academicJournals

Vol. 8(37), pp. 1836-1856, 9 October, 2013
DOI: 10.5897/IJPS11.1611
ISSN 1992 - 1950 © 2013 Academic Journals
http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Solitary wave solutions of fifth-order (1+1)-dimensional Caudrey-Dodd-Gibbon equation

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Accepted 26 September, 2013

The manuscript deals with the abundant travelling wave solutions of the Caudrey-Dodd-Gibbon (CDG) equation which have been obtained in a uniform way by using alternative (G'/G)–expansion method wherein the generalized Riccati equation is used. Moreover, a relatively new technique which is called ($\grave{U}/U$)-expansion is also used to find solitary wave solutions of CDG equation. The solutions obtained in this article may be imperative and significant for the explanation of some practical physical phenomena. Numerical results coupled with the graphical representation explicitly reveal the complete reliability and high efficiency of the proposed algorithms.

Key words: (G'/G)-expansion method, travelling wave solutions, ($\grave{U}/U$)-expansion method, Caudrey-Dodd-Gibbon (CDG) equation, nonlinear evolution equations.

INTRODUCTION

The rapid development of nonlinear sciences witnesses a wide range of reliable and efficient techniques which are of great help in tackling physical problems even of highly complex nature. After the observation of solitonary phenomena by John Scott Russell (Wazwaz, 2009) in 1834 and since the KdV equation was solved by Gardner et al. (1967) by the inverse scattering method, finding exact solutions of nonlinear evolution equations (NLEEs) has turned out to be one of the most exciting and particularly active areas of research. The appearance of solitary wave solutions in nature is quite common. Bell-shaped Sech-solutions and kink-shaped Tanh-solutions model wave phenomena in elastic media, plasmas, solid state physics, condensed matter physics, electrical circuits, optical fibers, chemical kinematics, fluids, bio-genetics etc. The travelling wave solutions of the KdV equation and the Boussinesq equation which describe water waves are well-known examples. Apart from their physical relevance, the closed-form solutions of NLEEs if available facilitate the numerical solvers in comparison, and aids in the stability analysis. In soliton theory, there are many methods and techniques to deal with the problem of solitary wave solutions for NLEEs, such as, Backlund transformation (Rogers and Shadwick, 1982), Hirota's bilinear transformation (Hirota, 1971), Variational Iteration (Mohyud-Din, 2008), homogeneous balance (Wang, 1996), Tanh-function (Malfliet, 1992), Jacobi elliptic function (Ali, 2011), F-expansion (Zhou et al., 2003),

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Mathematical Subject Classification: 35K99, 35P05, 35P99.

Homotopy Analysis (Liao, 1992a, b), Homotopy Perturbation (Mohyud-Din, 2007), Adomian's Decomposition (Adomian, 1994), First Integration (Taghizadeh and Mirzazadeh, 2011), Exp-function (He and Wu, 2006; Abdou et al., 2007; Akbar and Ali, 2011b; Mohyud-Din et al., 2010; Naher et al., 2011b), and others (Abbasbandy, 2007a, b; Mohyud-Din et al., 2009, 2011a, b; Usman et al., 2011).

In the similar context, Wang et al. (2008) established a widely used direct and concise technique which is called the (G'/G)-expansion method for obtaining the exact travelling wave solutions of NLEEs, where $G(\xi)$ satisfies the second order linear ordinary differential equation (ODE) $G''+\lambda G'+\mu G=0$, where λ and μ are arbitrary constants. In the articles, Abazari (2010), Akbar and Ali (2011a), Bekir (2008), Liu et al. (2010), Naher et al. (2011a), Zayed (2009a), Zayed and Gepreel (2009), Zhang et al. (2008a, b), Zayed and Al-Joudi (2010), the (G'/G)-expansion method is applied to investigate solutions of nonlinear partial differential equations in mathematical physics. It is to be highlighted that Zhang et al. (2010) presented an improved (G'/G)-expansion method to seek more general travelling wave solutions. Zayed (2009b) presented a new approach of the (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2=e_2G^4(\xi)+e_1G^2(\xi)+e_0$, e_2,e_1,e_0 are arbitrary constants, and obtained new exact solutions. Zayed (2011) again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi)=A+BG^2(\xi)$, where A and B are arbitrary constants. Inspired and motivated by the ongoing research in this area, we investigate ample new travelling wave solutions of the CDG equation in a uniform way by making use of the alternative (G'/G)–expansion method wherein the generalized Riccati equation is functioned. Moreover, we have also applied a relatively new technique namely (U`/U)-expansion Method to tackle the CDG equation. Numerical results coupled with the graphical representations explicitly reveal the complete reliability and high efficiency of the proposed algorithms.

