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Full Length Research Paper

Study on the frequency – amplitude relation of beam vibration

Hamid M. Sedighi*, Arash Reza and Jamal Zare

Department of Mechanical Engineering, Ramhormoz Branch, Islamic Azad University, Khuzestan, Iran.

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New analytical work on the well-known preload nonlinearity using the innovative equivalent function (EF) is presented in this paper. The nonlinear vibration of cantilever beam with nonlinear boundary condition in the presence of preload spring with cubic nonlinearity is studied. The powerful analytical method, called He's Parameter Expanding Method (HPEM) is used to obtain the exact solution of dynamic behavior of the mentioned system. It is shown that one term in series expansions is sufficient to obtain a highly accurate solution. Finally, we successfully compare our analysis with numerical solutions.

Key words: He's parameter expanding method, preload nonlinearity, exact equivalent function, nonlinear vibration of beam.

INTRODUCTION

Literature on the preload nonlinearity is very limited. Preloaded spring is encountered in many mechanical and structural systems, such as generators, automotive clutches, etc. However, approximation of this nonlinear condition to obtain analytical solution of behavior of the mentioned systems is always the major difficulty of engineer's computations. Babitsky (1998) has summarized the dynamics of typical systems and introduced semi-analytical approaches to such problems. Rogers et al. (2004) studied the joystick dynamics where the preload stiffness (as a stiff spring) was based on measured force-displacement profile. Aktiirk et al. (1997) performed an approximated theoretical investigation of the effect of varying the preload on the vibration characteristics of a shaft bearing system. Dynamics of a mechanical oscillator with preload nonlinearity is investigated by Chengwu and Rajendra (2007). They smoothen the preload nonlinearity with arctan function. Yoshitake and Sueoka (2000) conducted bifurcation and stability analyses for self-excited system with dry friction.

From many decades, the nonlinear flexural vibrations of straight beams have been studied by many reserachers (Aghababaei et al., 2011; Bayat et al., 2011a; Barari et al., 2011; Kovářová et al., 2007; Bhashyam and Prathap, 1980; Tsiatas, 2009; Li and Xu, 2009; Pielorz, 2004; Qiang and Xiangfeng, 2005; Wickert and Mote, 1998). The sources of nonlinearity of vibration systems are generally considered due to the following aspects: (1) the physical nonlinearity, (2) the geometric nonlinearity and (3) the nonlinearity of boundary conditions. In the case of discontinuous nonlinear boundary condition, the analytical solution of such problems becomes very complex. Preload nonlinearity, as a discontinuous nonlinear boundary condition, due to its inherent difficulty, has not been modeled exactly by researchers, till present. Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including perturbation methods (Nayfeh, 1985), form function approximations (Louand and Sikarskie, 1975), semi-analytical finite element (Patel et al., 2006), artificial small parameter (Lyapunov, 1992), energy balance method (Pakar and Bayat, 2011), Adomian's decomposition (Adomian, 1976), variational iteration method (Bayat et al., 2011b), frequency amplitude formulation (Fereidoon et al., 2011), HAM (Sedighi and Shirazi, 2011), multiple scales method (Michon 2008) and

*Corresponding author. E-mail: hmsedighi@gmail.com or h-msedighi@phdstu.scu.ac.ir.

