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Full Length Research Paper

Flow of an incompressible micropolar fluid through an elliptic pipe under a constant pressure gradient

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The study of the flow through pipes and channels either closed or open is a classical problem. In this paper, we make a theoretical investigation of the flow of an incompressible micropolar fluid through a straight elliptic pipe of uniform cross section under a constant pressure gradient. In any micropolar fluid flow problem, there is a need to determine two independent flow field vectors which includes, velocity vector $\overline{q}$ and microrotation vector $\overline{\upsilon}$ which are governed by a coupled system of partial differential equations. Herein the components of these vectors are determined in terms of Mathieu functions. The arbitrary constants that arise in the solution are determined by solving an infinite non homogeneous system of linear equations adopting a numerical procedure. The variation of the volume flow rate through a cross section of the pipe is numerically studied with respect to the material parameters, geometric parameters and the pressure gradient. The results are presented through graphs.

Key words: Flow, micropolar fluid, elliptic pipe, constant pressure gradient, Mathieu functions, volume flow rate.

INTRODUCTION

The theory of micropolar fluids, introduced by Eringen (1966) is a sub class of the theory of simple microfluids introduced earlier by Eringen (1964) himself. Physically micropolar fluids represent fluids consisting of rigid randomly oriented particles suspended in a viscous medium. Polymeric solutions, colloidal suspensions, animal blood, etc. can be modelled through micropolar fluid theory. During the last four and half decades, diverse fluid flow problems which were studied in viscous fluid and several non Newtonian fluid regimes have been investigated by several researchers in micropolar regime also. The micropolar fluid theory provides for micro rotational effects and sustenance of surface as well as body couples. The stress tensor is no longer symmetric. The theory constitutes a substantial generalization of the Navier Stokes model of classical hydrodynamics. An excellent exposition of the theory is available in Lukaszewicz (1999).

The study of flow through pipes, and channels either closed or open is a classical problem. Let there be an incompressible fluid flow through a circular pipe of uniform cross section under a constant pressure gradient. This flow is referred to as “Hagen-Poiselle flow “. Several authors have studied this problem with respect to diverse non-newtonian fluids. Eringen (1966) in his introductory paper on micropolar fluids discussed the flow of an incompressible micropolar fluid through a circular tube. To the extent the authors have surveyed, the flow of a micropolar fluid through an elliptic pipe of uniform cross section under a constant pressure gradient, has not been studied so far. The aim of the present paper is to study this classical problem. This problem can be of relevance in some context where flows occur through pipes of this shape. For example, flow problems through tubes of elliptic cross section are of fundamental importance in Biomedical engineering as observed by Haslam and Zamir (1998). The flow of blood in arteries and veins that occurs in human or animal bodies is essentially through tubes which are almost of elliptic cross section. The flow is in general oscillatory in character. This flow, taking the flow to be Newtonian viscous fluid, has been studied by modeling the tubes as those of circular cross sections by Sexl (1930), Uchida (1956) and Womersley (1955).But there are a number of important places, particularly in

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heart, where the surrounding tissue compresses blood vessels which results in making them non circular (Haslam and Zamir, 1998). In view of this, the study of flows through tubes of noncircular cross sections as well, can be of importance. This is a motivating factor for our present study of flow through a pipe (or tube) of elliptic cross section. Further modeling blood as a micropolar fluid is more appropriate (Eringen, 1966; Lukaszewicz, '1999) and this is yet another. This is a motivating factor for our present study. In the present paper, we have undertaken the study of the flow of a micropolar fluid through a tube of elliptic cross section. There seems to be no considerable work in recent years with respect to flows through elliptic tubes except the one by Haslam and Zamir (1998).

The field equations of micropolar fluids are presentable in terms of the velocity vector $\bar{q}$ and the micro rotation vector $\bar{\upsilon}$ associated with each particle in the fluid medium. The components of these vectors are evaluated in terms of series of Mathieu functions (McLachlan, 1947; Abramowitz and Stegun 1965). The determination of the arbitrary constants present in the expressions of the velocity and micro rotation is through the solution of an infinite system of nonlinear homogeneous equations. The volume flow rate across the pipe is evaluated and its variation is studied numerically with respect to the material parameters of the fluid, geometric parameter and the pressure gradient. Langlois (1964), in his treatise "Slow Viscous Flow" has discussed in a succinct form the problem of flow of a non polar viscous liquid (Newtonian viscous liquid) through an elliptic pipe under a constant pressure gradient. The present problem is considerably involved, when compared with the nonpolar fluid case. The flow of a micropolar fluid through a tube of elliptic cross section under a periodic pressure gradient is under study by the present authors.

FORMULATION OF THE PROBLEM

The field equations of micro-polar fluid flow as in Eringen (1966) are

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{q}) = 0 \tag{1}$$

$$\rho \frac{d\bar{q}}{dt} = \rho \bar{f} - \nabla p + k \nabla \times \bar{\upsilon} - (\mu + k)(\nabla \times \nabla \times \bar{q}) + (\lambda_1 + 2\mu + k)\nabla(\nabla \bullet (\bar{q})) \tag{2}$$

$$\rho j \frac{d\bar{\upsilon}}{dt} = \rho \bar{l} - 2k\bar{\upsilon} + k\nabla \times \bar{q} - \gamma \nabla \times \nabla \times \bar{\upsilon} + (\alpha + \beta + \gamma)\nabla(\nabla \bullet (\bar{\upsilon})) \tag{3}$$

in which $\bar{q}, \bar{\upsilon}$ are velocity and micro rotation vectors, $\bar{f}$, $\bar{l}$ are body force per unit mass, body couple per unit mass respectively. p is the fluid pressure at any point and ρ and j are the density of the fluid and gyration parameters respectively and are assumed to be constants. The material constants (λ_1, μ, k) are viscosity coefficients and (α, β, γ) are gyro viscosity coefficients. These constants confirm to the inequalities,

$$k \geq 0 \,;\, 2\mu + k \geq 0 \,;\, 3\lambda_1 + 2\mu + k \geq 0$$

$$\upsilon \geq 0 \,;\, |\beta| \leq \upsilon \,;\, 3\alpha + \beta + \gamma \geq 0 \tag{4}$$

