms was established on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption, we set:

$$f(x) = f_1(x) + f_2(x). \quad (11)$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 can be assigned to the zeroth component u_0, whereas the remaining part f_2 can be combined with the other terms given in Equation 11 to define u_1. Consequently, the modified recursive relation was developed:

$$u_0 = f_1(x),$$
$$u_1 = f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0),$$
$$\vdots$$
$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \qquad n \geq 1, \quad (12)$$

To obtain the approximation solution of Equation 1, according to the MADM, we can write the iterative formula of Equation 12 as follows:

$$u_0 = f_1(x)$$
$$u_1 = f_2(x) - \alpha \int_0^t A_0(x,t)dt - \beta \int_0^t B_0(x,t)dt - \gamma \int_0^t L_0(x,t)dt,$$
$$\vdots$$
$$u_{n+1} = -\alpha \int_0^t A_n(x,t)dt - \beta \int_0^t B_n(x,t)dt - \gamma \int_0^t L_n(x,t)dt, \qquad n \geq 1 \quad (13)$$

The operators $D^j(u)$ ($j = 3, 5$) and $F(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$F(u) = \sum_{n=0}^{\infty} Ai,$$
$$D^3(u) = \sum_{n=0}^{\infty} Bi,$$
$$D^5(u) = \sum_{n=0}^{\infty} Li,$$

where A_i, B_i and L_i are the Adomian polynomials. Also, we can use the following formula for the Adomian polynomials (Fariborz and Sadigh, 2011b):

$$A_n = F(s_n) - \sum_{i=0}^{n-1} Ai,$$
$$B_n = D^3(s_n) - \sum_{i=0}^{n-1} Bi,$$
$$L_n = D^5(s_n) - \sum_{i=0}^{n-1} Li, \qquad (14)$$

where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

Description of the VIM and MVIM

In the VIM (Fariborz et al 2010; Ghasemi et al., 2007; Golbabai and Keramati, 2009; Hunter and cheurle, 1988; He and Wu, 2006), the following nonlinear differential equation has been considered:

$$Lu + Nu = g1, \qquad (15)$$

where L is a linear operator, N is a nonlinear operator and g_1 is a known analytical function. In this case, the functions u_n may be determined recursively by:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,t)\left\{L\left(u_n(x,t)\right) + N\left(u_n(x,t)\right) - g_1(x,t)\right\} dt \qquad n \ge 0 \qquad (16)$$

where λ is a general Lagrange multiplier which can be computed using the variational theory. Here, the function $u_n(x, \tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x, t)$, $n \ge 0$ of the solution u(x, t) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be any selected function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x, t)$, $n \ge 0$ follow immediately.

Consequently, the exact solution may be obtained by using:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \qquad (17)$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. To obtain the approximation solution of Equation 1, according to the VIM, we can write iteration formula (Equation 16) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}\left(\lambda\left[u_n(x,t) - f(x) + \alpha\int_0^t [F(u_n(x,t))dt + \beta\int_0^t [D^3(u]_n(x,t))dt + \gamma\int_0^t [D^5(u_n(x,t))dt]\right).]\right] \quad n \ge 0 \qquad (18)$$

where

$$L_t^{-1}(.) = \int_0^t (.)\, d\tau$$

To find the optimal λ, we proceed as:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta L_t^{-1}\left(\lambda\left[u_n(x,t) - f(x) + \alpha\int_0^t [F\left(u_n(x,t)\right)dt + \beta\int_0^t [D^3(u]_n(x,t))dt + \gamma\int_0^t [D^5\left(u_n(x,t)\right)dt]\right).]\right] \qquad (19)$$

From Equation 19, the stationary conditions can be obtained as follows:

$\lambda' = 0$ and $1 + \lambda = 0$.

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in Equation 18, the following iteration formula is obtained.

$$u_0(x,t) = f(x),$$

$$u_{n+1}(x,t) =$$

$$u_n(x,t) - L_t^{-1}\left(u_n(x,t) - f(x) + \int_0^t F(u_n(x,t))dt + \beta\int_0^t [D^3(u]_n(x,t))dt + \gamma\int_0^t [D^5(u_n(x,t))dt\,),\,]\right. \quad n \ge 0 \qquad (20)$$

To obtain the approximation solution of Equation 1, based on the MVIM (He, 2004; He and Shu-Qiang, 2007), we can write the following iteration formula:

$$u_0(x,t) = f(x),$$

$$\alpha\int_0^t F\left(u_n(x,t) - u_{n-1}(x,t)\right)dt + \beta\int_0^t \left[D^3\left(u_n(x,t) - u_{n-1}(x,t)\right)\right]dt + \gamma\int_0^t \left[D^5\left(u_n(x,t) - u_{n-1}(x,t)\right)\right]dt, \quad n \geq 0 \qquad (21)$$

Relations (Equation 20) and (Equation 21) will enable us to determine the components $u_n(x, t)$ recursively for $n \geq 0$.