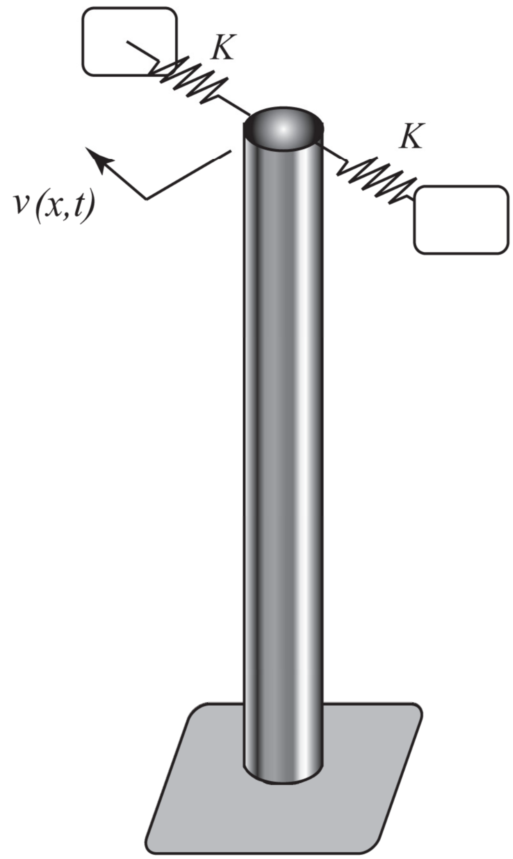

Figure 1. Cantilever beam with preload nonlinear boundary condition.

homotopy perturbation method (HPM) (Bayat et al., 2010; Marinca, 2006) are used to solve nonlinear problems. He's parameter expanding method (HPEM) is the most effective and convenient method to analytically solve nonlinear differential equations. HPEM has been shown to effectively, easily and accurately solve large nonlinear problems with components that converge rapidly to accurate solutions. Tao (2009) suggested He's parameter expanding method for strongly nonlinear oscillators and propose frequency-amplitude relationship of nonlinear oscillators. The application of new equivalent function for deadzone nonlinearity on the dynamical behavior of beam vibration has been investigated (Sedighi et al.., 2012).

In this paper, based on a Galerkin theory, nonlinear ordinary differential equation of beam vibration is extracted from partial differential equation with first mode approximation. Then, preload nonlinear boundary condition of beam is modeled using new introduced equivalent function (EF). The results presented in this paper demonstrate that this EF is very effective and convenient for nonlinear oscillators for which the highly nonlinear boundary conditions, such as deadzone, preload and saturation nonlinearities exist. The exact analytical solution of the mentioned system is obtained using HPEM and demonstrates that one term in series expansions is sufficient to obtain a highly accurate solution of the problem.

GOVERNING EQUATION

Consider the system as shown in Figure 1, where the cantilever beam is subjected to preload spring at free end. When the shearing deformations and out-of-plane motion for the beam are neglected, the governing partial differential equation for the nonlinear flexural vibration of the beam is, as follows (Malatkar, 2003):

$$m\ddot{v} + EIv^{iv} + EI\left[v'\left(v'v''\right)'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}' = 0 \quad (1)$$

Here, x is the axial coordinate measured from the origin, v denotes the lateral vibration in y direction, m is the mass per unit length of the beam, E is Young's modulus and I is the moment of inertia of area. The boundary conditions for the beam of length L are:

$$v(0,t) = \frac{\partial v}{\partial x}(0,t) = 0,\ \frac{\partial^2 v}{\partial x^2}(L,t) = 0,\ EI\frac{\partial^3 v}{\partial x^3}(L,t) = F_{pl}(L,t) \quad (2)$$

where $F_{pl}(L,t)$ is the boundary condition at free end, and is described by the following nonlinear preload formula with cubic nonlinearity:

$$F_{pl}(u) = \begin{cases} F_0 + Ku^3 & u > 0 \\ -F_0 + Ku^3 & u < 0 \end{cases} \quad (3)$$

where K is the constant of nonlinear spring. Assuming $v(x,t)=q(t)\varphi(x)$, where $\varphi(x)$ is the first eigenmode of the clamped-free beam and can be expressed as:

$$\varphi(x) = \cosh(\lambda x) - \cos(\lambda x) - \frac{\cosh(\lambda L) + \cos(\lambda L)}{\sinh(\lambda L) + \sin(\lambda L)}\left(\sinh(\lambda x) - \sin(\lambda x)\right) \quad (4)$$

where λ=1.875 is the root of characteristic equation for first eigenmode. Applying the weighted residual Bubnov-Galerkin method yields:

$$\int_0^L\left(m\ddot{v} + EIv^{iv} + EI\left[v'\left(v'v''\right)'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}'\right)\varphi(x)\,dx = 0 \quad (5)$$