The constitutive equations for the stress tensor t_{ij} and the couple stress tensor m_{ij} are given by

$$t_{ij} = (-p + \lambda_1 \nabla \bullet (\bar{q}))\delta_{ij} + (2\mu + k)e_{ij} + k\,\varepsilon_{ijm}(w^m - \upsilon^m) \tag{5}$$

$$m_{ij} = (\alpha \ \nabla \bullet \bar{\upsilon})\delta_{ij} + \beta \upsilon_{i,j} + \gamma \upsilon_{i,j} \tag{6}$$

where υ_i and $2\,w_i$ are the components of micro rotation and the vorticity vector respectively. The quantities e_{ij} denote the rate of deformation components, δ_{ij} denotes the Kronecker symbol and comma denotes the covariant differentiation.

Let us consider the steady flow of an incompressible micropolar fluid through a straight elliptic tube of uniform cross-section under a constant pressure gradient. Let the centre of a typical cross sectional ellipse be taken as origin and the major axis as x-axis. Let the axis of the tube in the flow direction be the positive z-axis.

Let us assume that,

$$\bar{q} = (0, 0, w(x, y))$$

$$\bar{\upsilon} = (A(x, y), B(x, y), 0) \tag{7}$$

$$\frac{dp}{dz} = -G$$

where G is a constant. Let the body force $\bar{f}$ and body couple $\bar{l}$ be absent. In view of the incompressibility of the fluid and the above assumptions, the equations governing the fluid flow are seen to be

$$\nabla \bullet \bar{q} = 0 \tag{8}$$

$$\nabla p + k \ \nabla \times \bar{\upsilon} - (\mu + k) \ \nabla \times \nabla \times \bar{q} = 0 \tag{9}$$

$$-2k\bar{\upsilon} + k \ \nabla \times \bar{q} - \gamma \ \nabla \times \nabla \times \bar{\upsilon} + (\alpha + \beta + \gamma) \ \nabla(\nabla \bullet \bar{\upsilon}) = 0 \tag{10}$$

Let (ξ, η, z) denote an elliptic coordinate system introduced through

$$(x + i\,y) = c \cosh(\xi + i\eta), \quad z = z \tag{11}$$

with $(\bar{e}_\xi, \bar{e}_\eta, \bar{e}_z)$ as the unit base vectors and (h_1, h_2, h_3)

as the scale factors of the system. We note that

$$h_1 = h_2 = c\sqrt{(\cosh 2\xi - \cos 2\eta)}\,;\quad h_3 = 1 \tag{12}$$

where c is the semi-focal distance of the cross sectional ellipse.
Let us write

$$\nabla \bullet \overline{\upsilon} = f(\xi, \eta)\,;\quad \nabla \times \overline{\upsilon} = g(\xi, \eta)\ \ \overline{e_z} \tag{13}$$

Taking curl of (2), we have

$$k\nabla \times (\nabla \times \overline{\upsilon}) - (\mu + k)\ \nabla \times \{\nabla \times (\nabla \times \overline{q})\} = 0$$

$$\nabla \times (\nabla \times \overline{\upsilon}) = \left(\frac{\mu + k}{k}\right) \nabla \times \{\nabla \times (\nabla \times \overline{q})\}$$

Taking curl of (3), we get,

$$-2k\ \nabla \times \overline{\upsilon} + k\nabla \times (\nabla \times \overline{q}) - \gamma\ \nabla \times \{\nabla \times (\nabla \times \overline{\upsilon})\} = 0$$

Taking curl of (9) and (10), we can express 'g' in terms of 'w' through the equation

$$g = -\frac{1}{2}\nabla^2 w - \frac{\gamma(\mu + k)}{2k^2}\nabla^4 w \tag{14}$$

Further we get

$$\nabla^2\left(\nabla^2 - \frac{\lambda^2}{c^2}\right) w = 2G \tag{15}$$

and

$$\left(\nabla^2 - \frac{p^2}{c^2}\right) f = 0 \tag{16}$$

where

$$\left.\begin{aligned} \frac{\lambda^2}{c^2} &= \frac{k(2\mu + k)}{\gamma(\mu + k)} \\ \frac{p^2}{c^2} &= \frac{2k}{\alpha + \beta + \gamma} \end{aligned}\right\} \tag{17}$$

and

$$\nabla^2 = \frac{2}{c^2(\cosh\ 2\xi - \cos\ 2\eta)}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \tag{18}$$

The equations for the determination of A, B are given by

$$2kA = \frac{k}{h}\frac{\partial w}{\partial \eta} - \frac{\gamma}{h}\frac{\partial g}{\partial \eta} + \frac{(\alpha + \beta + \gamma)}{h}\frac{\partial f}{\partial \xi} \tag{19}$$

$$2kB = -\frac{k}{h}\frac{\partial w}{\partial \xi} + \frac{\gamma}{h}\frac{\partial g}{\partial \xi} + \frac{(\alpha + \beta + \gamma)}{h}\frac{\partial f}{\partial \eta} \tag{20}$$

Let the boundary Γ of the cross-sectional ellipse be taken as

$$\Gamma:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{21}$$

and let this be given by $\xi = \xi_0$.

The hyperstick boundary conditions that are to be satisfied by w, A and B are given by

$$w(\xi_0, \eta) = 0\,;\ A(\xi_0, \eta) = 0\,;\ B(\xi_0, \eta) = 0\,. \tag{22}$$

Also the flow quantities are to be finite in the flow regime and must satisfy

$$w(\xi\ , \eta) = w(\xi, -\eta) = \quad w(\pi - \eta) = w(\pi + \eta)$$

$$a(\xi, \pi - \eta) = a(\xi, -\eta) = -a(\xi, \eta)\,; \tag{23}$$

$$b(\xi, \eta) = b(\xi, -\eta)$$

Thus the velocity component w and the micro rotation components A, B can be completely determined by solving

$$\nabla^2\left(\nabla^2 - \frac{\lambda^2}{c^2}\right) w = 2G \tag{24}$$

$$\left(\nabla^2 - \frac{p^2}{c^2}\right) f = 0 \tag{25}$$

and using Equations (14), (19) and (20) subject to the boundary conditions given in (22).