Description of the HAM

Consider $N[u] = 0$, where N is a nonlinear operator, $u(x, t)$ is an unknown function and x is an independent variable. Let $u_0(x, t)$ denote an initial guess of the exact solution $u(x, t)$, $h \neq 0$ an auxiliary parameter, $H_1(x, t) \neq 0$ an auxiliary function and L an auxiliary linear operator with the property $L[s(x, t)] = 0$ when $s(x, t) = 0$. Then, using $q \in [0, 1]$ as an embedding parameter, we construct homotopy as follows:

$$(1 - q)L[\varphi(x, t; q) - u_0(x, t)] - qhH_1(x, t)N[\varphi(x, t; q)] = \hat{H}[\varphi(x,t;q); u_0(x, t), H_1(x, t), h, q]. \qquad (22)$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x, t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h and the auxiliary function $H_1(x, t)$.
Enforcing the homotopy (Equation 22) to be zero, that is:

$$\widehat{H}_1[\varphi(x,t;q); u_0(x, t), H_1(x, t), h, q] = 0, \qquad (23)$$

We have the so-called zero-order deformation equation:

$$(1 - q)L[\varphi(x, t; q) - u_0(x, t)] = qhH_1(x, t)N[\varphi(x, t; q)]. \qquad (24)$$

When $q = 0$, the zero-order deformation (Equation 24) becomes:

$$\varphi(x; 0) = u_0(x, t), \qquad (25)$$

and when $q = 1$, since $h \neq 0$ and $H_1(x, t) \neq 0$, the zero-order deformation (Equation 24) is equivalent to:

$$\varphi(x, t; 1) = u(x, t). \qquad (26)$$

Thus, according to Equations 25 and 26, as the embedding parameter q increases from 0 to 1, $\varphi(x, t; q)$ varies continuously from the initial approximation $u_0(x, t)$ to the exact solution $u(x, t)$. Such kind of continuous variation is called deformation in homotopy (He, 2007; Javidi, 2009; Kawahara, 1972; Wazwaz, 2001; Yuan and Wu, 2008).
Due to Taylor's theorem, $\varphi(x, t; q)$ can be expanded in a power series of q as follows:

$$\varphi(x, t: q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \qquad (27)$$

where,

$$u_m(x, t) = \frac{1}{m!}\frac{\partial^m \varphi(x, t; q)}{\partial q^m}\Big|_{q=0}.$$

Let the initial guess $u_0(x, t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x, t)$ be properly chosen so that the power series (Equation 27) of $\varphi(x, t; q)$ converges at $q = 1$, then, we have under these assumptions the solution series:

$$u(x,t) = \varphi(x,t:1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \qquad (28)$$

From Equation 27, we can write Equation 24 as follows:

$$(1-q)L\left[\varphi(x,t:q) - u_0(x,t)\right] = (1-q)L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] = q\,h\,H_1(x,t)N[\varphi(x,t:q)] \Rightarrow$$
$$L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] - q\;L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] = q\,h\,H_1(x,t)N[\varphi(x,t:q)] \qquad (29)$$

By differentiating Equation 29 m times with respect to q, we obtain:

$$\left\{L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] - q\;L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right]\right\}^{(m)} = \left\{q\,h\,H_1(x,t)N[\varphi(x,t:q)]\right\}^{(m)} =$$
$$m!\;L\left[u_m(x,t) - u_{m-1}(x,t)\right] = h\,H_1(x,t)m\,\frac{\partial^{m-1}N[\varphi(x,t:q)]}{\partial q^{m-1}}\;\Big|q = 0$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\, R_m(u_{m-1}(x,t)), \qquad (30)$$

where

$$R_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \qquad (31)$$

and

$$X_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation (Equation 30) is governing the linear operator L, and the term $\square_m(u_{m-1}(x,t))$ can be expressed simply by Equation 30 for any nonlinear operator N.