To implement the end nonlinear boundary condition, applying integration by part on Equation 5, it is converted to the following:

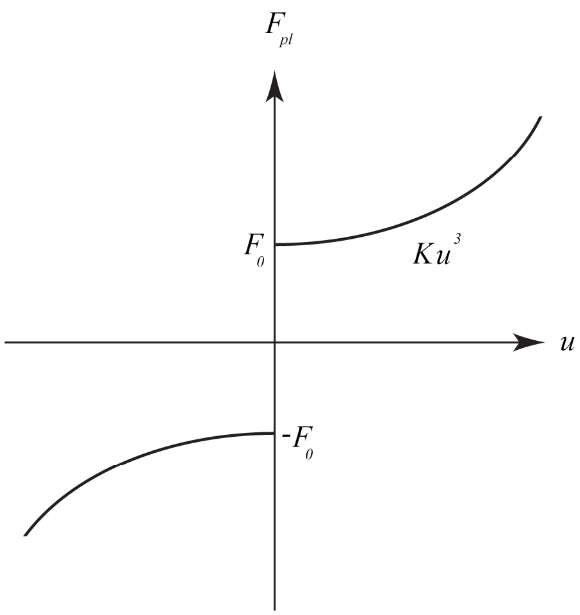

Figure 2. Plot of EF preload nonlinearity.

$$\int_0^L \left(m\ddot{v} + EI\left[v'\left(v'v''\right)' \right]' + \frac{1}{2} m \left\{ v' \int_L^x \left[\frac{\partial^2}{\partial t^2} \int_0^x v'^2 dx \right] dx \right\} \right) \varphi(x)\, dx + \int_0^L EI v^{iv} \varphi(x)\, dx = 0 \quad (6)$$

$$\int_0^L \left(m\ddot{v} + EI\left[v'\left(v'v''\right)' \right]' + \frac{1}{2} m \left\{ v' \int_L^x \left[\frac{\partial^2}{\partial t^2} \int_0^x v'^2 dx \right] dx \right\} \right) \varphi(x)\, dx \quad (7)$$
$$+ EIv'''\varphi(x)\Big|_0^L - \int_0^L EIv''' d\left(\varphi(x)\right) = 0$$

In the aforementioned equation, the boundary condition term *EIdv/dx(L,t)* is replaced by $F_{pl}(L,t)$. So, we can obtain the nonlinear equation in terms of the time-dependent variables as:

$$\ddot{q} + \beta_1 q + \beta_2 q^3 + \beta_4 q\dot{q}^2 + \beta_5 q^2 \ddot{q} + F_{pl}\, \varphi(L) = 0 \quad (8)$$

where

$$\beta_1 = 12.3624 EI/mL^4, \ \beta_2 = 40.44\, EI/mL^6 + 16K/mL, \ \beta_4 = \beta_5 = 4.6/L^2 \quad (9)$$

To solve nonlinear ordinary Equation 8 analytically, the preload condition F_{pl}, must be formulated, properly. We introduce suitable and novel exact equivalent function for this nonlinearity as:

$$F_{pl}(u) = \left(\frac{1}{2} + \frac{1}{2}\frac{|u|}{u} \right)\left(F_0 + Ku^3 \right) + \left(\frac{1}{2} - \frac{1}{2}\frac{|u|}{u} \right)\left(-F_0 + Ku^3 \right) \quad (10)$$

Figure 2 shows the equivalent function for F_{pl}, graphically.