SOLUTION OF THE PROBLEM

A solution of the Equation (24) can be obtained by the superposition of the general solutions of

$$\nabla^2 w = 0\,, \tag{26}$$

$$\left(\nabla^2 - \frac{\lambda^2}{c^2}\right) w = 0 \tag{27}$$

and the particular integral that satisfies the Equation (24). Consider

$$\nabla^2 w = 0$$

which is equivalent to

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = 0\,\cdot$$

Using the method of separation of variables, it can be seen that a typical solution of this equation is a linear combination of

$\{ e^{q\xi}, e^{-q\xi}\}$, $\{\cos q\xi, \sin q\xi\}$

In view of the periodicity, symmetry, regularity and finiteness of the solution within the flow regime, the most general solution of (26) appropriate for the present problem is

$$w_0 = \sum_{n=1}^{\infty} B_{2n} e^{-2n\xi} \cos 2n\eta \tag{28}$$

Solution of $\left(\nabla^2 - \frac{\lambda^2}{c^2}\right) w = 0$

Using the method of separation of variables and writing, W = R(ξ). S(η) we get

$$R''(\xi) - \left(\lambda^* + \frac{\lambda^2}{2}\cosh 2\xi\right) R(\xi) = 0 \tag{29}$$

$$S''(\eta) + \left(\lambda^* + \frac{\lambda^2}{2}\cosh 2\eta\right) s(\eta) = 0 \tag{30}$$

where λ^* is a constant of separation. We note that these are Mathieu equations (Clachlan, 1947). Solutions of (30) which are periodic corresponding to a discrete set of characteristic values of λ^* are the Mathieu functions.

$$ce_n\left(\eta, \frac{-\lambda^2}{4}\right), \; se_n\left(\eta, \frac{-\lambda^2}{4}\right).$$

Solutions of (29) corresponding to the above functions are respectively

$$Ce_n\left(\xi, \frac{-\lambda^2}{4}\right), \; Se_n\left(\xi, \frac{-\lambda^2}{4}\right).$$

In view of the comments made concerning the symmetry, periodicity, finiteness and regularity in the flow regime earlier, the appropriate solution to be chosen for the present equation is given by

$$w_1(\xi, \eta) = \Sigma C_{2n}^* Ce_{2n}\left(\xi, \frac{-\lambda^2}{4}\right) ce_{2n}\left(\eta, \frac{-\lambda^2}{4}\right) \tag{31}$$

We can directly verify that

$$w_p = \frac{Gc^2}{\lambda^2}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{32}$$

satisfies the equation

$$\nabla^2\left(\nabla^2 - \frac{\lambda^2}{c^2}\right) w = 2G.$$

using

$x = c\cosh\xi\,\cos\eta$, $y = c\sinh\xi\,\sin\eta$

The expression for w_p in (31) can be written in terms of ξ and η. Hence the most general solution appropriate for the present problem is

$$w = \frac{Gc^2}{\lambda^2}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) + \sum_{n=1}^{\infty} B_{2n} e^{-2n\xi}\cos 2n\eta$$

$$+ \sum_{n=0}^{\infty} C_{2n} Ce_{2n}\left(\xi, \frac{-\lambda^2}{4}\right) ce_{2n}\left(\eta, \frac{-\lambda^2}{4}\right) \tag{33}$$

Solution of $\left(\nabla^2 - \frac{p^2}{c^2}\right) f = 0$

Using the method of separation of variables as before and the requirements mentioned in (22) to (23), the solution appropriate for $f(\xi, \eta)$ for the equation (25) is given by

$$f(\xi, \eta) = \sum_{n=0}^{\infty} D_n \; Se_n\left(\xi, \frac{-p^2}{4}\right) se_n\left(\eta, \frac{-p^2}{4}\right) \tag{34}$$

adopting the notation of McLachlan (1947).Using (14) and (33), we have

$$g(\xi, \eta) = \frac{Gc^2}{\lambda^2} - \left(\frac{2\mu + k}{\gamma}\right)\Sigma C_{2n} Ce_{2n}\left(\xi, \frac{-\lambda^2}{4}\right) ce_n\left(\eta, \frac{-\lambda^2}{4}\right) \tag{35}$$

The expressions (33) to (35) when used in (19) and (20) give rise to

$$2kAh = \frac{kGc^4}{\lambda^2(a^2+b^2)}\left\{b^2\cosh^2\xi - a^2\sinh^2\xi\right\}\sin 2\eta$$

$$+ \sum_n \left[kB_{2n}e^{-2n\xi} + (2\mu + 2k)\sum_r (-1)^{n+r} A_{2n}^{2r} C_{2r} Ce_{2r}\left(\xi, \frac{\lambda^2}{4}\right)\right](-2n)\sin 2n\eta$$

$$+(\alpha+\beta+\gamma)\sum_{n=0}^{\infty}\left[\sum_{r=0}^{\infty}(-1)^{n+r}B_{2n+2}^{2r+2}D_{2r+2}^{*}Se'_{2r+2}\left(\xi,\frac{-p^2}{4}\right)\right]\text{Sin}\,(2n+2)\,\eta \qquad (36)$$

and

$$2kBh=\frac{kGc^4}{\lambda^2\left(a^2+b^2\right)}\left[\left(b^2+a^2\right)Sinh\xi Cosh\xi-\left(a^2-b^2\right)Sinh\xi Cosh\xi Cos2\eta\right]$$

$$+\sum_n\left[2knB_{2n}e^{-2n\xi}-(2\mu+2k)\sum_r(-1)^{n+r}A_{2n}^{2r}c_{2r}ce'_{2r}(\xi,\frac{\lambda^2}{4})\right]\text{Cos}2n\,\eta+(\alpha+\beta+\gamma)\sum_{n=0}^{\infty}\left[\sum_{r=0}^{\infty}(-1)^{n+r}B_{2n+2}^{2r+2}D_{2r+2}^{*}Se_{2r+2}\left(\xi,\frac{-p^2}{4}\right)\right](2n+2)Cos(2n+2)\eta \qquad (37)$$

