academicJournals

Vol. 9(9), pp. 208-213, 16 May, 2014
DOI: 10.5897/IJPS2013.4067
Article Number: 638CA8144589
ISSN 1992 - 1950

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Properties of Bertrand curves in dual space

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Received 8 November, 2013; Accepted 19 February, 2014

Starting from ideas and results given by Özkaldi, İlarslan and Yaylı in (2009), in this paper we investigate Bertrand curves in three dimensional dual space D^3. We obtain the necessary characterizations of these curves in dual space D^3. As a result, we find that the distance between two Bertrand curves and the dual angle between their tangent vectors are constant. Also, well known characteristic property of Bertrand curve in Euclid space E^3 which is the linear relation between its curvature and torsion is satisfied in dual space as $\widehat{\lambda}.\widehat{\kappa}(s)+\widehat{\mu}.\widehat{\tau}(s)=1.$ We show that involute curves, which are the curves whose tangent vectors are perpendicular, of a curve constitute Bertrand pair curves.

Key words: Bertrand curves, involute-evolute curves, dual space.

2000 Mathematics Subject Classification: 53A04 , 53A25 , 53A40.

INTRODUCTİON

In the study of differential geometry, the characterizations of the curves and the corresponding relations between the curves are significant problems. It is well known that many important results in the theory of the curves in E^3 were given by G. Monge and then G. Darboux detected the idea of moving frame. After this, Frenet defined moving frame and special equations which are used in mechanics, kinematics and physics.

A set of orthogonal unit vectors can be built, if a curve is differentiable in an open interval, at each point. These unit vectors are called Frenet frame. The Frenet vectors along the curve, define curvature and torsion of the curve. The frame vectors, curvature and torsion of a curve constitute Frenet apparatus of the curve.

It is certainly well known that a curve can be explained by its curvature and torsion except as to its position in space. The curvature (κ) and torsion (τ) of a regular curve help us to specify the shape and size of the curve. Such as If $\kappa=\tau=0$, then the curve is a geodesic. If $\kappa\neq 0$ (constant) and $\tau=0$, then the curve is a circle with radius $\frac{1}{\kappa}$. If $\kappa\neq 0$ (constant) and $\tau\neq 0$ (constant), then the curve is a helix (Guggenheimer, 1963; Struik, 1988).

Bertrand curves can be given as another example of that relation. Bertrand curves are discovered in 1850, by J. Bertrand who is known for his applications of differential equations to physics, especially thermodynamics. A Bertrand curve in E^3 is a curve such that its principal normal vectors are the principal normal vectors of another curve. It is proved in most studies on the subject that the characteristic property of a Bertrand

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curve is the existence of a linear relation between its curvature and torsion as:

$$\lambda\kappa + \mu\tau = 1$$

with constants $\lambda, \ \mu$ where $\lambda \neq 0$ (Kühnel, 2006).

Dual numbers were defined by Clifford (1849, 1879). After him E. Study used dual numbers and dual vectors to clarify a mapping from dual unit sphere to three dimensional Euclidean space E^3. This mapping is called Study mapping. Study mapping corresponds the dual points of a dual unit sphere to the oriented lines in E^3. So the set of oriented lines in Euclidean space E^3 is one to one correspondence with the points of dual space in D^3.

In this paper, we study Bertrand curves in dual space D^3.

PRELİMİNARİES

We now recall some basic notions about dual space and apparatus of curves.

The set D is called the dual number system and the elements of this set are in type of $\widehat{a} = a + \varepsilon a^*$. Here a and a^* are real numbers and $\varepsilon^2 = 0$ which is called a dual unit. The elements of the set D are called dual numbers. The set D is given by

$$\mathbb{D} = \{\widehat{a} = a + \varepsilon a^* \mid a, a^* \epsilon \ \mathbb{R}\}.$$

For the dual number $\widehat{a} = a + \varepsilon a^*$, $a \in R$ is called the real part of $\widehat{a}$ and $a^* \in \mathbb{R}$ is called the dual part of $\widehat{a}$. Two inner operations and equality on D are defined for $\widehat{a} = a + \varepsilon a^*$ and $\widehat{b} = b + \varepsilon b^*$ as ;

1) $+ : D \times D \to D$

$$\widehat{a} + \widehat{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

is called the addition in D.

