International Journal of the Physical Sciences Vol. 7(7), pp. 1025 - 1034, 9 February, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.1670
ISSN 1992 - 1950 © 2012 Academic Journals

Full Length Research Paper

Modified homotopy perturbation method for stiff delay differential equations (DDEs)

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Accepted 4 January, 2012

In this paper, an efficient modification is introduced into the well-known homotopy perturbation method which proves very effective to control the convergence region of approximate solution. By using this scheme, explicit exact solution is calculated in the form of a convergent power series with easily computable components. The proposed algorithm is tested on some neutral stiff functional-differential equations with proportional delays (DDEs). Numerical results explicitly reveal the complete reliability, efficiency and accuracy of the suggested technique. It is observed that the approach may be implemented on other models of physical nature.

Key words: Homotopy perturbation method, systems of stiff delay differential equations, Pade approximation.

INTRODUCTION

Consider the delay differential equation (DDE) by Zhu and Xiao (2009):

$$\begin{cases} y'(t) = f\big(t, y(t), y(t-\tau)\big) & ,t \geq 0 \\ y(t) = g(t) & -\tau \leq t \leq 0 \end{cases} \tag{1}$$

where $f: [0, +\infty) \times \mathcal{C}^N \times \mathcal{C}^N \to \mathcal{C}^N$ and $g: [\tau, 0] \to \mathcal{C}^N$ are any given functions and $\tau > 0$ is any given constant. In recent years, stability and convergence of numerical methods for DDEs have been frequently studied (Bartoszewski and Jackiewicz, 2002; Cong et al., 2004; Zennaro, 2003). Especially, two-step Runge-Kutta (RK) methods were frequently investigated. For example, a class of two-step RK methods for ordinary differential equations (ODEs) was introduced by Jackiewicz and Tracogna (1995). Moreover, order conditions and errors for two-step RK methods were investigated and some methods with good behaviors were constructed by Tracogna and Welfert (2000). Some numerical tests on a parallel computer showed the efficiency of special parallel explicit two-step RK method for non-stiff problems (Cong et al., 1998).

System of stiff delay differential equations (SDDE) is often used in mathematical modeling of immune response. The use of mathematical models to study physiological processes has provided significant insight that was not possible through experimental study. Most codes available in solving DDEs do not cater for stiff DDEs. Most of them used explicit Runge-Kutta methods to solve DDEs, for instance, Hayashi (1996). The classical efforts on stiff DDEs are the work of Roth (1980). He solved stiff DDEs using three methods, which are backward differentiation method (BDF), the Adams method and the Runge-Kutta Fehlberg method (Suleiman and Ismail, 2001). Also, Zhu and Xiao (2009) discussed for stiff DDEs, parallel two-step Rosenbrock-Wanner (ROW)-methods.

In recent years, the homotopy perturbation method (HPM) has successfully been applied to solve many types of linear and nonlinear functional equations. This method which is the combination of homotopy in topology and classic perturbations techniques is useful for obtaining exact and approximate solution of linear and nonlinear differential equations. In spite of the efficiency of this method, it cannot solve stiff DDEs. In this paper, we illustrate the ability of the new form of the homotopy

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perturbation method (NHPM) on stiff DDEs, introduced by Aminikhah and Hemmatnezhad (2011) to solve stiff ODEs. Also, we apply the Pade approximation technique for improving the result obtained. The numerical results show that NHPM-PADE technique gives the approximate solution with higher accuracy than using NHPM.

STIFF DELAY DIFFERENTIAL EQUATIONS (DDEs)

Formulation in 1975 of the simplest mathematical model of an infectious disease marked the beginning of applications of DDEs for studying the immune system and dynamics of infections. This motivated our subsequent research in developing efficient computational algorithms for solving stiff DDEs.

A system of DDEs is considered stiff when it contains processes of widely different time scales. From a computational point of view, the stiffness implies that, while solving numerically, the corresponding initial value problem by a given method with assigned tolerance, a step size is restricted by stability requirements rather than by the accuracy demands. For example, the response of an immune system cannot be represented correctly without the hereditary phenomena being taken into account: cell division, differentiation, etc. The kinetic parameters of the models represent high-rate (molecular) and slow-rate (cellular) interactions in the immune system that span a time scale from seconds to days. Therefore, the systems of DDEs appearing in immune response modeling are typically stiff (Bocharov et al., 1996).

Another example of a real-life modeling stiff problem is the dynamics of hepatitis B virus infection over 130 days interval. The disease dynamics is governed by the system of ten stiff DDEs (Bocharov et al., 1996).