METHODOLOGY

Suppose the general nonlinear partial differential equation

$$F(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\cdots)=0 \qquad (1)$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the alternative (G'/G)-expansion method combined with the generalized Riccati equation are as follows:

Step 1: The travelling wave variable

$$u(x,t)=u(\xi),\ \xi=x-Vt \qquad (2)$$

where V is the speed of the travelling wave, and permits us to transform the Equation (1) into an ODE:

$$Q(u,u',u'',\cdots)=0 \qquad (3)$$

where the superscripts stands for the ordinary derivatives with respect to ξ.

Step 2: If Equation (3) is integrable, integrate term by term one or more times, yields constant(s) of integration.

Alternative (G`/G)-expansion method with generalized Riccati equation

Step 3: Suppose the traveling wave solution of Equation (3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi)=\sum_{n=0}^{m}a_n\left(\frac{G'}{G}\right)^n,\quad a_m\neq 0 \qquad (4)$$

where $G=G(\xi)$ satisfies the generalized Riccati equation,

$$G'=r+pG+qG^2, \qquad (5)$$

where $a_n\ (n=0,1,2,\cdots,m)$, r, p and q are arbitrary constants to be determined later.

The generalized Riccati Equation (5) has the following twenty seven solutions (Zhu, 2008).

Family 1: When $p^2-4qr<0$ and $pq\neq 0$ (or $rq\neq 0$), the solutions of Equation (5) are:

$$G_1=\frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\tan\left(\frac{1}{2}\sqrt{4qr-p^2}\,\xi\right)\right],$$

$$G_2=-\frac{1}{2q}\left[p+\sqrt{4qr-p^2}\cot\left(\frac{1}{2}\sqrt{4qr-p^2}\,\xi\right)\right],$$

$$G_3=\frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\left(\tan\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sec\left(\sqrt{4qr-p^2}\,\xi\right)\right)\right],$$

$$G_4 = -\frac{1}{2q}\left[p + \sqrt{4qr - p^2}\left(\cot\left(\sqrt{4qr - p^2}\xi\right) \pm \csc\left(\sqrt{4qr - p^2}\xi\right)\right)\right],$$

$$G_5 = \frac{1}{4q}\left[-2p + \sqrt{4qr - p^2}\left(\tan\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right) - \cot\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right)\right)\right],$$

$$G_6 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2}\cos\left(\sqrt{4qr - p^2}\xi\right)}{A\sin\left(\sqrt{4qr - p^2}\xi\right) + B}\right],$$

$$G_7 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2}\cos\left(\sqrt{4qr - p^2}\xi\right)}{A\sin\left(\sqrt{4qr - p^2}\xi\right) + B}\right],$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 > 0$.

$$G_8 = \frac{-2r\cos\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)}{\sqrt{4qr - p^2}\sin\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right) + p\cos\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)},$$

$$G_9 = \frac{2r\sin\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)}{-p\sin\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right) + \sqrt{(4qr - p^2)}\cos\left(\frac{1}{2}\sqrt{4qr - p^2}\xi\right)},$$

$$G_{10} = \frac{-2r\cos\left(\sqrt{4qr - p^2}\xi\right)}{\sqrt{(4qr - p^2)}\sin\left(\sqrt{4qr - p^2}\xi\right) + p\cos\left(\sqrt{4qr - p^2}\xi\right) \pm \sqrt{(4qr - p^2)}},$$

$$G_{11} = \frac{2r\sin\left(\sqrt{4qr - p^2}\xi\right)}{-p\sin\left(\sqrt{4qr - p^2}\xi\right) + \sqrt{(4qr - p^2)}\cos\left(\sqrt{4qr - p^2}\xi\right) \pm \sqrt{(4qr - p^2)}},$$

$$G_{12} = \frac{4r\sin\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right)}{-2p\sin\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right) + 2\sqrt{(4qr - p^2)}\cos^2\left(\frac{1}{4}\sqrt{4qr - p^2}\xi\right) - \sqrt{(4qr - p^2)}}.$$