The constants $\{B_{2n}\},\{C_{2n}\},\{D_{2n}\}$ in (33), (36) and (37) are to be determined using the boundary conditions

$w(\xi_0,\eta)=0$;
$A(\xi_0,\eta)=0$;
$B(\xi_0,\eta)=0$;

Given in (22) and using the following standard expressions from McLachlan (1947):

(1) $c_{e_{2n}}\left(\xi,\frac{-\lambda^2}{4}\right)=(-1)^n\sum_{r=0}^{\infty}(-1)^r A_{2r}^{2n}\text{Cosh}2r\xi \quad (a_{2n})$

(2) $c_{e_{2n}}\left(\eta,\frac{-\lambda^2}{4}\right)=(-1)^n\sum_{r=0}^{\infty}(-1)^r A_{2r}^{2n}\text{Cos}2r\xi \quad (a_{2n})$

(3) $s_{e_{2n+2}}\left(\xi,\frac{-p^2}{4}\right)=(-1)^n\sum_{r=0}^{\infty}(-1)^r B_{2r+2}^{2n+2}\text{Sinh}(2r+2)\xi \quad (b_{2n+2})$

(4) $s_{e_{2n+2}}\left(\eta,\frac{-p^2}{4}\right)=(-1)^n\sum_{r=0}^{\infty}(-1)^r B_{2r+2}^{2n+2}\text{Sin}(2r+2)\eta \quad (b_{2n+2})$

NON DIMENSIONALIZATION SCHEME

Let us introduce the following non dimensionalization scheme.

$W=U\tilde{w}$;
$A=\frac{U}{c}\tilde{A}$;
$B=\frac{U}{c}\tilde{B}$;
$B_{2n}=U\tilde{B}_{2n}$;
$C_{2n}=U\tilde{C}_{2n}$;
$D_{2n}=\frac{U}{c^2}\tilde{D}_n$; (38)

and then drop tildes. We get

$$w=\frac{Gc^2a^2b^2}{U\lambda^2\left(a^2+b^2\right)}\left[1-\frac{c^2}{2}\left(\frac{Cosh^2\xi}{a^2}+\frac{Sinh^2\xi}{b^2}\right)-\frac{c^2}{2}\left(\frac{Cosh^2\xi}{a^2}-\frac{Sinh^2\xi}{b^2}\right)Cos2\eta\right]+\sum_{n=0}^{\infty}\left[B_{2n}e^{-2n\xi}+\sum_{r=0}^{\infty}(-1)^{n+r}A_{2n}^{2r}Ce_{2r}\left(\xi,\frac{-\lambda^2}{4}\right)c_{2r}\right]\text{Cos}\,2n\,\eta \qquad (39)$$

where $B_0=0$;

$$2A\sqrt{(\text{Cosh}\,2\xi-\text{Cos}\,2\eta)}=\frac{Gc^4}{U\lambda^2\left(a^2+b^2\right)}\left[\left(b^2\text{Cosh}^2\xi-a^2\text{Sinh}^2\xi\right)\text{Sin}2\eta\right]\sum_{n=0}^{\infty}\left[B_{2n}e^{-2n\xi}+\frac{2(\mu+k)}{k}\sum_{r=0}^{\infty}A_{2n}^{2r}(-1)^{n+r}Ce_{2r}\left(\xi,\frac{-\lambda^2}{4}\right)c_{2r}\right](-2n)\,\text{Sin}\,2n\,\eta$$

$$+\frac{2}{p^2}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{\infty}(-1)^{n+r}B_{2n+2}^{2r+2}Se'_{2r+2}\left(\xi,\frac{-p^2}{4}\right)D_{2r+2}\right]\text{Sin}(2n+2)\eta \qquad (40)$$

and

$$2B\sqrt{(\cosh 2\xi-\cos 2\eta)}=\frac{Gc^4}{U\lambda^2\left(a^2+b^2\right)}\left[\text{Cosh}\xi\text{Sinh}\xi\left(a^2+b^2\right)+\text{Cosh}\xi\text{Sinh}\xi\left(b^2-a^2\right)\text{Cos}2\eta\right]$$

$$+\sum_n\left[2nB_{2n}e^{-2n\xi}-\frac{2(\mu+k)}{k}\sum_{r=0}^{\infty}(-1)^{n+r}A_{2n}^{2r}Ce'_{2r}\left(\xi,\frac{-\lambda^2}{4}\right)c_{2r}\right]\text{Cos}2n\,\eta \qquad (41)$$

$$+\frac{2}{p^2}\sum_n\left[\sum_r(-1)^{n+r}B_{2n+2}^{2r+2}Se_{2r+2}\left(\xi,\frac{-p^2}{4}\right)D_{2r+2}\right](2n+2)\ \text{Cos}\ (2n+2)\,\eta$$

DETERMINATION OF THE ARBITRARY CONSTANTS

Let us now use the boundary conditions (22) to determine $\{B_n\}$, $\{C_n\}$ and $\{D_n\}$. The boundary of the cross-sectional ellipse is $\Gamma: \xi = \xi_0$.