HOMOTOPY PERTURBATION METHOD

The HPM, proposed first by He (1999) was further developed and improved by scientists and engineers (Chun, 2007; He, 2006; Yusufoglu, 2009). The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. The HPM is applied to nonlinear oscillators (He, 2004), to bifurcation of nonlinear problems (He, 2005), to the differential-difference equations (Yıldırım, 2008), to the system of linear equations (Liu and Xiao, 2007), to boundary value problems (He, 2006; Noor and Mohyud-Din, 2008; Yildirim, 2008) and to other fields (Jafari et al., 2010; Koçak and Yıldırım, 2009; Mohyud-Din and Noor, 2009). Since this method is an iterative method, so the Banach's fixed point theorem can be applied for convergence study of the series solution (Biazar and Ghazvini, 2000).

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(\boldsymbol{u}) - f(t) = 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad t \in \Omega \tag{2}$$

where A is a general differential operator, $\boldsymbol{u}_0$ is an initial approximation of Equation 2, $f(t)$ is a known analytical function on the domain Ω. The operator A can be divided into two parts, which are L and N, where L is a linear, but N is nonlinear. Equation 2 can be, therefore, rewritten as follows:

$$L(\boldsymbol{u}) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy

$\boldsymbol{U}(r,p): \Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(\boldsymbol{U},p) = (1-p)[L\boldsymbol{U}(t) - L\boldsymbol{u}_0(t)] + p[A\boldsymbol{U}(t) - f(t)] = 0, \quad p \in [0,1], \ t \in \Omega \tag{3}$$

$$H(\boldsymbol{U},p) = L\boldsymbol{U}(t) - L\boldsymbol{u}_0(t) + pL\boldsymbol{u}_0(t) + p[N\boldsymbol{U}(t) - f(t)] = 0, \ \ p \in [0,1], \ t \in \Omega \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Equations 3 or 4, we will have:

$$H(\boldsymbol{U},0) = L\boldsymbol{U}(t) - L\boldsymbol{u}_0(t) = 0,$$
$$H(\boldsymbol{U},1) = A\boldsymbol{U}(t) - f(t) = 0 \tag{5}$$

The changing process of p from zero to unity is just that of $\boldsymbol{U}(t,p)$ from $\boldsymbol{u}_0(t)$ to $\boldsymbol{u}(t)$. In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Equations 3 or 4 can be written as a power series in p:

$$\boldsymbol{U} = \sum_{n=0}^{\infty} p^n \boldsymbol{U}_n = \boldsymbol{U}_0 + p\boldsymbol{U}_1 + p^2\boldsymbol{U}_2 + p^3\boldsymbol{U}_3 + \cdots \tag{6}$$

Setting $p = 1$, results in the approximate solution of Equation 2:

$$\boldsymbol{u}(t) = \lim_{p\to 1} \boldsymbol{U} = \boldsymbol{U}_0 + \boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 + \cdots \tag{7}$$

Applying the inverse operator $L^{-1} = \int_0^t (\cdot)dt$ to both sides of Equation 4, we obtain:

$$\boldsymbol{U}(t) = \boldsymbol{U}(0) + \int_0^t L\boldsymbol{u}_0(t)dt - p\int_0^t L\boldsymbol{u}_0(t)dt - p\left[\int_0^t (N\boldsymbol{U}(t) - f(t))\,dt\right] \tag{8}$$

where $U(0) = u_0$. Now, suppose that the initial approximations to the solutions $L\boldsymbol{u}_0(t)$ have the form:

$$L\boldsymbol{u}_0(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n P_n(t) \tag{9}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \ldots$ are specific functions.

Substituting Equations 6 and 9 into 8 and equating the coefficients of p with the same power leads to:

$$\begin{cases} p^0: \boldsymbol{U}_0(t) = \boldsymbol{u}_0 + \sum_{n=0}^{\infty} \boldsymbol{\alpha}_n \int_0^t P_n(t)dt \\ p^1: \boldsymbol{U}_1(t) = -\sum_{n=0}^{\infty} \boldsymbol{\alpha}_n \int_0^t P_n(t)dt - \int_0^t (N\boldsymbol{U}_0(t) - f(t))\, dt \\ p^2: \boldsymbol{U}_1(t) = -\int_0^t (N\boldsymbol{U}_0(t))\, dt \\ p^j: \boldsymbol{U}_j(t) = -\int_0^t \left(N\boldsymbol{U}_{j-1}(t)\right)\, dt \qquad j = 3,4,5, \ldots \end{cases} \quad (10)$$

Now, if these equations are solved in such a way that $\boldsymbol{U}_1(t) = 0$, then Equation 10 results in, $\boldsymbol{U}_1(t) = \boldsymbol{U}_2(t) = \boldsymbol{U}_3(t) = \cdots = 0$ and therefore, the exact solution can be obtained by using:

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt$$

It is worth noting that, if $\boldsymbol{U}(t)$ is analytic at $t = t_0$, then their Taylor series:

$$\boldsymbol{U}(t) = \sum_{n=0}^{\infty} \boldsymbol{a}_n (t - t_0)^n$$

can be used in Equation 10, where $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots$ are known coefficients and $\boldsymbol{\alpha}_n$ are unknown ones, which must be computed.