Family 2: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $rq \neq 0$), the solutions of Equation (5) are:

$$G_{13} = -\frac{1}{2q}\left[p + \sqrt{p^2 - 4qr}\tanh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right)\right],$$

$$G_{14}=-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)\right],$$

$$G_{15}=-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{16}=-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}\xi\right)\pm \mathrm{csch}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{17}=-\frac{1}{4q}\left[2p+\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+\coth\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{18}=\frac{1}{2q}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

$$G_{19}=\frac{1}{2q}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4qr)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

where A and B are two non-zero real constants and satisfies the condition $B^2-A^2>0$.

$$G_{20}=\frac{2r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{21}=\frac{2r\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{22}=\frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}},$$

$$G_{23}=\frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}},$$

$$G_{24}=\frac{4r\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)}{-2p\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+2\sqrt{p^2-4qr}\cosh^2\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)-\sqrt{p^2-4qr}}.$$

Family 3: When $r=0$ and $pq \neq 0$, the solutions of Equation (5) are:

$$G_{25} = \frac{-p\,d}{q\left[d+\cosh(p\xi)-\sinh(p\xi)\right]},$$

$$G_{26} = -\frac{p\left[\cosh(p\xi)+\sinh(p\xi)\right]}{q\left[d+\cosh(p\xi)+\sinh(p\xi)\right]},$$

where d is an arbitrary constant.

Family 4: When $q \neq 0$ and $r=p=0$, the solution of Equation (5) is:

$$G_{27} = -\frac{1}{q\xi + c_1},$$

where c_1 is an arbitrary constant.

Step 4: To determine the positive integer m, substitute solution Equation (4) along with Equation (5) into Equation (3) and then consider homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equation (3).

Step 5: Substituting Equation (4) together with Equation (5) into Equation (3) together with the value of m obtained in step 3, we obtain polynomials in G^i and $G^{-i}\ (i=0,1,2,3\cdots)$ and vanishing each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for a_n p, q, r and V.

Step 6: Suppose the value of the constants a_n p, q, r and V can be obtained by solving the set of algebraic equations obtained in step 5. Since the general solutions of Equation (5) are known for us, substituting, a_n p, q, r and V together with the general solution of Equation (5) into Equation (4), we obtain new exact traveling wave solutions of the nonlinear evolution Equation (1).

New approach of (*G`*/*G*)-expansion method

Step 3: According to new approach of (*G`*/*G*)-expansion method, we assume that the wave solution can be expressed in the following form

$$u(\xi) = a_0 + \sum_{n=1}^{M} a_n \left(\frac{G(\xi)}{G(\xi)}\right)^M, \qquad (6)$$

where $G(\xi)$ is the solution of first order nonlinear equation in the form

$$G(\xi)G''(\xi) - \delta_1 G^2(\xi) + \delta_2 (G'(\xi))^2 = 0, \qquad (7)$$

$$\left(\frac{G(\xi)}{G(\xi)}\right)' = \delta_1 - \delta_2\left(\frac{G(\xi)}{G(\xi)}\right)^2 - \left(\frac{G(\xi)}{G(\xi)}\right)^2,$$

$$\left(\frac{G(\xi)}{G(\xi)}\right)'' = -2\delta_2\delta_1\left(\frac{G(\xi)}{G(\xi)}\right)' + 2\delta_2^2\left(\frac{G(\xi)}{G(\xi)}\right)^3 + 4\delta_2\left(\frac{G(\xi)}{G(\xi)}\right)^3 - 2\delta_1\left(\frac{G(\xi)}{G(\xi)}\right) + 2\left(\frac{G(\xi)}{G(\xi)}\right)^3,$$

where δ_1 and δ_2 are real constants. The Riccati Equation (5) plays important role in manipulating nonlinear equations to get exact solutions by the (*G′*/*G*)-expansion method. It has the following type of exact solutions.