To obtain the approximation solution of Equation 1, according to HAM, let:

$$N[u(x,t)] = u(x,t) - f(x) + \alpha \int_0^t F(u(x,t)dt + \beta \int_0^t D^3(u(x,t)dt + \gamma \int_0^t D^5(u(x,t)dt$$

so,

$$R_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - f(x) + \alpha \int_0^t F(u_{m-1}(x,t)dt + \beta \int_0^t D^3(u_{m-1}(x,t)dt + \gamma \int_0^t D^5(u_{m-1}(x,t)dt. \qquad (32)$$

Substituting Equation 32 into Equation 30:

$$L[u_m(x,t) - X_m u_{m-1}(x,t)] = h\,H_1(x,t)[u_{m-1}(x,t) + \alpha \int_0^t F(u_{m-1}(x,t))dt + \beta \int_0^t D^3(u_{m-1}(x,t))dt + \gamma \int_0^t D^5\big(u_{m-1}(x,t)\big)dt + (1 - X_m)\, f(x)] \qquad (33)$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into Equation 33 to give the recurrence relation:

$$u_0(x,t) = f(x)$$
$$u_{n+1}(x,t) = -\alpha \int_0^t F\big(u_n(x,t)\big)dt - \beta \int_0^t D^3(u_n(x,t))\,dt - \gamma \int_0^t D^5\big(u_n(x,t)\big)dt]\,), \qquad n \ge 0 \qquad (34)$$

Therefore, the solution u(x, t) becomes:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= f(x) + \sum_{n=1}^{\infty}(\,\alpha \int_0^t F\big(u_n(x,t)\big)dt - \beta \int_0^t D^3(u_n(x,t))\,dt - \gamma \int_0^t D^5\big(u_n(x,t)\big)dt]\,) \qquad (35)$$

which is the method of successive approximations. If $u_n(x, t) \mid < 1$, then the series solution (Equation 35) convergence uniformly.

Description of the HPM and MHPM

To explain HPM (Liao, 2003, 2009), we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u) \qquad (36)$$

with initial conditions $u(x,0) = f(x)$.

According to HPM, we construct a homotopy which satisfies the following relation:

$$H(u,p) = Lu - Lv_0 + p\,Lv_0 + p\,[Nu - f(u)] = 0, \qquad (37)$$

where $p \in [0, 1]$ is an embedding parameter and v0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Equation 37 is expressed as:

$$u(x,t) = u_0(x,t) + p\,u_1(x,t) + p^2\,u_2(x,t) + \ldots \qquad (38)$$

Hence, the approximate solution of Equation 36 can be expressed as a series of the power of p, that is:

$$\lim_{p\to 1} u = u_0 + u_1 + u_2 + \ldots$$

where,

$u_0(x,t) = f(x),$

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$$u_m(x,t) = \sum_{k=0}^{m-1} - \alpha \int_0^t F(u_{m-k-1}(x,t))dt - \beta \int_0^t D^3(u_{m-k-1}(x,t))\,dt - \gamma \int_0^t D^5(u_{m-k-1}(x,t))dt]\,), \quad m \geq 1. \tag{39}$$

To explain MHPM (Matinfar et al., 2008; Tutalar, 2006; Wazwaz, 1997), we consider Equation 1 as:

$$L(u) = u(x,t) - f(x) + \alpha \int_0^t F(u(x,t))dt + \beta \int_0^t D^3(u(x,t))dt + \gamma \int_0^t D^5(u(x,t))dt$$

where $F(u(x,t)) = g1(x)h1(t)$, $D^3(u(x,t)) = g2(x)h2(t)$ and $D^5(u(x,t)) = g3(x)h3(t)$. We can define homotopy $H(u, p, m)$ by $H(u, 0, m) = f(u)$, $H(u, 1, m) = L(u)$, where, m is an unknown real number and $f(u(x,t)) = u(x,t) - f(x)$.

Typically, we may choose a convex homotopy by:

$$H(u, p, m) = (1 - p)f(u) + p\,L(u) + p\,(1 - p)[m(g1(x) + g2(x) + g3(x))] = 0, \quad 0 \leq p \leq 1. \tag{40}$$

where *m* is called the accelerating parameters, and for *m* = 0, we define H(u, p, 0) = H(u, p), which is the standard HPM.

The convex homotopy (Equation 40) continuously trace an implicity defined curve from a starting point H(u(x, t) − f (u), 0, m) to a solution function H(u(x, t), 1, m). The embedding parameter p monotonically increase from 0 to 1 as trivial problem f (u) = 0 is continuously deformed to original problem L(u) = 0.