PADE APPROXIMANT

The results demonstrate that PADE technique gives the approximate solution with faster convergence rate and higher accuracy than using the standard HPM or NHPM.

Definition 1

Let F be a function that is analytic in a neighborhood of zero, with $F(0) \neq 0$ (Partington, 2004). Then, an $[n, n]$ Padé approximant to F is a function $G = \frac{P_m(s)}{Q_n(s)}$, where P_m and Q_n are polynomials of degree at most n, such that $Q_n(0) = 1$ and

$$F(s) = \frac{P_m(s)}{Q_n(s)} + O(s^{m+n+1}) \quad as\ s \to 0$$

The basic idea is to match the series coefficients as far as possible. Evan though the series has finite region of convergence, we can obtain the limit of the function as $x \to \infty$ if $m = n$ (Dehghan et al., 2007). This is an alternative to truncating the Taylor expansion of F (taking a polynomial approximation) and in many cases it is a better behaved method of approximation. Note that having chosen $Q_n(0) = 1$, we must determine the remaining $m + n + 1$ coefficients of P_m and Q_n, in order to solve the $m + n + 1$ simultaneous equations implied by the identity:

$$F(s)Q_n(s) - P_m(s) = O(s^{m+n+1})\ as\ s \to 0.$$

A collection of Pade approximants formed by using a suitable set of values of m and n often provides a means of obtaining information about the function outside its circle of convergence, and of more rapidly, evaluating the function within its circle of convergence. Every power series has a circle of convergence $|z| = R$. If the given power series converges to the same function for $|z| < R$ with $0 < R < 1$, then a sequence of Pade approximants may converge for $z \in D$, where D is a domain larger than $|z| < R$. Employing higher order Pade approximations produces more efficient results. In this work, we used well known software Maple to calculate the series and the rational functions obtained from the proposed techniques. We explain this method by considering several examples.

ILLUSTRATIVE EXAMPLES

Here, we demonstrate the effectiveness of the proposed NHPM-Pade technique with three illustrative examples. It will be shown that the NHPM-Pade method is very efficient for solving the systems of stiff delay differential equations. The algorithms are performed by Maple 12.

Example 1

Consider the systems of stiff delay differential equations taken from (Suleiman and Ismail, 2001):

$$\begin{cases} y_1'(t) = -2e^2 y_1(t) + 4y_2(t) \\ y_2'(t) = y_1(t) - y_1(t-1) \end{cases} \qquad \tau = 1 \qquad (11)$$

Where

$$y_1(t) = e^{-2t} \qquad t \leq 0$$
$$y_2(t) = \frac{1}{2}(e^{-2(t-1)} - e^{-2t}) \qquad t \leq 0$$

The exact solution is $y_1(t) = e^{-2t}$ and $y_2(t) = \frac{1}{2}(e^{-2(t-1)} - e^{-2t})$.

Results are given for $t \in [0,10]$. For solving system (Equation 11) by NPHM, we construct the following homotopy:

$$\begin{cases} (1-p)[LU_1(t) - Lu_{1,0}(t)] + p[LU_1(t) + 2e^2 U_1(t) - 4U_2(t)] = 0, \\ (1-p)[LU_2(t) - Lu_{2,0}(t)] + p[LU_2(t) - U_1(t) + U_1(t-1)] = 0, \end{cases}$$

or

$$\begin{cases} LU_1(t) - Lu_{1,0}(t) + pLu_{1,0}(t) + p[2e^2U_1(t) - 4U_2(t)] = 0, \\ LU_2(t) - Lu_{2,0}(t) + pLu_{2,0}(t) + p[-U_1(t) + U_1(t-1)] = 0, \end{cases} \quad (12)$$

where $L = \frac{d}{dt}$ and $p \in [0,1]$ is an embedding parameter.