2) $\cdot : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$

$$\widehat{a} \cdot \widehat{b} = (a + \varepsilon a^*).(b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*)$$

is called the multiplication in D.

3) $\widehat{a} = \widehat{b}$ if and only if $a = b$ and $a^* = b^*$. (Köse et al., 1988; Veldkamp, 1976)

Also, the set $\mathbb{D} = \{\widehat{a} = a + \varepsilon a^* \mid a, a^* \epsilon \ \mathbb{R}\}$ forms a commutative ring with the following operations

$$i) \ (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

$$ii) \ (a + \varepsilon a^*).(b + \varepsilon b^*) = a.b + \varepsilon(ab^* + ba^*).$$

The division of two dual numbers $\widehat{a} = a + \varepsilon a^*$ and $\widehat{b} = b + \varepsilon b^*$ provided $b \neq 0$ can be defined as

$$\frac{\widehat{a}}{\widehat{b}} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^* b - a b^*}{b^2}.$$

The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{ \begin{array}{c} \vec{\widehat{a}} \mid \vec{\widehat{a}} = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) \\ = (a_1, a_2, a_3) + \varepsilon (a_1^*, a_2^*, a_3^*) \\ = \vec{a} + \varepsilon \vec{a^*}, \quad \vec{a} \in \mathbb{R}^3 \ , \ \vec{a^*} \in \mathbb{R}^3 \end{array} \right\}$$

is a module on the ring D which is called D. Module and the elements are dual vectors consisting of two real vectors (Çöken and Görgülü, 2002; Güven et al., 2011).

The inner product and vector product of $\vec{\widehat{a}} = \vec{a} + \varepsilon \vec{a^*} \in \mathbb{D}^3$ and $\vec{\widehat{b}} = \vec{b} + \varepsilon \vec{b^*} \in \mathbb{D}^3$ are given by

$$\begin{aligned} \left\langle \vec{\widehat{a}}, \vec{\widehat{b}} \right\rangle &= \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b^*} \right\rangle + \left\langle \vec{a^*}, \vec{b} \right\rangle \right) \\ \vec{\widehat{a}} \times \vec{\widehat{b}} &= \left(\widehat{a}_2 \widehat{b}_3 - \widehat{a}_3 \widehat{b}_2 \ , \widehat{a}_3 \widehat{b}_1 - \widehat{a}_1 \widehat{b}_3 \ , \ \widehat{a}_1 \widehat{b}_2 - \widehat{a}_2 \widehat{b}_1 \right) \end{aligned}$$

where $\widehat{a}_i = a_i + \varepsilon a_i^* \ , \ \widehat{b}_i = b_i + \varepsilon b_i^* \in \mathbb{D} \ , \ 1 \leq i \leq 3.$

The norm $\left\| \vec{\widehat{a}} \right\|$ of $\vec{\widehat{a}}$ is defined by

$$\left\| \vec{\widehat{a}} \right\| = \sqrt{\left\langle \vec{\widehat{a}}, \vec{\widehat{a}} \right\rangle} = \| \vec{a} \| + \varepsilon \frac{\left\langle \vec{a}, \vec{a^*} \right\rangle}{\| \vec{a} \|}$$

where $a \neq 0$.

If the norm of $\vec{\widehat{a}}$ is 1, then $\vec{\widehat{a}}$ is called a dual unit vector.

Let

$$\begin{aligned} \widehat{\alpha} : I \subset \mathbb{D} \quad &\longrightarrow \quad \mathbb{D}^3 \\ \lambda \quad &\longrightarrow \quad \vec{\widehat{\alpha}}(\lambda) = \vec{\alpha}(\lambda) + \varepsilon \vec{\alpha^*}(\lambda) \end{aligned}$$

be a dual space curve with differentiable vectors $\vec{\alpha}(\lambda)$ and $\vec{\alpha^*}(\lambda)$. The dual arc-length parameter of $\vec{\widehat{\alpha}}(\lambda)$ is defined as

$$s = \int_{t_1}^{t} \left\| \overrightarrow{\widehat{\alpha}}(\lambda)' \right\| d\lambda.$$