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{41}$$

when p → 1, Equation 37 corresponds to the original one and Equation 41 becomes the approximate solution of Equation 1, that is:

$$u = \lim_{p\to 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$u_0(x,t) = f(x)$$

$$u_1(x,t) = -\alpha \int_0^t F(u_0(x,t))dt - \beta \int_0^t D^3(u_0(x,t))\,dt - \gamma \int_0^t D^5(u_0(x,t))dt - m(g_1(x) + g_2(x) + g_3(x)),$$

$$u_2(x,t) = -\alpha \int_0^t F(u_1(x,t))dt - \beta \int_0^t D^3(u_1(x,t))\,dt - \gamma \int_0^t D^5(u_1(x,t))dt - m(g_1(x) + g_2(x) + g_3(x)),$$

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$$u_m(x,t) = \sum_{k=0}^{m-1} - \alpha \int_0^t F(u_{m-k-1}(x,t))dt - \beta \int_0^t D^3(u_{m-k-1}(x,t))\,dt - \gamma \int_0^t D^5(u_{m-k-1}(x,t))dt, \quad m \geq 3 \tag{42}$$

EXISTENCE AND CONVERGENCE OF ITERATIVE METHODS

Assume that,

$\alpha_1 := T\ (|\alpha| L_1 + |\beta| L_2 + |\gamma| L_3),$

$\beta_1 := 1 - T\ (1 - \alpha_1), \qquad \gamma_1 := 1 - T\ \alpha_1$

Theorem 1

Let $0 < \alpha_1 < 1$, then Kawahara Equation 1, has a unique solution.

Proof

Let u and u∗ be two arbitrary different solutions of Equation 1, then

$$|u-u^*| = \left|-\alpha\int_0^t [F(u(x,t)) - F(u^*(x,t))]dt - \beta\int_0^t [D^3(u(x,t)) - D^3(u^*(x,t))]dt - \gamma\int_0^t [D^5(u(x,t)) - D^5(u^*(x,t))]dt.\right| \le$$
$$|\alpha|\int_0^t |F(u(x,t)) - F(u^*(x,t))|dt + |\beta|\int_0^t |D^3(u(x,t)) - D^3(u^*(x,t))|dt + |\gamma|\int_0^t |D^5(u(x,t)) - D^5(u^*(x,t))|dt.$$
$$\le T\,(|\alpha|L_1 + |\beta|L_2 + |\gamma|L_3)\,|u-u^*| = \alpha_1|u-u^*|.$$

From which we get $(1-\alpha_1)\,|u-u^*| \le 0$. Since $0 < \alpha_1 < 1$, then $|u-u^*| = 0$. It implies that $u = u^*$ and completes the proof.

Theorem 2

The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (Equation 1) using MADM convergence when $0 < \alpha_1 < 1$, $|u_1(x,t)| < \infty$.

Proof

The Banach space of all continuous functions on J with the norm $\|f(t)\| = \max_t |f(t)|$, for all t in J is denoted as $(C[J], \|.\|)$. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \ge m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\|s_n - s_m\| = \max\forall t\epsilon j\ |s_n - s_m| = \max\forall t\epsilon j\ |\textstyle\sum_{i=m+1}^{n} u_i(x,t)| =$$
$$\max\forall t\epsilon j\left|-\alpha\int_0^t (\sum_{i=m}^{n-1} A_i)dt - \beta\int_0^t (\sum_{i=m}^{n-1} B_i)dt - \gamma\int_0^t (\sum_{i=m}^{n-1} L_i)dt\right|.$$

From (Fariborzi and Sadigh, 2011b), we have:

$$\sum_{i=m}^{n-1} A_i = F(s_{n-1}) - F(s_{m-1}),$$
$$\sum_{i=m}^{n-1} B_i = D^3(s_{n-1}) - D^3(s_{m-1}),$$
$$\sum_{i=m}^{n-1} L_i = D^5(s_{n-1}) - D^5(s_{m-1}).$$

So,

$$\|s_n - s_m\| = \max_{\forall t\epsilon j}\left|-\alpha\int_0^t [F(s_{n-1}) - F(s_{m-1})]dt - \beta\int_0^t [D^3(s_{n-1}) - D^3(s_{m-1})]dt - \gamma\int_0^t [D^5(s_{n-1}) - D^5(s_{m-1})]dt.\right|$$
$$\le |\alpha|\int_0^t |F(s_{n-1}) - Fs_{m-1})|dt + |\beta|\int_0^t |D^3(s_{n-1}) - D^3(s_{m-1})|dt + |\gamma|\int_0^t |D^5(s_{n-1}) - D^5(s_{m-1})|dt \le \alpha_1\|s_n - s_m\|.$$