Assume that $u_{1,0}(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t)$, $u_{2,0}(t) = \sum_{n=0}^{\infty} \beta_n P_n(t)$, $P_i(t) = t^i$ and from the initial conditions, $U_1(0) = 1$, $U_2(0) = \frac{1}{2}(e^2 - 1)$. Applying the inverse operator $L^{-1} = \int_0^t (\cdot)dt$ to Equation 12, we have:

$$\begin{cases} U_1(t) = U_1(0) + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} - p \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} - p \int_0^t \left(2e^2U_1(t) - 4U_2(t)\right)dt, \\ U_2(t) = U_2(0) + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} - p \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} - p \int_0^t \left(-U_1(t) + U_1(t-1)\right)dt, \end{cases} \quad (13)$$

Suppose the solutions of the system (Equation 13) to be in the following form:

$$U_i = U_{i,0} + pU_{i,1} + p^2U_{i,2} + p^3U_{i,3} + \cdots \quad ,i = 1,2 \quad (14)$$

where in $U_{i,j}$ $i = 1,2$ and $j = 0,1,2,\ldots$ are functions which should be determined.

Substituting Equation 14 into Equation 13, and equating the coefficients of p with the same powers leads to:

$$p^0: \begin{cases} U_{1,0}(t) = U_1(0) + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} = 1 + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} \\ U_{2,0}(t) = U_2(0) + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} = \frac{1}{2}(e^2 - 1) + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} \end{cases}$$

$$p^1: \begin{cases} U_{1,1}(t) = -\sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} - \int_0^t \left(2e^2U_{1,0}(t) - 4U_{2,0}(t)\right) dt \\ U_{2,1}(t) = -\sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} - \int_0^t \left(-U_{1,0}(t) + U_{1,0}(t-1)\right) dt \end{cases}$$

$$p^j: \begin{cases} U_{1,j}(t) = -\int_0^t \left(2e^2U_{1,j-1}(t) - 4U_{2,j-1}(t)\right) dt \\ U_{2,j}(t) = -\int_0^t \left(-U_{1,j-1}(t) + U_{1,j-1}(t-1)\right) dt \end{cases} \quad j = 2,3,\ldots$$

Now, if we set the Taylor series of $U_{1,1}(t)$ at $t = 0$ equal to zero, we have:

$$U_{1,1}(t) = (-\alpha_0 - 2)t + \left(-\frac{1}{2}\alpha_1 - e^2\alpha_0 + 2\beta_0\right)t^2 + \left(-\frac{1}{3}\alpha_2 + \frac{2}{3}\beta_1 - \frac{1}{3}e^2\alpha_1\right)t^3 + \left(-\frac{1}{4}\alpha_3 - \frac{1}{6}e^2\alpha_2 + \frac{1}{3}\beta_2\right)t^4 + \cdots = 0$$

In a similar manner, setting the Taylor series of $U_{2,1}(t)$ at $t = 0$ equal to zero, leads to:

$$\begin{aligned} U_{2,1}(t) = {} & \left(-\frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2 - \frac{1}{6}\alpha_5 - \frac{1}{4}\alpha_3 + \alpha_0 - \beta_0 + \frac{1}{5}\alpha_4 + \cdots\right)t \\ & + \left(-\frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_4 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_5 - \frac{1}{2}\beta_1 + \cdots\right)t^2 \\ & + \left(\frac{1}{3}\alpha_2 - \frac{1}{2}\alpha_3 + \frac{2}{3}\alpha_4 - \frac{5}{6}\alpha_5 - \frac{1}{3}\beta_2 + \cdots\right)t^3 + \cdots = 0 \end{aligned}$$

Table 1. Absolute error for example 1.

t	NHPM		NHPM-Pade [15,15]	
	$\lvert y_{1E}-u_1 \rvert$	$\lvert y_{2E}-u_2 \rvert$	$\lvert y_{1E}-u_1 \rvert$	$\lvert y_{2E}-u_1 \rvert$
0	0	0	**0**	**0**
1	1.09e-18	4.50e-18	1.02e-18	4.40e-18
2	3.49e-16	2.82e-17	1.13e-19	4.75e-18
3	1.22e-10	1.13e-11	4.25e-16	4.82e-15
4	1.02e-06	1.26e-07	1.94e-13	2.98e-12
5	0.00112	0.000173	1.40e-11	2.62e-10
6	0.338	0.0626	3.32e-10	7.00e-09
7	42.1	9.06	3.81e-09	8.56e-08
8	2.74e+03	671	2.65e-08	6.10e-07
9	1.09e+05	2.97e+04	1.29e-07	2.96e-06
10	2.94e+06	8.80e+05	4.76e-07	1.08e-05

It can easily be shown that:

$$\alpha_0 = -2, \quad \alpha_1 = 4.00000, \quad \alpha_2 = -4.00002, \quad \alpha_3 = 2.66666, \quad \alpha_4 = -1.33335,$$
$$\alpha_5 = 0.533333, \quad \alpha_6 = -0.177779, \quad \ldots$$

$$\beta_0 = -6.38906, \quad \beta_1 = 12.7782, \quad \beta_2 = -12.7781, \quad \beta_3 = 8.51873, \quad \beta_4 = -4.25933,$$
$$\beta_5 = 1.70375, \quad \beta_6 = -0.567921, \quad \ldots$$