Using this new definition of F_{pl}, Equation 9 is written as follows:

$$\ddot{q} + \beta_1 q + 1.\left[\beta_2 q^3 + \beta_3 |q|/q + \beta_4 q\dot{q}^2 + \beta_5 q^2 \ddot{q} \right] = 0 \quad (11)$$

where

$$\beta_3 = 2F_0/mL \quad (12)$$

ANALYTICAL SOLUTION PROCEDURE

Consider the Equation 11 for the vibration of a cantilever Euler-Bernoulli beam with the following general initial conditions:

$$q(0) = A\,, \ \ \dot{q}(0) = 0 \quad (13)$$

The limit-cycles of oscillating systems are periodic motions with the period $T=2\pi/\omega$, and thus *q(t)* can be expressed by such a set of base functions:

$$\cos(m\omega t), \ m = 1, 2, 3, ... \quad (14)$$

We denote the angular frequency of oscillation by ω and note that one of our major tasks is to determine $\omega(A)$, that is, the functional behavior of ω as a function of the initial amplitude *A*. In the HPEM, an artificial perturbation equation is constructed by embedding an artificial parameter *p*I [0,1] which is used as an expanding parameter.

According to HPEM, the solution of Equation 11 is expanded into a series of *p* in the form:

$$q(t) = q_0(t) + pq_1(t) + p^2 q_2(t) + ... \quad (15)$$

The coefficients *1* and β_1' in Equation 11 are expanded in a similar way:

$$\begin{aligned} 1 &= 1 + pa_1 + p^2 a_2 + ... \\ \beta_1 &= \omega^2 - pb_1 - p^2 b_2 + ... \\ 1 &= pc_1 + p^2 c_2 + ... \end{aligned} \quad (16)$$

where a_i, b_i, c_i *(i=1,2,3,…)* are to be determined. When *p=0*, Equation 11 becomes a linear differential equation for which an exact solution can be calculated for *p=1*. Substituting Equations 15 and 16 into Equation 11, we have:

$$\begin{aligned} &(1+pa_1)(\ddot{q}_0 + p\ddot{q}_1) + \left(\omega^2 - pb_1\right)(q_0 + pq_1) \\ &+\left(pc_1 + p^2 c_2\right)\Big[\beta_2 (q_0 + pq_1)^3 + \beta_4 (q_0 + pq_1)(\dot{q}_0 + p\dot{q}_1)^2 + ... \\ &... + \beta_5 (q_0 + pq_1)^2 (\ddot{q}_0 + p\ddot{q}_1) + \beta_3 f_{pl}(q_0 + pq_1) \Big] = 0 \end{aligned} \quad (17)$$

where

$$f_{pl}(q)=|q|/q \tag{18}$$

In Equation 18, we have taken into account the following expression:

$$f_{pl}(q)=f_{pl}(q_0+pq_1+p^2q_2+...)=f_{pl}(q_0)+pq_1f'_{pl}(q_0)+O(p^2) \tag{19a}$$

where

$$f'_{pl}(q)=\frac{df_{pl}}{dq}=f''_{pl}(q)=f'''_{pl}(q)=...=0 \tag{19b}$$

Therefore,

$$f_{pl}(q)=f_{pl}(q_0+pq_1+p^2q_2+...)=f_{pl}(q_0) \tag{20}$$

Collecting the terms of the same power of p in Equation 17, we obtain a series of linear equations which the first equation is:

$$\ddot{q}_0(t)+\omega^2q_0(t)=0, \qquad q_0(0)=A, \quad \dot{q}_0(0)=0$$

With the solution:

$$q_0(t)=A\cos(\omega t), \tag{22}$$

Substitution of this result into the right-hand side of second equation gives:

$$\ddot{q}_1(t)+\omega^2q_1(t)=$$
$$\left(b_1A-\frac{8}{\pi}c_1\beta_3F_0-\frac{3}{4}c_1\beta_2A^3+\frac{3}{4}c_1\beta_5A^3\omega^2-\frac{1}{4}c_1\beta_4A^3\omega^2+a_1A\omega^2\right)\cos(\omega t) \tag{23}$$
$$\frac{1}{4}c_1A^3(\beta_4\omega^2+\beta_5\omega^2-\beta_2)\cos(3\omega t),$$