Let $n = m+1$, then

$$\|s_n - s_m\| \le \alpha_1\|s_m - s_{m-1}\| \le \alpha_1^2\|s_{m-1} - s_{m-2}\| \le \ldots \le \alpha_1^m\|s_1 - s_0\|.$$

From the triangle inequality, we have:

$$\|s_n - s_m\| \le \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\| \le [\alpha_1^m + \alpha_1^{m+1} + \cdots + \alpha_1^{n-m-1}]\|s_1 - s_0\| \le \alpha_1^m[1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-m-1}]\|s_1 - s_0\| \le \alpha_1^m\left[\frac{1-\alpha_1^{n-m}}{1-\alpha_1}\right]\|u_1(x,t)\|.$$

Since $0 < \alpha_1 < 1$, we have $(1-\alpha_1^{n-m}) < 1$, then

$$\|s_n - s_m\| \le \frac{\alpha_1^m}{1-\alpha_1}\max_{\forall t\in J}|u_1(x,t)|$$

But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore, the series is convergence and the proof is complete.

Theorem 3

The maximum value of truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ for problem (Equation 1) by using MADM is estimated to be:

$$\max\left|u(x,t) - \sum_{i=0}^{m} u_i(x,t)\right| \le \frac{k\alpha_1^m}{1-\alpha_1}. \qquad (44)$$

Proof

From inequality (Equation 43), when $n \to \infty$, then $s_n \to u$ and $\max|u_1(x,t)|$

$$\max|u_1(x,t)| \le T(|\alpha|\max_{\forall t\in J}|F(u_0(x,t))| + |\beta|\max_{\forall t\in J}|D^3(u_0(x,t))| + |\gamma|\max_{\forall t\in J}|D^5(u_0(x,t))|).$$

Therefore,

$$\|u(x,t) - s_m\| \le \frac{\alpha_1^m}{1-\alpha_1} T(|\alpha|\max_{\forall t\in J}|F(u_0(x,t))| + |\beta|\max_{\forall t\in J}|D^3(u_0(x,t))| + |\gamma|\max_{\forall t\in J}|D^5(u_0(x,t))|).$$

Finally, the maximum value of truncation error in the interval J is obtained by Equation 44.

Theorem 4

The solution $u_n(x, t)$ obtained from the relation (Equation 20) using VIM converges to the exact solution of the problem (Equation 1), when $0 < \beta_1 < 1$.

Proof

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) - f(x) + \alpha\int_0^t [F(u_n(x,t))dt + \beta\int_0^t [D^3(u]_n(x,t))dt + \gamma\int_0^t [D^5(u_n(x,t))dt]\right)]\,]\quad (45)$$

$$u(x,t) = u(x,t) - L_t^{-1}([u(x,t) - f(x) + \alpha\int_0^t F(u(x,t))dt + \beta\int_0^t D^3(u(x,t))\,dt + \gamma\int_0^t D^5(u(x,t))dt])\quad (46)$$

By subtracting relation (Equation 45) from (Equation 46), we have:

$$u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - L_t^{-1}(u_n(x,t) - u(x,t) + \alpha\int_0^t[F(u_n(x,t)) - F(u(x,t))]dt + \beta\int_0^t[D^3(u_n(x,t)) - D^3(u(x,t))]\,dt + \gamma\int_0^t[D^5(u_n(x,t)) - D^5(u(x,t))]dt)$$

If we set, $e_{n+1}(x, t) = u_{n+1}(x, t) - u_n(x, t)$, $e_n(x, t) = u_n(x, t) - u(x, t)$, $|e_n(x, t^*)| = \max_t|e_n(x, t)|$, then since e_n is a decreasing function with respect to t from the mean value theorem, we can write:

$$e_{n+1}(x,t) = e_n(x,t) + L_t^{-1}\left(-e_n(x,t) - \alpha\int_0^t\left[F(u_n(x,t)) - F(u(x,t))\right]dt - \beta\int_0^t [[[D]^3(u]_n(x,t)) - D^3(u(x,t))\Big]dt - \gamma\int_0^t [[D^5(u_n(x,t)) - D]^5(u(x,t))\Big]dt\right)]$$

$\le e_n(x, t) + L_t^{-1}[-e_n(x,t) + L_t^{-1}|e_n(x, t)|(T(|\alpha|L_1 + |\beta|L_2 + |\gamma|L_3)] \le e_n(x, t) - T e_n(x, \eta) + T(|\alpha|L_1 + |\beta|L_2 + |\gamma|L_3)L_t^{-1}L_t^{-1}|e_n(x, t)| \le (1 - T(1 - \alpha_1)|e_n(x, t^*)|$,

where $0 \le \eta \le t$.