Therefore, the approximate solutions of the system of differential Equation 11, can be expressed as:

$$u_1(t) = U_{1,0}(t) = 1.-2.t + 2.00000t^2 - 1.33333t^3 + .666667t^4 - .266667t^5 + 0.888889 \times 10^{-1}t^6 - 0.253968 \times 10^{-1}\, t^7 + 0.634921 \times 10^{-2}\, t^8 - 0.141093 \times 10^{-2}t^9 + 0.282187 \times 10^{-3}t^{10} + \cdots$$

$$u_2(t) = U_{2,0}(t) = 3.19453 - 6.38906t + 6.38906t^2 - 4.25937t^3 + 2.12969t^4 - .851874t^5 + .283958t^6 - 0.811309 \times 10^{-1}t^7 + 0.202827 \times 10^{-1}t^8 - 0.450727 \times 10^{-2}t^9 + 0.901454 \times 10^{-3}t^{10} + \cdots$$

This results achieved for $N = 30$.

Now, we apply NHPM-Pade technique to approximate $u_1(t)$ and $u_2(t)$ using the rational approximation [15,15] (Table 1).

Example 2

Consider the stiff linear delay differential equation (Zhu and Xiao, 2009):

$$y'(t) = Ly(t) + My(t-0.1) + N, \; 0 \le t \le 10, \qquad \tau = 0.1, \quad (15)$$

$$y(t) = (1 + e^{-t}, 1 + e^{-2t})^T \qquad -0.1 \le t \le 0$$

where

$$L = \begin{bmatrix} -1001 & -125 \\ 8 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1000e^{-0.1} & 125e^{-0.2} \\ -8e^{-0.1} & -2e^{-0.2} \end{bmatrix}, \quad N = \begin{bmatrix} 1126 - 1000e^{-0.1} - 125e^{-0.2} \\ -8 + 8e^{-0.1} + 2e^{-0.2} \end{bmatrix}$$

The exact solution is:

$$y(t) = \left(y_1(t), y_2(t)\right)^T = (1 + e^{-t}, 1 + e^{-2t})^T.$$

For solving the system (Equation 15) by NPHM, we construct the following homotopy:

$$\begin{cases} LU_1(t) - Lu_{1,0}(t) + pLu_{1,0}(t) + p\begin{bmatrix} 1001U_1(t) + 125U_2(t) - 1000e^{-0.1}U_1(t-0.1) - \\ 125e^{-0.2}U_2(t-0.1) - N_1 \end{bmatrix} = 0, \\ LU_2(t) - Lu_{2,0}(t) + pLu_{2,0}(t) + p[-8U_1(t) + 8e^{-0.1}U_1(t-0.1) + 2e^{-0.2}U_2(t-0.1) - N_2] = 0, \end{cases} \quad (16)$$

$$U_1(0) = 2,\ U_2(0) = 2.$$

where $L = \frac{d}{dt}$ and $p \in [0,1]$ is an embedding parameter.

Assume that $u_{1,0}(t) = \sum_{n=0}^{\infty} \alpha_n t^n$, $u_{2,0}(t) = \sum_{n=0}^{\infty} \beta_n t^n$ and from the initial conditions,

Applying the inverse operator $L^{-1} = \int_0^t (\cdot)dt$ to Equation 16, we have:

$$\begin{cases} U_1(t) = U_1(0) + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} - p\sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} \\ -p\int_0^t (1001U_1(t) + 125U_2(t) - 1000e^{-0.1}U_1(t-0.1) - 125e^{-0.2}U_2(t-0.1) - N_1)dt, \\ U_2(t) = U_2(0) + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} - p\sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} \\ -p\int_0^t (-8U_1(t) + 8e^{-0.1}U_1(t-0.1) + 2e^{-0.2}U_2(t-0.1) - N_2)dt, \end{cases} \quad (17)$$

Suppose the solutions of the system (Equation 17) to be in the following form:

$$U_i = U_{i,0} + pU_{i,1} + p^2U_{i,2} + p^3U_{i,3} + \cdots \quad , i = 1,2, \quad (18)$$

where in $U_{i,j}$ $i = 1,2$ and $j = 0,1,2,\ldots$ are functions which should be determined.