Family 1: When $\delta_1, \delta_2 \neq 0$,

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}.$$

Family 2: When $\delta_1 < 0$, and $(1+\delta_2) > 0$, or $\delta_1 > 0$, and $(1+\delta_2) < 0$

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)-\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{-\delta_1}+\sqrt{-\delta_1}}{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)+\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}.$$

Family 3: When $\delta_1 \neq 0$, and $\delta_2 = 0$,

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]-1}.$$

Family 4: When $\delta_1 = 0$, and $\delta_2 \neq 0$,

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}\frac{1}{1+\delta_2}.$$

Family 5: When $\delta_1 = 0$, and $\delta_2 = 0$.

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}.$$

Step 4: Determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Equation (4).

Step 5: Substituting Equation (6) into Equation (4) with (7) will yields an algebraic equation involving power of (*G′*/*G*). Equating the coefficients of like power of (*G′*/*G*) to zero gives a system of algebraic equations for a_i, k, l, m and ω. Then, we solve the system with the aid of a computer algebra system (CAS), such as MAPLE 13, to determine these constants.

Step 6: Putting these constant into Equation (6), coupled with the well known solutions of Equation (7), we can obtained the more general type and new exact travelling wave solution of the nonlinear partial differential Equation (1).

(*U`*/*U*)-expansion method

Step 3: According to (*U`*/*U*)-expansion method, we assume that the wave solution can be expressed in the following form

$$u(\xi) = \sum_{n=0}^{M} a_n \left(\frac{U^{\alpha}}{U}\right)^n, \tag{8}$$

where U is the solution of first order nonlinear equation in the form

$$U' = AU + B, \tag{9}$$

where A and B are real constants, M is a positive integer to be determined and the Equation (9) has solution

$$\frac{U'(\xi)}{U(\xi)} = \left(\frac{Ac\exp[A\xi]}{-\frac{B}{A}+c\exp[A\xi]}\right).$$

Step 4: Determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Equation (4).

Step 5: Substituting (9) into ODE with (8) yields an algebraic equation involving power of *U*. Equating the coefficients of like power of *U* to zero gives a system of algebraic equations for a_i, k, l, m and ω. Then, we solve the system with the aid of a computer algebra system (CAS), such as MAPLE 13, to determine these constants.

Step 6: Putting these constant into Equation (8), coupled with the well known solutions of Equation (9), we obtained the more general type and new exact travelling wave solution of the nonlinear partial differential Equation (1).

New travelling wave solutions of Cuadrey-Dodd-Gibbon (CDG) equation

Here, the alternative (G'/G)-expansion method together with the generalized Riccati equation is employed to construct some new travelling wave solutions for the (1+1)-dimensional Cuadrey-Dodd-Gibbon (CDG) equation which is a very important nonlinear evolution equation in mathematical physics and engineering and have been paid attention by many researchers. Some exact solutions of the CDG equation are found in the literature. In general, the solutions of the CDG equation have been obtained by means of an Ansatz method. Included in the methods are the sin-cosine method and the rational Exp-function method (Abdollahzadeh et al., 2010), the Hirota’s bilinear transformation method (Jiang and Bi, 2010), the Exp-function method (Xu, 2008), the variational iteration method (Jin, 2010), the multi-wave method (Shi et al., 2010), and the variable separation method (Zheng, 2010). In this paper, we apply the alternative (G'/G)-expansion method together with generalized Riccati equation for searching its solitary wave solutions. Let us consider the CDG equation:

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0 \tag{10}$$

NUMERICAL RESULTS AND DISCUSSION

Alternative (*G`*/*G*)-expansion method using generalized Riccati equation

Now, we use the wave transformation equation into Equation (10), which yield:

$$-Vu' + u^{(5)} + 30uu''' + 30u'u'' + 180u^2u' = 0, \tag{11}$$

where $u^{(5)}$ denotes the fifth derivative of u with respect to ξ. Equation (11) is integrable, therefore, integrating we obtain

$$C - Vu + u^{(4)} + 30uu'' + 60u^3 = 0 \tag{12}$$

According to step 3, the solution of Equation (12) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2 + \cdots + a_m(\frac{G'}{G})^m,\ a_m \neq 0 \tag{13}$$

where $a_n, (n = 0,1,2,\cdots,m)$ are constants to be determined and $G = G(\xi)$ satisfies the generalized Riccati Equation (10). Considering the homogeneous balance between the highest order derivative and the nonlinear terms in Equation (12), we obtain $m = 2$.