Since w = 0 on $\xi = \xi_0$ we get the following equations:

$$\sum_{r=0}^{\infty} (-1)^r A_0^{2r} Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r} = 0 \qquad (42)$$

$$B_{2n} \; e^{-2n\xi_0} + \sum_{r=0}^{\infty} (-1)^{n+r} A_{2n}^{2r} Ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r} = 0$$

(n = 1, 2,……..) (43)

Since $A(\xi \quad , \quad \eta) = 0$ on $\xi = \xi_0$ we have

$$-2(n+1)B_{2n+2}\, e^{-(2n+2)\xi_0} + 2(n+1)\frac{(2\mu + 2k)}{k}\sum_{r=0}^{\infty} A_{2n+2}^{2r}(-1)^{n+r}$$

$$Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r} + \frac{2}{p^2}\sum_{r=0}^{\infty}(-1)^{n+r} B_{2n+2}^{2r+2}\, Se'_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right)$$

$$D_{2r+2} = 0$$

(n = 0,1, 2,……..) (44)

Further in view of $B(\xi \quad , \quad \eta) = 0$ on $\xi = \xi_0$ we have

$$\frac{2(\mu + k)}{k}\sum (-1)^r A_0^{2r} \; Ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r} = \frac{Gc^4 \mathrm{Cosh}\xi_0 \mathrm{Sinh}\xi_0}{U\lambda^2} \qquad (45)$$

$$2B_2 \, e^{-2\xi_0} + \frac{2(\mu + k)}{k}\sum (-1)^r A_2^{2r} \; Ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r}$$

$$+ \frac{4}{p^2}\sum_{r=0}^{\infty}(-1)^r B_2^{2r+2}\, Se_{2r+2}\left(\xi \quad , \quad \frac{-p^2}{4}\right) D_{2r+2}$$

$$= \frac{Gc^4 \mathrm{Cosh}\xi_0 \mathrm{Sinh}\xi_0}{U\lambda^2\left(\mathrm{Cosh}^2\xi_0 + \mathrm{Sinh}^2\xi_0\right)} \qquad (46)$$

And

$$2(n+1)b_{2n+2}\, e^{-2(n+1)\xi_0} + \frac{2(\mu + k)}{k}\sum_{r=0}^{\infty} A_{2n+2}^{2r}(-1)^{n+r}$$

$$ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) c_{2r} + (2n+2)\frac{2}{p^2}\sum_{r=0}^{\infty}(-1)^{n+r} B_{2n+2}^{2r+2}$$

$$se_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right) d_{2r+2} = 0$$

(n = 1, 2, …) (47)

Eliminating B_2 from (43) and (44) we have,

$$\frac{2(2\mu + k)}{k}\sum (-1)^r A_2^{2r} \; Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r}$$

$$+ \frac{2}{p^2}\sum_{r=0}^{\infty}(-1)^r B_2^{2r+2} \; Se'_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right) D_{2r+2} = 0 \qquad (48)$$

Eliminating B_{2n+2} from (43) and (44) we have,

$$2(n+1)\,\frac{(2\mu + k)}{k}\sum_{r=0}^{\infty} A_{2n+2}^{2r}(-1)^{n+r} \; Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) C_{2r}$$

$$+ \frac{2}{p^2}\sum_{r=0}^{\infty}(-1)^{n+r} B_{2n+2}^{2r+2} \; Se'_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right) D_{2r+2} = 0 \qquad (49)$$

Eliminating B_2 from (43) and (46) we have,

$$\sum_{r=0}^{\infty}(-1)^r A_2^{2r}$$

$$\left[\frac{2(\mu + k)}{k} Ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) + 2Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right)\right]$$

$$+ \frac{4}{p^2}\sum_{r=0}^{\infty}(-1)^r B_2^{2r+2} \; Se_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right) D_{2r+2}$$

$$= \frac{Gc^4 \mathrm{Cosh}\xi_0 \mathrm{Sinh}\xi_0}{U\lambda^2 \mathrm{Cosh}\, 2\xi_0} \qquad (50)$$

Eliminating B_{2n+2} from (43) and (47) we have,

$$\sum_{r=0}^{\infty} A_{2n+2}^{2r}(-1)^{n+r}$$

$$\left[\frac{2(\mu + k)}{k} Ce'_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right) + 2(n+1)Ce_{2r}\left(\xi_0 \quad , \quad \frac{-\lambda^2}{4}\right)\right] Ce_{2r}$$

$$+ \frac{4}{p^2}(n+1)\sum_{r=0}^{\infty}(-1)^{n+r} B_{2n+2}^{2r+2} \; Se_{2r+2}\left(\xi_0 \quad , \quad \frac{-p^2}{4}\right) D_{2r+2} = 0 \qquad (51)$$

(45), (48), (49), (50), (51) constitute an infinite set of non-homogeneous linear equations in the unknowns

$\{C_0, C_2, C_4, \ldots\ldots C_{2r}, \ldots\ldots; D_2, D_4, D_6, \ldots\ldots D_{2r}, \ldots\ldots\}$

and hence are to be solved by a numerical procedure. When once the C's and D's are determined, the B's can be evaluated using Equation (44).

VOLUME FLOW RATE

The volume flow rate across the pipe is given by,

$$Q = \int_{\eta=0}^{2\pi}\int_{\xi=0}^{\xi_0} W(\alpha \quad , \quad \beta)\frac{c^2}{2}[Cosh 2\xi - Cos 2\eta] d\xi d\eta$$

integrated over the area of the cross sectional ellipse and is seen to be

$$\frac{\pi G c^4 a^2 b^2}{U\lambda^2\left(a^2 + b^2\right)}$$

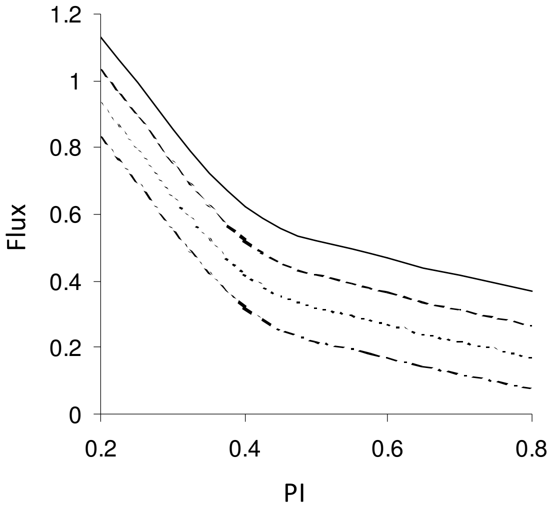

Figure 1. Variation of flux with respect to PI with varying p. lamda = 1.0 alpha = 0.1.