Substituting Equation 18 into Equation 17, and equating the coefficients of p with the same powers leads to:

$$p^0: \begin{cases} U_{1,0}(t) = U_1(0) + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} = 2 + \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} t^{n+1} \\ U_{2,0}(t) = U_2(0) + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} = 2 + \sum_{n=0}^{\infty} \frac{\beta_n}{n+1} t^{n+1} \end{cases}$$

$$p^j: \begin{cases} U_{1,j}(t) = -\int_0^t \left(1001U_{1,j-1}(t) + 125U_{2,j-1}(t) - 1000e^{-0.1}U_{1,j-1}(t-0.1) - 125e^{-0.2}U_{2,j-1}(t-0.1)\right) dt \\ U_{2,j}(t) = -\int_0^t \left(-8U_{1,j-1}(t) + 8e^{-0.1}U_{1,j-1}(t-0.1) + 2e^{-0.2}U_{2,j-1}(t-0.1)\right) dt \qquad j = 2,3,\ldots \end{cases}$$

Now, if we set the Taylor series of $U_{1,1}(t)$ at $t = 0$ equal to zero, we have:

$$\begin{aligned} U_{1,1}(t) = (&-91.4837\alpha_0 - 10.2341\beta_0 + 4.52418\alpha_1 + 0.511708\beta_1 - 0.204682\times 10^{-3}\beta_4 - 0.341138\times 10^{-1}\beta_2 \\ &- 0.301612\alpha_2 + 0.255854\times 10^{-2}\beta_3 - 0.180967\times 10^{-2}\alpha_4 + 0.150806\times 10^{-3}\alpha_5 - 118.827 \\ &+ 0.226209\times 10^{-1}\alpha_3 + 0.170569\times 10^{-4}\beta_5 + \cdots)t \\ &+ (-0.511705\times 10^{-3}\beta_5 + 0.511705\times 10^{-2}\beta_4 + 4.52418\alpha_2 - 0.511710\times 10^{-1}\beta_3 - 11.3295\beta_0 \\ &- 45.7418\alpha_1 - 48.0815\alpha_0 + 0.452418\times 10^{-1}\alpha_4 + 0.511705\beta_2 - .452418\alpha_3 - 5.11710\beta_1 - 0.452418 \\ &\times 10^{-2}\alpha_5 + \cdots)t^2 \\ &+ (0.852846\times 10^{-2}\beta_5 - 16.0273\alpha_1 - 3.77640\beta_1 - 0.603223\alpha_4 + .511706\beta_3 - 3.41136\beta_2 \\ &- 0.682273\times 10^{-1}\beta_4 - 30.4945\alpha_2 + 4.52416\alpha_3 + 0.754029\times 10^{-1}\alpha_5 + \cdots)t^3 \\ &+ (-22.8709\alpha_3 - 8.01368\alpha_2 - 1.88822\beta_2 - 0.852845\times 10^{-1}\beta_5 - 2.55855\beta_3 + .511705\beta_4 \\ &- 0.754030\alpha_5 + 4.52418\alpha_4 + \cdots)t^4 \\ &+ (-18.2967\alpha_4 + .511708\beta_5 - 4.80820\alpha_3 - 1.13292\beta_3 + 4.52418\alpha_5 - 2.04682\beta_4 + \cdots)t^5 + \cdots \end{aligned}$$

In a similar manner, setting the Taylor series of $U_{2,1}(t)$ at $t = 0$ equal to zero, leads to:

$$U_{1,2}(t) = (-0.00818732\ \beta_1 + 0.723870\ \alpha_0 - 0.836254\ \beta_0 - 0.0361934\ \alpha_1 - 0.000180967\ \alpha_3 - 0.0000409366\ \beta_3 + 0.0000144774\ \alpha_4 - 0.876159 + 0.000545820\ \beta_2 + 0.00241290\ \alpha_2 - 2.72910\ 10^{-7}\beta_5 + 0.00000327492\ \beta_4 - 0.00000120645\ \alpha_5 + \cdots)\ t + (-0.0000818730\ \beta_4 + 0.00000818730\ \beta_5 + 0.000818730\ \beta_3 - 0.000361935\ \alpha_4 - 0.00818730\ \beta_2 - 0.0361935\ \alpha_2 + 0.00361934\ \alpha_3 - 0.418127\ \beta_1 + 0.361934\ \alpha_1 + 0.0000361935\ \alpha_5 + 0.380650\ \alpha_0 - 0.818730\ \beta_0 + \cdots)t^2 + (-0.00818733\ \beta_3 - 0.000136455\ \beta_5 + 0.241290\ \alpha_2 + 0.126887\ \alpha_1 - 0.272910\ \beta_1 - 0.278751\ \beta_2 + 0.00109164\ \beta_4 + 0.00482580\ \alpha_4 - 0.000603226\ \alpha_5 - 0.0361933\ \alpha_3 + \cdots)t^3 + (-0.209064\ \beta_3 + 0.0634425\ \alpha_2 - 0.136455\ \beta_2 + 0.00136455\ \beta_5 + 0.180967\ \alpha_3 + 0.00603225\ \alpha_5 - 0.0361935\ \alpha_4 - 0.00818730\ \beta_4 + \cdots)t^4 + (-0.00818730\ \beta_5 + 0.0380660\ \alpha_3 - 0.0818732\ \beta_3 - 0.0361936\ \alpha_5 + 0.144774\ \alpha_4 - 0.167251\ \beta_4 + \cdots)t^5 + \cdots$$