Now we will give dual Frenet vectors of the dual curve

$$\begin{array}{cccc} \widehat{\alpha} : I \subset \mathbb{D} & \longrightarrow & \mathbb{D}^3 \\ s & \longrightarrow & \overrightarrow{\widehat{\alpha}}(s) = \overrightarrow{\alpha}(s) + \varepsilon \overrightarrow{\alpha^*}(s) \end{array}$$

with the dual arc-length parameter s. Then

$$\frac{d\overrightarrow{\widehat{\alpha}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{\alpha}}}{ds} \cdot \frac{ds}{d\widehat{s}} = \overrightarrow{\widehat{T}}$$

is called the unit tangent vector of $\overrightarrow{\widehat{\alpha}}(s)$. The norm of the vector $\frac{d\overrightarrow{\widehat{T}}}{ds}$ which is given by

$$\frac{d\overrightarrow{\widehat{T}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{T}}}{ds} \cdot \frac{ds}{d\widehat{s}} = \frac{d^2\overrightarrow{\widehat{\alpha}}}{d\widehat{s}^2} = \widehat{\kappa}\overrightarrow{\widehat{N}}$$

is called curvature function of $\overrightarrow{\widehat{\alpha}}(s)$. Here $\widehat{\kappa} : I \longrightarrow \mathbb{D}$ is nowhere pure-dual. Then the unit principal normal vector of $\overrightarrow{\widehat{\alpha}}(s)$ is defined as

$$\overrightarrow{\widehat{N}} = \frac{1}{\widehat{\kappa}} \cdot \frac{d\overrightarrow{\widehat{T}}}{d\widehat{s}}$$

The vector $\overrightarrow{\widehat{B}} = \overrightarrow{\widehat{T}} \times \overrightarrow{\widehat{N}}$ is called the binormal vector of $\overrightarrow{\widehat{\alpha}}(s)$. Also, we call the vectors $\overrightarrow{\widehat{T}}, \overrightarrow{\widehat{N}}, \overrightarrow{\widehat{B}}$ Frenet trihedron of $\overrightarrow{\widehat{\alpha}}(s)$ at the point $\widehat{\alpha}(s)$. The derivatives of dual Frenet vectors $\overrightarrow{\widehat{T}}, \overrightarrow{\widehat{N}}, \overrightarrow{\widehat{B}}$ can be written in matrix form as:

$$\begin{bmatrix} \overrightarrow{\widehat{T}}' \\ \overrightarrow{\widehat{N}}' \\ \overrightarrow{\widehat{B}}' \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{\widehat{T}} \\ \overrightarrow{\widehat{N}} \\ \overrightarrow{\widehat{B}} \end{bmatrix}$$

which are called Frenet formulas (Köse et al. (1988). The function $\widehat{\tau} : I \longrightarrow \mathbb{D}$ such that $\frac{d\overrightarrow{\widehat{B}}}{d\widehat{s}} = -\widehat{\tau}\overrightarrow{\widehat{N}}$ is called the torsion of $\overrightarrow{\widehat{\alpha}}(s)$ which is nowhere pure-dual.

For a general parameter t of a dual space curve $\overrightarrow{\widehat{\alpha}}$, the curvature and torsion of $\overrightarrow{\widehat{\alpha}}$ can be calculated as:

$$\widehat{\kappa} = \frac{\left\| \widehat{\alpha}' \times \widehat{\alpha}'' \right\|}{\left\| \widehat{\alpha}' \right\|^3} \quad , \quad \widehat{\tau} = \frac{\det\left(\widehat{\alpha}', \widehat{\alpha}'', \widehat{\alpha}''' \right)}{\left\| \widehat{\alpha}' \times \widehat{\alpha}'' \right\|}.$$

BERTRAND CURVES IN $\mathbb{D}^3$

Here, we define Bertrand curves in dual space $\mathbb{D}^3$ and give characterizations and theorems for these curves.