In the aforementioned equation, the possible following Fourier series expansion have been accomplished as:

$$f_{pl}(q_0)=f_{pl}(A\cos(\omega t))=\sum_{n=1}^{\infty}h_n\cos(n\omega t)=h_1\cos(\omega t)+h_2\cos(2\omega t)+... \tag{24}$$

where

$$h_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f_{pl}(A\cos\theta)\cos(n\theta)\,d\theta, \tag{25}$$

and the functions f_{pl} are substituted from Equations 18 and 20. The first terms of the expansion of Equation 25 is given by:

$$h_1=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f_{pl}(A\cos\theta)\cos(\theta)\,d\theta=\frac{8F_0}{\pi} \tag{26}$$

Solution of Equation 23 should not contain the so-called secular term $\cos(\omega t)$. To ensure so, the right-hand side of this equation should not contain the terms *cos*, that is, the coefficients of cos must be zero:

$$b_1A-\frac{8}{\pi}c_1\beta_3F_0-\frac{3}{4}c_1\beta_2A^3+\frac{3}{4}c_1\beta_5A^3\omega^2-\frac{1}{4}c_1\beta_4A^3\omega^2+a_1A\omega^2=0 \tag{27}$$

Equation 16 for one term approximation of series respect to p and for $p=1$ yields:

$$a_1=0,\ b_1=\omega^2-\beta_1,\ c_1=1 \tag{28}$$

From Equations 27 and 28, we can easily find that the solution ω is:

$$\omega(A)=\pm\sqrt{\frac{\beta_1+\frac{8}{\pi A}\beta_3F_0+\frac{3}{4}\beta_2A^2-4\beta_3}{1+\frac{3}{4}\beta_5A^2-\frac{1}{4}\beta_4A^2}} \tag{29}$$

Replacing ω from Equation 29 into Equation 22 yields:

$$q(t)\approx q_0(t)=A\cos\left(\sqrt{\frac{\beta_1+\frac{8}{\pi A}\beta_3F_0+\frac{3}{4}\beta_2A^2-4\beta_3}{1+\frac{3}{4}\beta_5A^2-\frac{1}{4}\beta_4A^2}}\,t\right) \tag{30}$$

DISCUSSION

To verify the soundness of the obtained analytical solution, the authors calculate the variation of non-dimensional amplitude A/L, for different values of β_1 versus $\tau=\omega t$, numerically. As can be seen in Figure 3, the obtained first order approximation of *q(t)* using HPEM and EF for preload nonlinearity, shows an excellent accordance with numerical results using fourth-order Runge-Kutta method. The results demonstrate that the novel introduced EF is very effective and convenient for nonlinear oscillators, where the highly nonlinear preload boundary condition exists. The exact HPEM analytical solution exhibits that first term in series expansions is sufficient to achieve a highly accurate solution of the problem. Typical amplitude of cantilever beam vibrations along its length is illustrated in Figure 4.