It can easily be shown that:

$$\alpha_0 = -1.000054835, \quad \alpha_1 = 1.000064451, \quad \alpha_2 = -.5000376805, \quad \alpha_3 = .1666815663, \quad \alpha_4 = -0.4166961591 \times 10^{-1}, \quad \alpha_5 = 0.8335184623 \times 10^{-2}, \quad \alpha_6 = -0.1389285832 \times 10^{-2}, \ldots$$

$$\beta_0 = -2.000048867, \beta_1 = 4.000100960, \beta_2 = -4.000093592, \beta_3 = 2.666719277, \beta_4 = -1.333358386, \beta_5 = 0.5333436597, \beta_6 = -0.1777807251, \ldots$$

Therefore, the approximate solutions of the system of differential Equation 1 can be expressed as:

$$u_1(t) = U_{1,0}(t) = -1.08105 \times 10^{-8} \times t - 2.46174 \times 10^{-8} \times t^2 - 3.9932 \times 10^{-8} \times t^3 + 7.5 \times 10^{-10} \times t^4 + 8.\times 10^{-10} \times t^5 + 4.00000 \times 10^{-10} \times t^6 + 1.00000 \times 10^{-10} \times t^7 + 5.00559 \times 10^{-12} \times t^8 - 1.16222 \times 10^{-11} \times t^9 - 1.10375 \times 10^{-12} \times t^{10} + \cdots$$

$$u_2(t) = U_{2,0}(t) = 8.55182 \times 10^{-10} \times t + 6.44545 \times 10^{-10} \times t^2 - 4.0908 \times 10^{-10} \times t^3 + 3.38 \times 10^{-11} \times t^4 + 3.1 \times 10^{-11} \times t^5 + 1.38622 \times 10^{-22} \times t^6 - 2.77511 \times 10^{-21} \times t^7 + 2.79268 \times 10^{-13} \times t^8 - 1.91722 \times 10^{-13} \times t^9 - 2.22247 \times 10^{-14} \times t^{10} + \cdots$$

This results achieved for $N = 30$.

Now, we apply NHPM-Pade technique to approximate $u_1(t)$ and $u_2(t)$ using the rational approximation [15,15] (Table 2).

Example 3

Now, consider the stiff delay equation (El-Hawary and Mahmoud, 2003):

$$y'(t) = Ay(t) + y\left(t - \frac{3\pi}{2}\right) \qquad (19)$$

where

$$y(t) = e^{pt} + \sin t, \quad t \in [-\tau, 0], \qquad p = -2, \qquad A = p - e^{-\tau p}$$

The exact solution is given by $y(t) = e^{pt} + \sin t$

For large negative values of p, the solution consists of a short transient of exponential decay, followed by a periodic sinusoidal oscillation. The parameter p also enters the delay equation exponentially; therefore, its effect on the stiffness of the equation is dramatic.

For solving Equation 19 by NPHM, we construct the following homotopy:

$$LU(t) - Lu_0(t) + pLu_0(t) + p\left[-AU(t) - U\left(t - \frac{3\pi}{2}\right) + A\sin t\right] = 0, \quad (20)$$

where $L = \frac{d}{dt}$ and $p \in [0,1]$ is an embedding parameter.

Assume that $u_0(t) = \sum_{n=0}^{\infty} \alpha_n t^i$ and from the initial conditions, $U(0) = 1$. Applying the inverse operator $L^{-1} = \int_0^t (\cdot)dt$ to Equation 20, we have:

Table 2. Absolute error for example 2.