Definition 1

Let $\mathbb{D}^3$ be the dual space with standard inner product $\langle , \rangle$ and $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be the dual space curves. If there exists a corresponding relationship between the dual space curves $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ so that the principal normal vectors of $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are linear dependent to each other at the corresponding points of the dual curves, then $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are called Bertrand curves in $\mathbb{D}^3$.

Let the curves $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be Bertrand curves in $\mathbb{D}^3$, parameterized by their arc-length s and $\widehat{s}$, respectively. Let $\left\{ \overrightarrow{\widehat{T}}(s), \overrightarrow{\widehat{N}}(s), \overrightarrow{\widehat{B}}(s) \right\}$ indicate the unit Frenet frame along $\overrightarrow{\widehat{\alpha}}$ and $\left\{ \overrightarrow{\widehat{T}^\circ}(s), \overrightarrow{\widehat{N}^\circ}(s), \overrightarrow{\widehat{B}^\circ}(s) \right\}$ the unit Frenet frame along $\overrightarrow{\widehat{\beta}}$. Also $\widehat{\kappa}(s) = \kappa(s) + \varepsilon\kappa^*(s)$ and $\widehat{\tau}(s) = \tau(s) + \varepsilon\tau^*(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\alpha}}$, respectively. Similarly, $\widehat{\kappa^\circ}(s) = \kappa^\circ(s) + \varepsilon\kappa^{\circ *}(s)$ and $\widehat{\tau^\circ}(s) = \tau^\circ(s) + \varepsilon\tau^{\circ *}(s)$ are the curvature and torsion of $\overrightarrow{\widehat{\beta}}$, respectively.

In the following theorems, we obtain the characterizations of a dual Bertrand curve.

Theorem 1

Let $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be two curves in $\mathbb{D}^3$. If $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are Bertrand curves, then

$$d\left(\overrightarrow{\widehat{\alpha}}(s), \overrightarrow{\widehat{\beta}}(s) \right) = \widehat{c}$$

where $s \in I \subset \mathbb{D}$ and $\widehat{c} \in \mathbb{D}$ (constant).

Proof: Let $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ be Bertrand curves.

If $\overrightarrow{\widehat{\alpha}}$ and $\overrightarrow{\widehat{\beta}}$ are Bertrand curves , then it can be written from Figure 1;

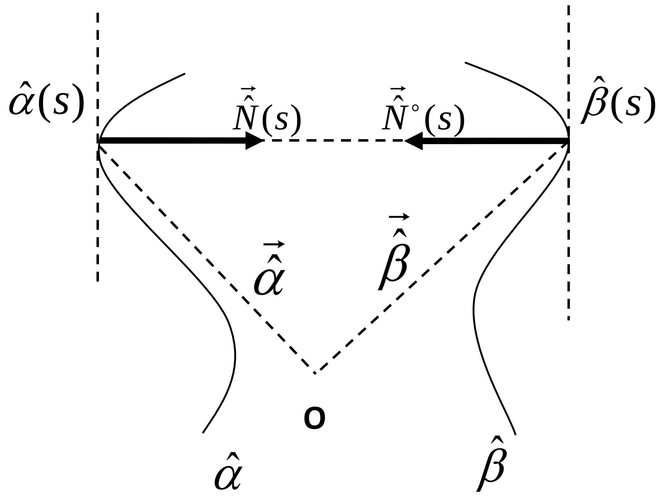

Figure 1. Bertrand pair curves.

$$\vec{\hat{\beta}}(s) = \vec{\hat{\alpha}}(s) + \hat{\lambda}(s)\vec{\hat{N}}(s) \tag{1}$$

for the vectors of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$

By differentiating the Equation 1 with respect to s and applying the Frenet formulas,

$$\frac{d\hat{s}}{ds}.\vec{\hat{T}^{\circ}}(s) = \left(1-\hat{\lambda}(s)\hat{\kappa}(s)\right).\vec{\hat{T}}(s) + \hat{\lambda}'(s).\vec{\hat{N}}(s) + \hat{\lambda}(s).\hat{\tau}(s).\vec{\hat{B}}(s). \tag{2}$$

is obtained.