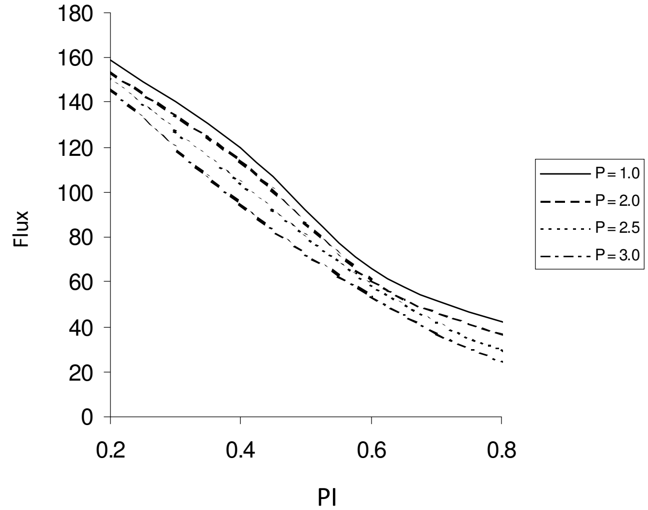

Figure 2. Variation of flux with respect to PI with varying p. lamda = 1.0 alpha = 0.6.

$$\left[\mathrm{Sinh}\,2\xi_0\left(\frac{1}{2}-\frac{c^2\left(b^2-a^2\right)}{16a^2b^2}\right)-\frac{c^2\left(a^2+b^2\right)}{32a^2b^2}\mathrm{Sinh}\,4\xi_0\right]$$
$$+\pi c^2\sum_{r=0}^{\infty}A_0^{2r}\,\frac{A_2^{2r}}{2}\left(\xi_0+\frac{\mathrm{Sinh}4\xi_0}{4}\right)$$
$$+\pi c^2\sum_{\substack{s=0\\ s\neq 1}}^{\infty}(-1)^s\frac{A_{2s}^{2r}}{2}\left(\frac{\sinh(2s+2)\xi_0}{2s+2}+\frac{\sinh(2s-2)\xi_0}{2s-2}\right)C_{2r}$$
$$+\pi c^2\,\frac{B_2e^{-2\alpha_0}}{4}$$
$$+\pi c^2\sum_{r=0}^{\infty}\frac{A_2^{2r}}{2}\left(A_0^{2r}\xi_0+\sum_{s=1}^{\infty}(-1)^sA_{2s}^{2r}\,\frac{\sinh 2s\xi_0}{2s}\right)C_{2r} \qquad (52)$$

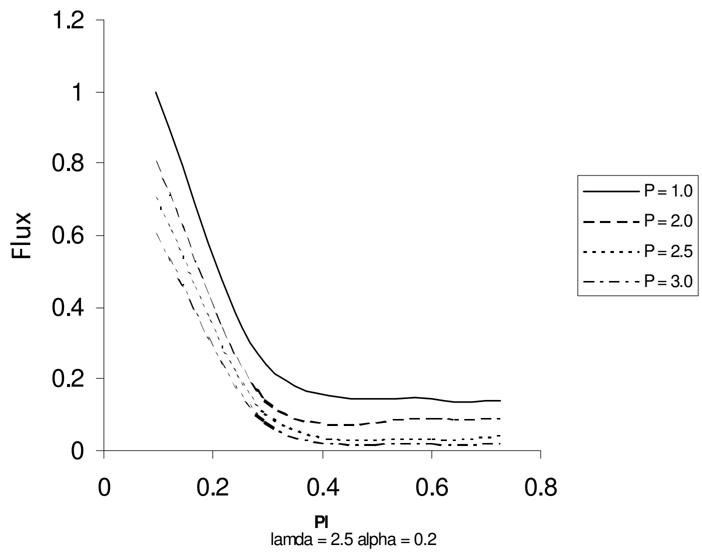

Figure 3. Variation of flux with respect to PI with varying p. lamda = 2.5 alpha = 0.2.

NUMERICAL WORK

The infinite system of equations given in (51) is truncated to a 10%10 system, taking 5 terms involving C's and 5 terms involving D's. This truncation is resorted to in view of the smallness of the subsequent terms. Using the available C's and D's, the required B_2 is evaluated using Equation (44). The volume flow rate (flux) across the cross section of the pipe is numerically estimated for a number of values of the micropolarity parameters λ, p and PI $= k/(\mu+k)$. It is estimated for a prescribed parameter representing the pressure gradient. In the figures, alpha stands for ξ_0 and λ, PI and p have their earlier meanings. Figures 1 to 4 depict the variation of flux with respect to PI with varying p for fixed values of λ and ξ_0. It is seen that the flux decreases as the micropolarity parameter increases. Further as p increases, the flux decreases. An increase in p implies an increase in k which is an extra microrotation viscosity parameter and this increase results in the reduction of the speed of a particle which naturally reduces the flux. Figures 5 to 8 show the variation of flux with respect to alpha (= ξ_0). It is seen that for fixed values of λ, p and PI, the flux decreases as the area of cross section increases for a prescribed pressure gradient.

The problem considered here deserves to be attempted for elliptic tubes with porous boundaries as well since it will be more realistic in the context of flows through arteries and veins.