Therefore, the solution Equation (13) takes the form,

$$u(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2,\ a_2 \neq 0 \tag{14}$$

Using Equation (5), Equation (14) can be rewritten as,

$$u(\xi) = a_0 + a_1(p + rG^{-1} + qG) + a_2(p + rG^{-1} + qG)^2 \tag{15}$$

Substituting Equation (15) into (12), the left hand side is converted into polynomials in G^{i} and $G^{-i},(i=0,1,2,3,\cdots)$. Setting each coefficient of these resulted polynomials to zero, we obtain a set of simultaneous algebraic equations (we will omit to display them for simplicity) for a_0,a_1,a_2, p, q, r and V.

Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple, we obtain

$$C=\frac{p^6}{9}-\frac{4}{3}p^4 r q+\frac{16}{3}p^2q^2r^2-\frac{64}{9}q^3r^3,\ V=p^4-8p^2q\mathrm{r}+16q^2r^2,\ a_2=-1,\ a_1=p,\ a_0=-\frac{p^2}{6}+\frac{2}{3}qr, \tag{16}$$

where p, q and r are arbitrary constants.

Now on the basis of the solutions of Equation (5), we obtain the following families of solutions of Equation (10).

Family 1: When $p^2-4qr<0$ and $pq\neq 0$ (or $rq\neq 0$), the periodic form solutions of Equation (10) are,

$$u_1=-\frac{p^2}{6}+\frac{2}{3}qr+p\left(\frac{2\Delta^2\sec^2(\Delta\xi)}{-p+2\Delta\tan(\Delta\xi)}\right)-\left(\frac{2\Delta^2\sec^2(\Delta\xi)}{-p+2\Delta\tan(\Delta\xi)}\right)^2,$$

where $\Delta=\frac{1}{2}\sqrt{4qr-p^2}$, $\xi=x-(p^4-8p^2q\mathrm{r}+16q^2r^2)t$ and p, q, r are arbitrary constants.

$$u_2=-\frac{p^2}{6}+\frac{2}{3}qr-p\left(\frac{2\Delta^2\csc^2(\Delta\xi)}{p+2\Delta\cot(\Delta\xi)}\right)-\left(\frac{2\Delta^2\csc^2(\Delta\xi)}{p+2\Delta\cot(\Delta\xi)}\right)^2,$$

$$u_3=-\frac{p^2}{6}+\frac{2}{3}qr+p\left(\frac{4\Delta^2\sec(2\Delta\xi)\left(1\pm\sin(2\Delta\xi)\right)}{-p\cos(2\Delta\xi)+2\Delta\sin(2\Delta\xi)\pm 2\Delta}\right)-\left(\frac{4\Delta^2\sec(2\Delta\xi)\left(1\pm\sin(2\Delta\xi)\right)}{-p\cos(2\Delta\xi)+2\Delta\sin(2\Delta\xi)\pm 2\Delta}\right)^2,$$

$$u_4=-\frac{p^2}{6}+\frac{2}{3}qr-p\left(\frac{4\Delta^2\csc(2\Delta\xi)\left(1\pm\cos(2\Delta\xi)\right)}{p\sin(2\Delta\xi)+2\Delta\cos(2\Delta\xi)\pm 2\Delta}\right)-\left(\frac{4\Delta^2\csc(2\Delta\xi)\left(1\pm\cos(2\Delta\xi)\right)}{p\sin(2\Delta\xi)+2\Delta\cos(2\Delta\xi)\pm 2\Delta}\right)^2,$$

$$u_5=-\frac{p^2}{6}+\frac{2}{3}qr-p\left(\frac{2\Delta^2\csc(\Delta\xi)}{p\sin(\Delta\xi)+2\Delta\cos(\Delta\xi)}\right)-\left(\frac{2\Delta^2\csc(\Delta\xi)}{p\sin(\Delta\xi)+2\Delta\cos(\Delta\xi)}\right)^2,$$