Hence, $e_{n+1}(x, t) \le \beta_1|e_n(x, t^*)|$. Therefore, $\|e_{n+1}\| = \max_{\forall t\in J}|e_{n+1}| \le \beta_1 \max_{\forall t\in J}|e_n| \le \beta_1\|e_n\|$. Since $0 < \beta_1 < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete.

Theorem 5

The solution $u_n(x, t)$ obtained from the relation (Equation 21) using MVIM for the problem (Equation 1) converges when $0 < \gamma_1 < 1$.

Proof

The proof is similar to the proof of Theorem 4.

Theorem 6

The maximum value of truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem (Equation 1) by using VIM is estimated to be:

$$\|e_n\| \le \frac{B_1^n k'}{1-\beta_1} \qquad k' = \max|u_1(x,t)|,\ 0 < \beta_1 < 1$$

Proof

$$u_{n+1} - u_n = (u_{n+1} - u) + (u - u_n) = e_n - e_{n+1}$$
$$\rightarrow e_n = e_{n+1} + (u_{n+1} - u_n)$$
$$\|e_n\| = \|e_{n+1} + (u_{n+1} - u_n)\| \leq \|e_{n+1}\| + \|u_{n+1} - u_n\| \leq \beta_1\|e_n\| + \|u_{n+1} - u_n\|$$
$$\rightarrow \|e_n\| \leq \frac{\|u_{n+1} - u_n\|}{1-\beta_1} \leq \frac{\beta_1^n k'}{1-\beta_1}.$$

Theorem 7

If the series solution (Equation 34) of problem (Equation 1) using HAM, is convergent, then it converges to the exact solution of the problem (Equation 1).

Proof

From Equation 33 assume that:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t),$$
$$\hat{F}(u(x,t)) = \sum_{m=0}^{\infty} F\ (u_m(x,t)),$$
$$\hat{D}^3(u(x,t)) = \sum_{m=0}^{\infty} D^3\ (u_m(x,t)),$$
$$\hat{D}^5(u(x,t)) = \sum_{m=0}^{\infty} D^5\ (u_m(x,t)).$$

where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x,t) - x_m u_{m-1}(x,t)] = u_1 + (u_2 - u_1) + \ldots + (u_n - u_{n-1}) = u_n(x,t). \qquad (47)$$

Hence, from Equation 47:

$$\lim_{n\to\infty} u_n(x,t) = 0. \qquad (48)$$

So, using Equation 48 and the definition of the linear operator L, we have:

$$\sum_{m=1}^{\infty} L[u_m(x,t) - x_m u_{m-1}(x,t)] = L[\sum_{m=1}^{\infty}[u_m(x,t) - x_m u_{m-1}(x,t)]] = 0.$$

Therefore, from Equation 30, we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t) - x_m u_{m-1}(x,t)] = hH_1(x,t)\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have:

$$\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x,t)) = 0. \qquad (49)$$

By substituting $R_{m-1}(u_{m-1}(x,t))$ into the relation (Equation 49) and simplifying it, we have:

$$\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x,t)) = \sum_{m=1}^{\infty}[u_{m-1}(x,t) + \alpha\int_0^t F(u_{m-1}(x,t)dt + \beta\int_0^t D^3(u_{m-1}(x,t)dt + \gamma\int_0^t D^5(u_{m-1}(x,t)dt + (1-x_m)f(x)] = u(x,t) - f(x) + \alpha\int_0^t \hat{F}(u(x,t))dt + \beta\int_0^t \hat{D}^3(u(x,t))\,dt + \gamma\int_0^t \hat{D}^5(u(x,t))dt. \qquad (50)$$

From Equations 49 and 50, we have:

$$u(x,t) = f(x) - \alpha\int_0^t \hat{F}(u(x,t))dt - \beta\int_0^t \hat{D}^3(u(x,t))\,dt - \gamma\int_0^t \hat{D}^5(u(x,t))dt.$$