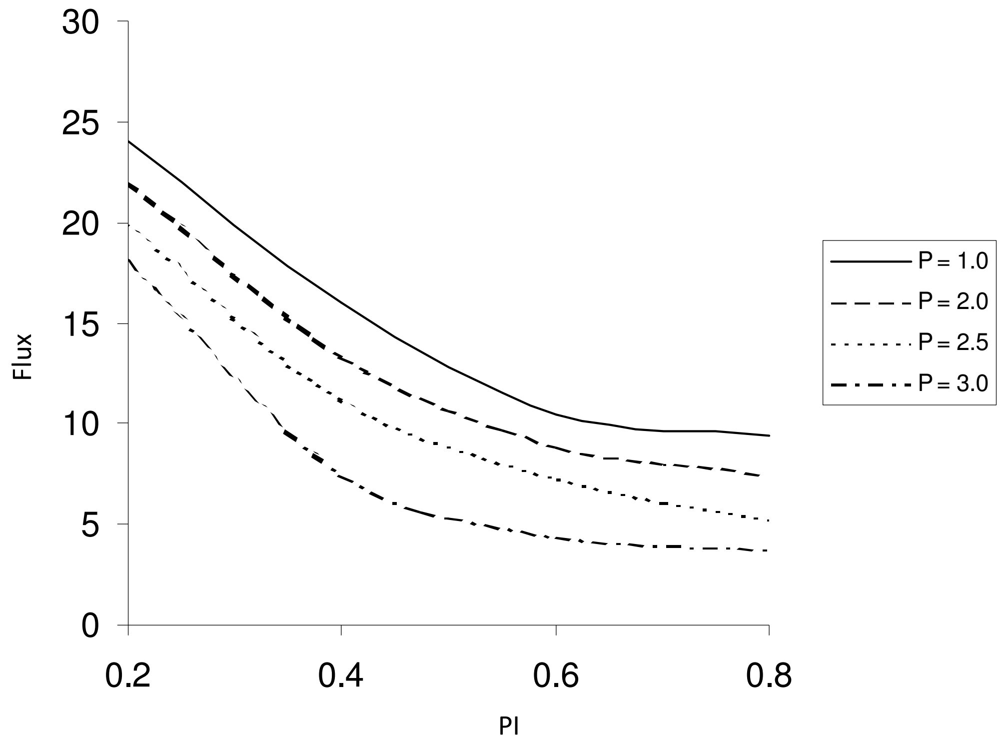

Figure 4. Variation of flux with respect to PI with varying p. lamda = 2.2 alpha = 0.1.

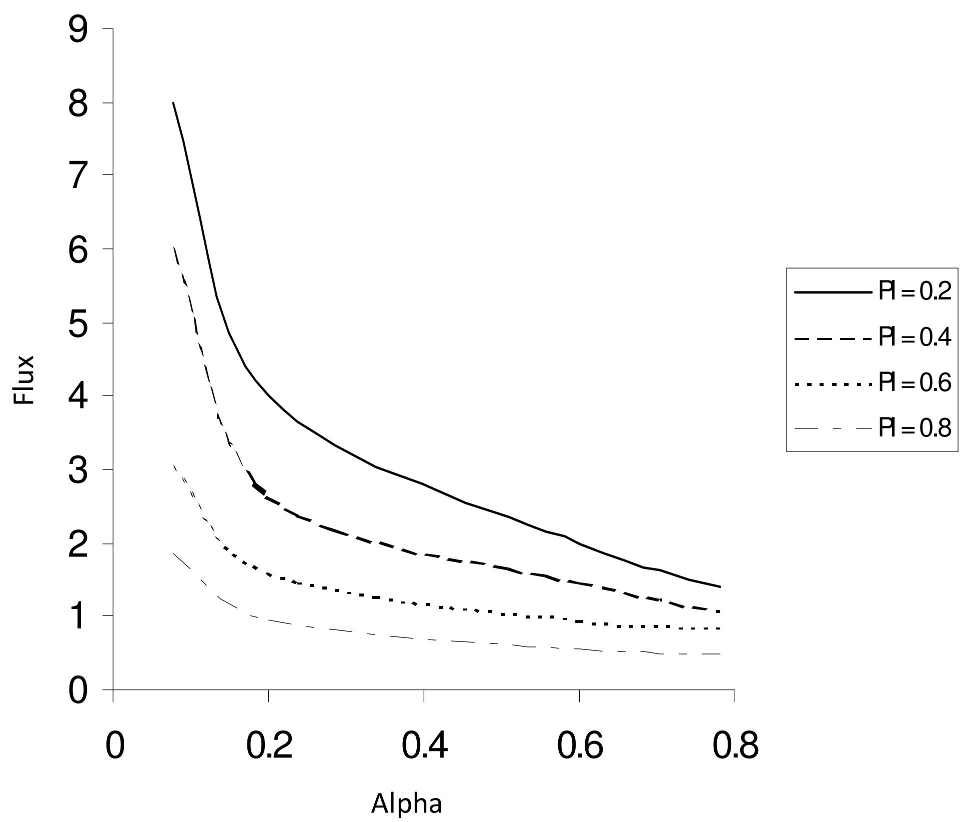

Figure 5. Variation of flux with respect to alpha with varying PI. lamda = 1.8 P = 1.0.