If we take the inner product of the Equation 2 with $\vec{\hat{N}}(s)$ both sides,

$$0 = \hat{\lambda}'(s)$$

is found.Thus, we notice that

$$\hat{\lambda}(s) = \hat{c}$$

where $\hat{c} \in \mathbb{D}$ (constant). If we use

$$d\left(\vec{\hat{\alpha}}(s), \vec{\hat{\beta}}(s)\right) = \left\|\vec{\hat{\beta}}(s) - \vec{\hat{\alpha}}(s)\right\|$$

and the Equation 1 we get

$$d\left(\vec{\hat{\alpha}}(s), \vec{\hat{\beta}}(s)\right) = \hat{c}$$

where $s \in I \subset \mathrm{D}$ and $\hat{c} \in \mathbb{D}$ (constant).

Theorem 2

Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two curves in D^3. If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ are Bertrand curves, then the dual angle between the tangent vectors at the corresponding points of the dual Bertrand curves is constant.

Proof: Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two Bertrand curves in D^3.

If the dual angle between the tangent vectors $\vec{\hat{T}}(s)$ and $\vec{\hat{T}^{\circ}}(s)$ at the corresponding points of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ is

$$\phi = \varphi + \varepsilon\varphi^{*} \in \mathrm{D},$$

then we write

$$\vec{\hat{T}^{\circ}}(s) = \cos\phi\,\vec{\hat{T}}(s) + \sin\phi\,\vec{\hat{B}}(s). \tag{3}$$

If we differentiate the last equation of the above, we obtain

$$\hat{\kappa}^{\circ}(s)\vec{\hat{N}^{\circ}}(s)\frac{d\hat{s}}{ds} = \frac{d\cos\phi}{ds}\vec{\hat{T}}(s) + \left(\hat{\kappa}(s)\cos\phi - \hat{\tau}(s)\sin\phi\right)\vec{\hat{N}}(s) + \frac{d\sin\phi}{ds}\vec{\hat{B}}(s). \tag{4}$$

If we take the inner product of the last equation of the above with $\vec{\hat{T}}(s)$ both sides and we use the Frenet formulas, we have

$$\frac{d\cos\phi}{ds} = 0.$$

So

$$\cos\phi = \text{constant}$$

is found where $\phi = \varphi + \varepsilon\varphi^{*} \in \mathrm{D}$.

This completes the proof.

Conclusion 1

If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be dual Bertrand curves, then the distance between the corresponding points of them and the dual angle between the tangent vectors are constant.

Theorem 3

Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be two curves in D^3. If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ are Bertrand curves, $\hat{\kappa}(s)$ and $\hat{\tau}(s)$ are the curvature and

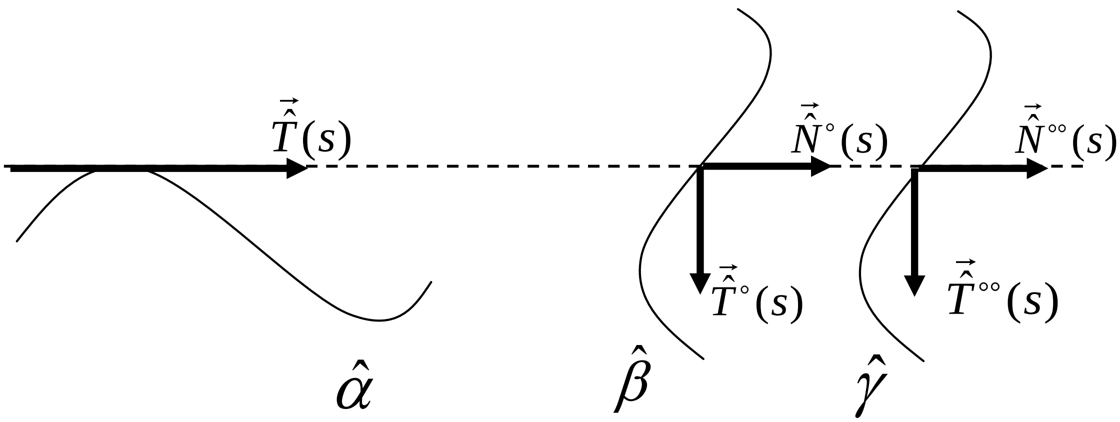

Figure 2. Involute curves of α .

torsion of $\vec{\hat{\alpha}}$, $\widehat{\kappa^{\circ}}(s)$ and $\widehat{\tau^{\circ}}(s)$ are the curvature and torsion of $\vec{\hat{\beta}}$, respectively, then

$$\hat{\lambda}.\hat{\kappa}(s)+\hat{\mu}.\hat{\tau}(s)=1 \tag{5}$$

where $\hat{\lambda},\hat{\mu}\in\mathbb{D}$ are constant.