t	NHPM		NHPM-Pade [15,15]	
	$\lvert y_{1E}-u_1\rvert$	$\lvert y_{2E}-u_2\rvert$	$\lvert y_{1E}-u_1\rvert$	$\lvert y_{2E}-u_1\rvert$
0	0	0	0	0
1	2.0e-19	3.5e-19	2.0e-19	2.1e-19
2	8.29e-18	5.90e-17	1.1e-19	3.06e-19
3	2.33e-12	2.44e-11	4.21e-19	2.73e-17
4	1.69e-08	2.32e-07	3.21e-17	6.86e-14
5	1.68e-05	0.000281	1.95e-15	5.28e-12
6	0.00473	0.092	5.88e-14	7.09e-11
7	0.558	12.3	9.50e-13	7.44e-10
8	34.8	845	9.58e-12	5.40e-09
9	1.34e+03	3.52e+04	6.76e-11	2.86e-08
10	3.49e+04	9.88e+05	3.61e-10	1.18e-07

$$U(t)=U(0)+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+1}t^{n+1}-p\sum_{n=0}^{\infty}\frac{\alpha_n}{n+1}t^{n+1}-p\int_0^t\left(-AU(t)-U\left(t-\frac{3\pi}{2}\right)+A\sin t\right)dt, \qquad (21)$$

Suppose the solutions of the system (Equation 21) to be in the following form:

$$U=U_0+pU_1+p^2U_2+p^3U_3+\cdots, \qquad (22)$$

where in $U_{i,j}$ $i=1,2$ and $j=0,1,2,\ldots$ are functions which should be determined.

Substituting Equation 22 into Equation 21, and equating the coefficients of p with the same powers leads to:

$$p^0: U_0(t)=U(0)+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+1}t^{n+1}=1+\sum_{n=0}^{\infty}\frac{\alpha_n}{n+1}t^{n+1}$$

$$p^1: U_1(t)=-\sum_{n=0}^{\infty}\frac{\alpha_n}{n+1}t^{n+1}-\int_0^t\left(-AU_0(t)-U_0\left(t-\frac{3\pi}{2}\right)+A\sin t\right)dt$$

$$p^j: U_j(t)=-\int_0^t\left(-AU_{j-1}(t)-U_{j-1}\left(t-\frac{3\pi}{2}\right)\right)dt \qquad j=2,3,\ldots$$

Now, if we set the Taylor series of $U_1(t)$ at $t=0$ equal to zero, we have:

$$\begin{aligned}U_1(t)=&(-464.7673966\alpha_4-34.88206128\alpha_2+11.10330495\alpha_1+123.2833809\alpha_3-5.712388981\alpha_0\\&-12392.64782)t\\&+(-6196.323910\alpha_0-2.856194490\alpha_1+11.10330496\alpha_2+246.5667618\alpha_4\\&-52.32309190\alpha_3+6196.823910)t^2\\&+(-1.904129660\alpha_2-2065.441303\alpha_1+11.10330495\alpha_3-69.76412256\alpha_4)t^3\\&+(-1032.720652\alpha_2-1.428097245\alpha_3-516.4019925+11.10330496\alpha_4)t^4+\cdots=0\end{aligned}$$

It can easily be shown that:

$$\alpha_0=-0.999860,\quad \alpha_1=3.99927,\quad \alpha_2=-4.49878,\quad \alpha_3=2.66563,\quad \alpha_4=-1.29116,$$
$$\alpha_5=0.533207,\quad \alpha_6=-0.179172,\ldots$$

Therefore, the approximate solutions of the system of differential Equation 1 can be expressed as:

$$\begin{aligned}u(t)=U_0(t)=&1.-0.999860\times t+1.99964\times t^2-1.49959\times t^3+.666408\times t^4-.258233\times t^5+0.888678\times 10^{-1}\\&\times t^6-0.255960\times 10^{-1}\times t^7+0.63514\times 10^{-2}\times t^8-0.140907\times 10^{-2}\times t^9+0.282387\times 10^{-3}\\&\times t^{10}+\cdots\end{aligned}$$

Table 3. Absolute error for example 3.

t	NHPM	NHPM-Pade [25,25]
	$\lvert y_E\text{-}u \rvert$	$\lvert y_E\text{-}u \rvert$
0	0	0
$\frac{\pi}{2}$	2.17e-06	2.17e-06
π	4.33e-07	4.33e-07
$\frac{3\pi}{2}$	9.11e-13	9.11e-13
2π	8.71e-10	1.75e-10
$\frac{5\pi}{2}$	6.10e-05	4.69e-11
3π	0.666	1.10e-07

This results achieved for $N = 50$.

Now, we apply NHPM-Pade technique to approximate $u_1(t)$ and $u_2(t)$ using the rational approximation [25,25] (Table 3).

Conclusion

In this work, we proposed an efficient modification of the HPM based on NHPM and Pade approximation which achieves to the exact or approximate solution of the systems of stiff delay differential equations. The present technology provides a simple way to adjust and control the convergence region of approximate solution for any values of t. Numerical results explicitly reveal the complete reliability, efficiency and accuracy of the suggested technique.

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