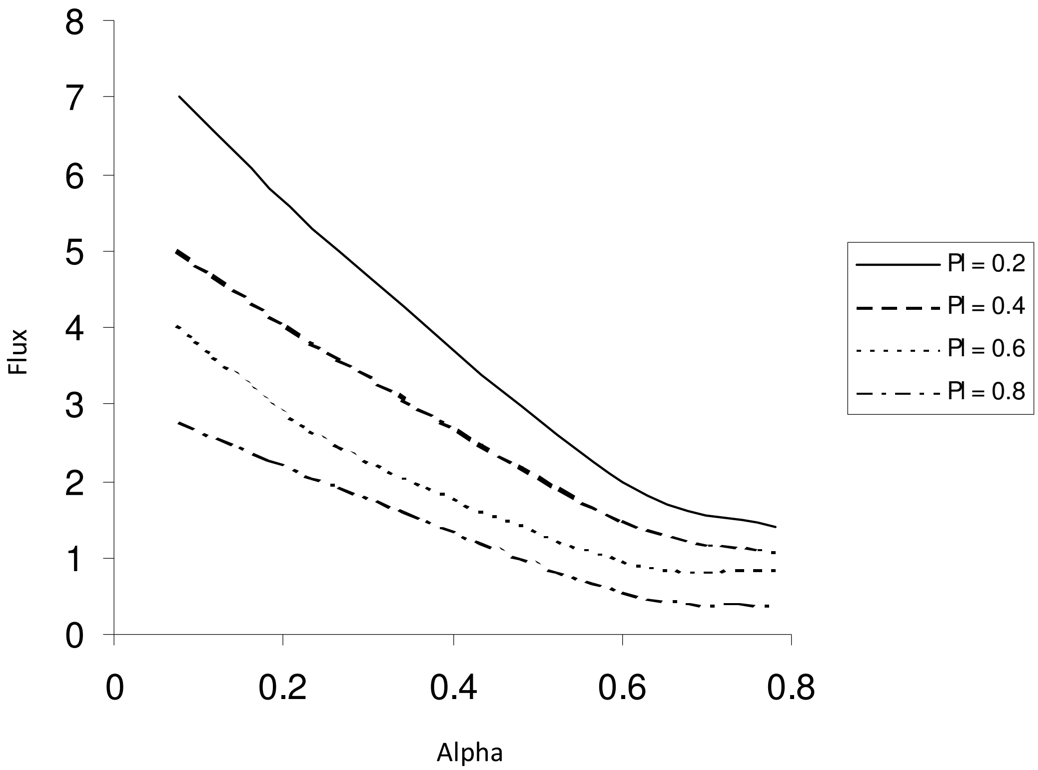

Figure 6. Variation of flux with respect to alpha with varying Pl. lamda = 1.8 P = 2.0.

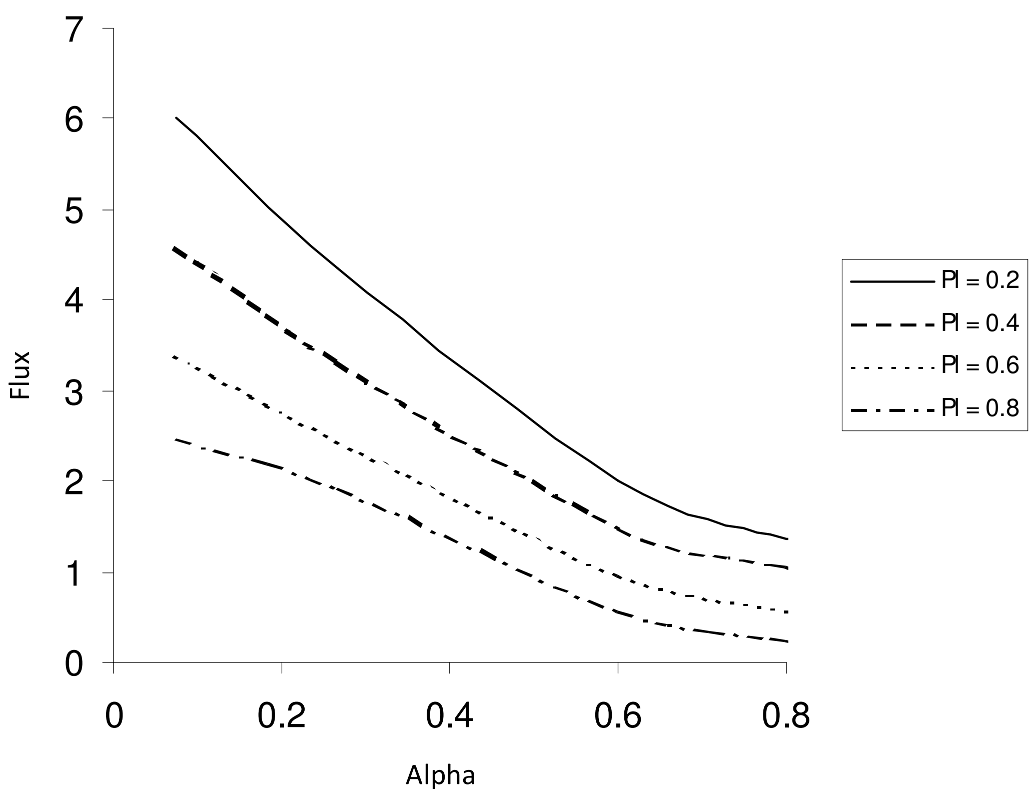

Figure 7. Variation of flux with respect to alpha with varying Pl. lamda = 1.8 P = 2.5.

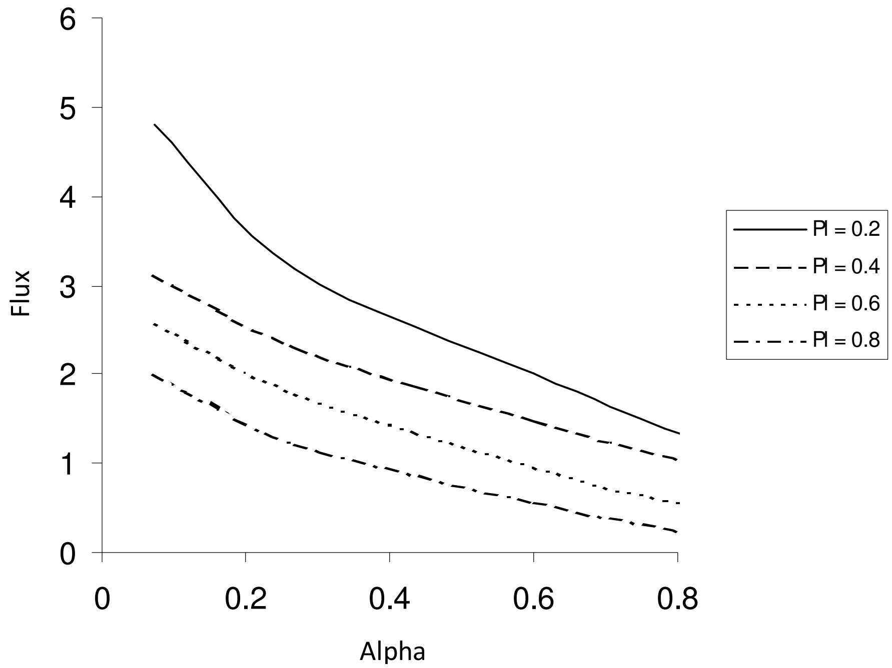

Figure 8. Variation of flux with respect to alpha with varying Pl. lamda = 1.8 P = 3.0.

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