Proof: Let $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ are Bertrand curves.

If the dual angle between the tangent vectors $\vec{\hat{T}}(s)$ and $\vec{\hat{T}^{\circ}}(s)$ at the corresponding points of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ is

$\phi=\varphi+\varepsilon\varphi^{*}\in$ D,

then from the previous proof we have

$$\vec{\hat{T}^{\circ}}(s)=\cos\phi\,\vec{\hat{T}}(s)+\sin\phi\,\vec{\hat{B}}(s) \tag{6}$$

From the Equation (2) we write

$$\frac{d\hat{s}}{ds}.\vec{\hat{T}^{\circ}}(s)=\left(1-\hat{\lambda}.\hat{\kappa}(s)\right).\vec{\hat{T}}(s)+\hat{\lambda}.\hat{\tau}(s).\vec{\hat{B}}(s). \tag{7}$$

In above equations, if we take into account $\frac{d\hat{s}}{ds}=\hat{a}$ (constant)

then we get

$$1-\lambda.\kappa(s)=\cot\phi.\lambda.\hat{\tau}(s). \tag{8}$$

We can write

$$\mu=\lambda.\cot\phi \text{ (constant)}. \tag{9}$$

Finally, from the Equations 8 and 9, we find

$$\hat{\lambda}.\hat{\kappa}(s)+\hat{\mu}.\hat{\tau}(s)=1.$$

Conclusion 2

If $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$ be dual Bertrand curves, then the characterization of being Bertrand curve in dual space is same with Euclid space.

Theorem 4

Let $\vec{\hat{\alpha}}$ be a plane curve in D^3. If $\vec{\hat{\beta}}$ and $\vec{\hat{\gamma}}$ are the involutes of $\vec{\hat{\alpha}}$, then $\vec{\hat{\beta}}$ and $\vec{\hat{\gamma}}$ are Bertrand curves in D^3 .

Proof: Let $\vec{\hat{\beta}}$ and $\vec{\hat{\gamma}}$ be the involutes of $\vec{\hat{\alpha}}$ as in Figure 2.

It can be written from Caliskan et al. (in press)

$$\widehat{\tau^{\circ}}(s)=\frac{\hat{\kappa}(s).\hat{\tau}'(s)-\hat{\kappa}'(s).\hat{\tau}(s)}{\hat{\kappa}(s).(\hat{c}-s).\left(\hat{\kappa}^{2}(s)+\hat{\tau}^{2}(s)\right)} \tag{10}$$

where $\hat{\kappa}(s)$, $\hat{\tau}(s)$ and $\widehat{\kappa^{\circ}}(s)$, $\widehat{\tau^{\circ}}(s)$ are the curvature and torsion of $\vec{\hat{\alpha}}$ and $\vec{\hat{\beta}}$, respectively, and $\hat{c}\in\mathbb{D}$ (constant) .

If $\vec{\hat{\alpha}}$ is a plane curve, then

$$\hat{\tau}(s)=0. \tag{11}$$

From the last two equation given above, we have

$$\widehat{\tau}^{\circ}(s) = 0.$$

So $\overrightarrow{\widehat{\beta}}$ is a plane curve. Similarly, $\overrightarrow{\widehat{\gamma}}$ is a plane curve. Consequently, the principal normal vectors of $\overrightarrow{\widehat{\beta}}$ and $\overrightarrow{\widehat{\gamma}}$ are linearly dependent to each other at the corresponding points of the dual curves. In that case $\overrightarrow{\widehat{\beta}}$ and $\overrightarrow{\widehat{\gamma}}$ curves are Bertrand curves in D^3.

Conflict of Interest

The author(s) have not declared any conflict of interests.

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