Therefore, u(x, t) must be the exact solution. We set,

$$\emptyset_{n+1}(x,t) = \sum_{i=1}^{n+1} u_i(x,t).$$

$$|\emptyset_{n+1}(x,t) - \emptyset_n(x,t)| = D(\emptyset_{n+1}(x,t), \emptyset_n(x,t)) = D(\emptyset_n + u_n, \emptyset_n) = D(u_n, 0)$$
$$\leq \int_0^t |\alpha||F(u_n(x,t))|dt + \int_0^t |\beta||D^3(u_n(x,t))|dt + \int_0^t |\beta||D^5(u_n(x,t))|dt.$$

$$\rightarrow \sum_{n=0}^{\infty} \|\emptyset_{n+1}(x,t) - \emptyset_n(x,t)\| \leq \alpha_1 |f(x)| \sum_{n=0}^{\infty} \alpha_1^n.$$

Therefore,

$$\lim_{n\to\infty} u_n(x,t) = u(x,t).$$

Theorem 8

The maximum value of truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem (Equation 1) by using HAM is estimated to be

$$\|e_n\| \leq \frac{\alpha_1^n k'}{1-\alpha_1} \quad k' = \max |u_1(x,t)|.$$

Proof

The proof is similar to the proof of Theorem 6.

Theorem 9

If | $u_m(x, t)$ | ≤ 1, then the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (Equation 1) converges to the exact solution by using HPM.

Proof

We set,

$$\emptyset_n(x,t) = \sum_{i=1}^{n} u_i(x,t),$$

$$\emptyset_{n+1}(x,t) = \sum_{i=1}^{n+1} u_i(x,t).$$

$$|\emptyset_{n+1}(x,t) - \emptyset_n(x,t)| = D(\emptyset_{n+1}(x,t), \emptyset_n(x,t)) = D(\emptyset_n + u_n, \emptyset_n) = D(u_n, 0)$$
$$\leq \sum_{k=0}^{m-1} |\alpha| \int_0^t |F(u_{m-k-1}(x,t))|dt + |\beta| \int_0^t |D^3(u_{m-k-1}(x,t))|dt + |\gamma| \int_0^t |D^5(u_{m-k-1}(x,t))|dt.$$

$$\rightarrow \sum_{n=0}^{\infty} \|\emptyset_{n+1}(x,t) - \emptyset_n(x,t)\| \leq m\alpha_1 |f(x)| \sum_{n=0}^{\infty} (m\alpha_1)^n.$$

Therefore,

$$\lim_{n\to\infty} u_n(x,t) = u(x,t).$$

Theorem 10

If | $u_m(x, t)$ | ≤ 1, then the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem (Equation 1) converges to the exact solution by using MHPM.

Proof

The proof is similar to the proof of Theorem 9.

Theorem 11

The maximum value of truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem (Equation 1) by using HPM is estimated to be: $\|e_n\| \leq \frac{(n\alpha_1)^n nk'}{1-\alpha_1} \quad k' = \max |u_1(x,t)|.$

Proof

The proof is similar to the proof of Theorem 6.

NUMERICAL EXAMPLE

Here, we compute a numerical example which is solved

Table 1. Numerical results for Example 1.

(x, t)	Errors			
	ADM (n = 14)	MADM (n = 11)	VIM (n = 7)	MVIM (n = 6)
(0.1, 0.15)	0.080437	0.072436	0.051789	0.042675
(0.2, 0.17)	0.081779	0.073765	0.052447	0.043168
(0.3, 0.20)	0.082556	0.074158	0.052879	0.043721
(0.4, 0.23)	0.082799	0.074788	0.053262	0.044256
(0.5, 0.25)	0.083472	0.075237	0.053791	0.044749
(0.7, 0.30)	0.084185	0.075864	0.054187	0.045363
(0.9, 0.35)	0.085708	0.076332	0.054673	0.045887
(x, t)	**Errors**			
	HPM (n = 7)	**MHPM (n = 6)**	**HAM (n = 4)**	
(0.1, 0.15)	0.060854	0.031654	0.023567	
(0.2, 0.17)	0.062743	0.034893	0.023941	
(0.3, 0.20)	0.063385	0.035172	0.024557	
(0.4, 0.23)	0.063847	0.035681	0.024839	
(0.5, 0.25)	0.064295	0.035897	0.02512	
(0.7, 0.30)	0.064673	0.036356	0.02561	
(0.9, 0.35)	0.065127	0.367952	0.025986	
(1.0, 0.40)	0.065682	0.037254	0.026263	

by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM.

Lemma 1

The computational complexity of the ADM and MADM are $O(n^3)$, HAM is $O(6n)$, VIM and MVIM are $O(9n)$, HPM and MHPM are $O(n^2)$.