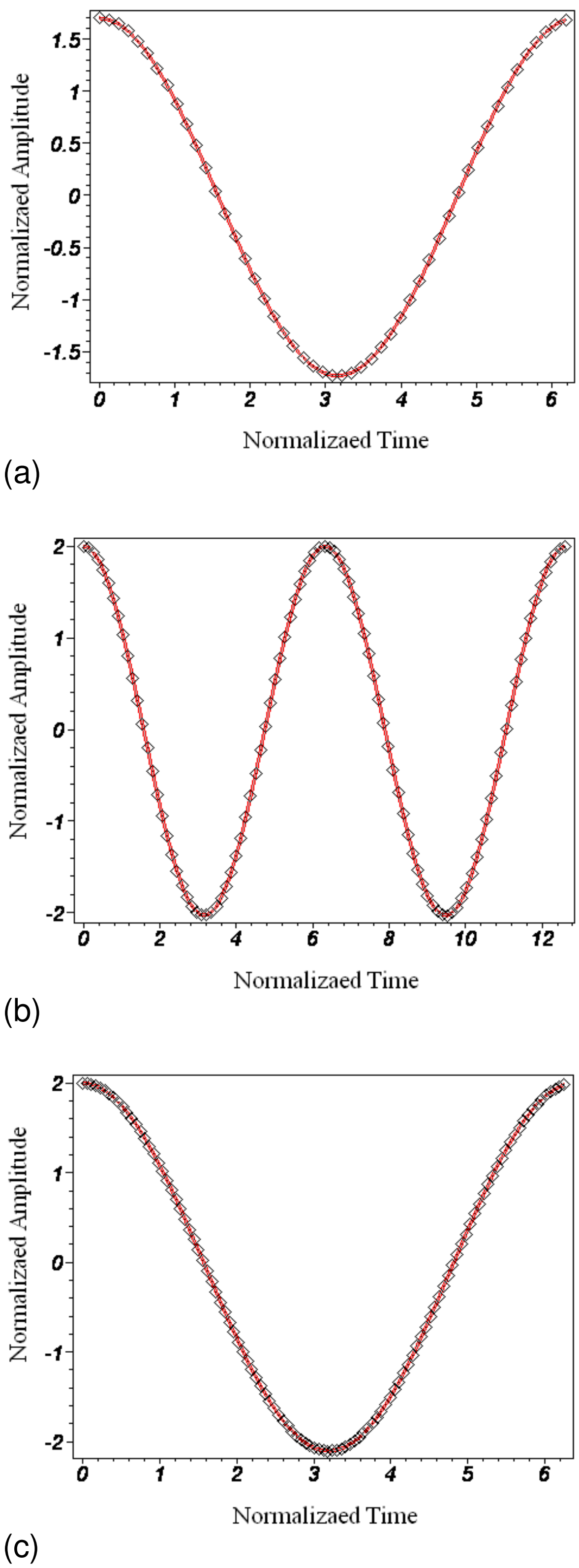

Figure 3. Comparison of the approximate first order periodic solution (continuous line) with the numerical solution (diamond) with a: $A/L = 0.17$ and $\beta_1 = 6.5\times10^3$, b: $A/L = 0.2$ and $\beta_1 = 6.5\times10^3$, c: $A/L = 0.2$ and $\beta_1 = 13\times10^3$.

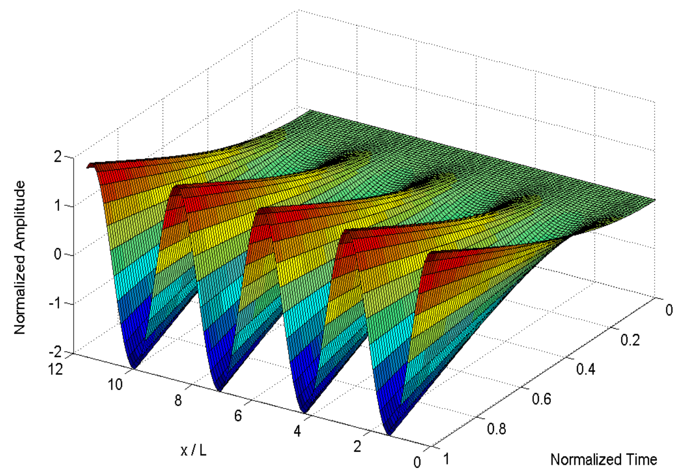

Figure 4. Typical vibration amplitude along beam length with $A/\delta = 2$.

Conclusion

Dynamics of a structural beam with preload nonlinearity is introduced in this article. Novel EF for discontinues preload nonlinearity has been employed to predict analytical response of nonlinear cantilever beam vibration in the time domain. The preload nonlinearity, as a boundary condition of cantilever beam, redefined exactly using the continuous functions. This new EF is implemented in nonlinear vibration of cantilever beam and an excellent first-order analytical approximate solution by HPEM was obtained. It appears from the present work, that the introduced EF can significantly make the analytical investigation of the nonlinear problems to be estimated quite easily. The authors believe that the introduced procedure has special potential to be applied to other strong nonlinearities such as preload, deadzone and saturation discontinues.

Nomenclature: Cross section area, A = 25 × 10^{-4} m^2; spring coefficient, K = 5000 N/m; modulus of elasticity, E = 200 GPa; beam length, L = 1 m; moment of inertia, I = 52×10^{-8} m^4; mass per unit length, m = 19.65 kg/m.

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