Example 1

Consider the Kawahara equation as follows:

$$u_t + uu_x + u_{xxx} + u_{xxxxx} = 0,$$

Subject to the initial conditions:

$$u(x,0) = -\frac{25}{78} + \frac{57}{120} sech^2(kx), k = \frac{1}{5\sqrt{10}}.$$

Table 1 shows that, approximate solution of the Kawahara equation converges with 4 iterations by using the HAM. By comparing the results of Table 1, we can observe that the HAM more rapid converges than the ADM, MADM, VIM, MVIM, HPM and MHPM.

Conclusion

In this paper, several methods to solve Kawahara equation were proposed. It was shown that, the HAM in comparison with ADM, MADM, VIM, MVIM, HPM and MHPM is more effective, rapid, easy and accurate. Also, the complexity of HAM is less than the other methods.

REFERENCES

Abbasbandy S (2010). Homotopy analysis method for the Kawahar equation. Nonlinear Analysis: Real World Appl., 11: 307-312.

Assas LMB (2009). New exact solutions for the Kawahar equation using Exp-function method. J. Comput. Appl. Math., 23: 397-102.

Bongsoo J (2009). New exact traveling wave solutions for the Kawahara equations, Non. Ana: Theory, Methods Appl., 70: 10-515.

Biazar J, Gholamian D, Hosseini K (2008). Variational iteration and Adomian decomposition methods for solving Kawahara Equation and modified Kawahara equation, Appl. Math. Sci., pp. 22705-2712.

Behriy SH, Hashish H, E-Kalla IL, Elsaid A (2009).A new algorithm for the decomposition solution of nonlinear differential equations. App. Math. Comput., 54(459-466.

Fariborz AMA, Sadigh BS (2011a). Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Homotopy analysis method J. Appl. Math. Comput., In Press.

Fariborzi AMA, Sadigh BS (2011b). Numerical solution for solving Burger's-Fisher equation by using Iterative Methods. Mathematical and Computational Applications, In Press.

Fariborzi A, Sadigh MA, Behzadi S (2010). Solving nonlinear Volterra-Fredholm integro-differential equations using He's Variational iteration method. Int. J. Comput. Math. In Press.

Ghasemi M, Tavasoli M, Babolian E (2007). Application of He's homotopy perturbation method of nonlinear integro-differential equation. Appl. Math. Comput., 188: 538-548.

Golbabai A Keramati B (2009). Solution of non-linear Fredholm integral equations of the first kind using modified homotopy Perturbation method. Chaos, Solitons and Fractals, pp. 516-2321.

Hunter JK, Scheurle J (1988). Existence of perturbed solitary wave solutions to a model equation for water waves. Phys. D., 32253-268.

He JH, Wu XH (2006). Exp-function method for nonlinear wave equations .Chaos, Solitons and Fractals, 30: 700-708.

He JH (2004). Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos, Solitons and Fractals, 19: 847-851.

He JH, Shu-Qiang W (2007). Variational iteration method for solving integro-differential equations. Physics Letters A, 367: 188-191.

He JH (2007) Variational iteration methods recent results and new interpretations. J. Comput. Appl. Math., 207: 3-17.

Javidi M (2009). Modified homotopy perturbation method for solving linear Fredholm integral equations. Chaos, Solitons Fractals, 50: 159-165.

Kawahara T (1972). Oscillatory solitary waves in dispersive media. J. Phys. Soc. Jpn., 33: 260-264.

Liao SJ (2003). Beyond Perturbation: Introduction to the Homotopy Analysis Method. Chapman and Hall/CRC Press, Boca Raton.

Liao SJ (2009). Notes on the homotopy analysis method: some definitions and theorems. Communication in Nonlinear Science and Numerical Simulation. (14): 983-997.

Matinfar M, Hosseinzadeh H, Ghanbari M (2008). Exact and numerical solution of Kawahara equation by the variational iteration method. Appl. Math. Sci., 2: 2119-2126.

Tutalar HI (2006). Aanalytic and numerical solution to a modified Kawahar a equation and a convergence an alysi sof the method. Appl. Math. Comput,, 181: 193-199.

Wazwaz AM (2001). Construction of solitary wave solution and rational solutions for the KdVequation by ADM. Chaos, Solution and fractals, 12: 2283-2293.

Wazwaz AM (1997). A first course inintegral equations, WSPC, New Jersey.

Yuan.JM, Wu SJ (2008). A Dual-Perov-Galerkin method for the Kawahara-Type equations. J. Sci. Comput., 34: 48-63.