$$u_6=-p\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)-B\sin(2\Delta\xi)-A\right\}\left\{A\sin(2\Delta\xi)+B\right\}}{\left\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\right\}\left\{pA\sin(2\Delta\xi)+2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)$$
$$-\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)-B\sin(2\Delta\xi)-A\right\}\left\{A\sin(2\Delta\xi)+B\right\}}{\left\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\right\}\left\{pA\sin(2\Delta\xi)+2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\frac{p^2}{6}+\frac{2}{3}qr,$$

$$u_7=-p\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)+B\sin(2\Delta\xi)+A\right\}\left\{A\sin(2\Delta\xi)+B\right\}}{\left\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\right\}\left\{pA\sin(2\Delta\xi)-2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)$$
$$-\left(\frac{4A\Delta^2\left\{\sqrt{A^2-B^2}\cos(2\Delta\xi)+B\sin(2\Delta\xi)+A\right\}\left\{A\sin(2\Delta\xi)+B\right\}}{\left\{A^2\cos^2(2\Delta\xi)-A^2-B^2-2AB\sin(2\Delta\xi)\right\}\left\{pA\sin(2\Delta\xi)-2A\Delta\cos(2\Delta\xi)+pB-2\Delta\sqrt{A^2-B^2}\right\}}\right)^2$$
$$-\frac{p^2}{6}+\frac{2}{3}qr,$$

where A and B are two non-zero real constants satisfies the condition $A^2 - B^2 > 0$.

$$u_8 = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{2\Delta^2 \sec(\Delta\xi)\{p\cos(\Delta\xi) + 2\Delta\sin(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) + 4\Delta^2}\right)$$
$$-\left(\frac{2\Delta^2 \sec(\Delta\xi)\{p\cos(\Delta\xi) + 2\Delta\sin(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) + 4\Delta^2}\right)^2,$$

$$u_9 = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)$$
$$-\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)^2,$$

$$u_{10} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{2\Delta^2 \sec(2\Delta\xi)\{1 \pm \sin(2\Delta\xi)\}\{p\cos(2\Delta\xi) + 2\Delta\sin(2\Delta\xi) \pm 2\Delta\}}{(p^2 - 2rq)\cos^2(2\Delta\xi) + 2\Delta\{1 \pm \sin(2\Delta\xi)\}\{2\Delta \pm p\cos(2\Delta\xi)\}}\right)$$
$$-\left(\frac{2\Delta^2 \sec(2\Delta\xi)\{1 \pm \sin(2\Delta\xi)\}\{p\cos(2\Delta\xi) + 2\Delta\sin(2\Delta\xi) \pm 2\Delta\}}{(p^2 - 2rq)\cos^2(2\Delta\xi) + 2\Delta\{1 \pm \sin(2\Delta\xi)\}\{2\Delta \pm p\cos(2\Delta\xi)\}}\right)^2,$$

$$u_{11} = -\frac{p^2}{6} + \frac{2}{3}qr \pm p\left(\frac{2\Delta^2 \csc(2\Delta\xi)\{-p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta\}}{(2rq - p^2)\cos(2\Delta\xi) - 2p\Delta\sin(2\Delta\xi) \pm 2qr}\right)$$
$$-\left(\frac{2\Delta^2 \csc(2\Delta\xi)\{-p\sin(2\Delta\xi) + 2\Delta\cos(2\Delta\xi) \pm 2\Delta\}}{(2rq - p^2)\cos(2\Delta\xi) - 2p\Delta\sin(2\Delta\xi) \pm 2qr}\right)^2,$$

$$u_{12} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)$$
$$-\left(\frac{2\Delta^2 \csc(\Delta\xi)\{p\sin(\Delta\xi) - 2\Delta\cos(\Delta\xi)\}}{2(p^2 - 2rq)\cos^2(\Delta\xi) + 4\Delta\, p\sin(\Delta\xi)\cos(\Delta\xi) - p^2}\right)^2.$$

Family 2: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $rq \neq 0$), the soliton and soliton-like solutions of Equation (10) are,

$$u_{13} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{2\Omega^2 \operatorname{sec}h^2(\Omega\xi)}{p + 2\Omega\tanh(\Omega\xi)}\right) - \left(\frac{2\Omega^2 \operatorname{sec}h^2(\Omega\xi)}{p + 2\Omega\tanh(\Omega\xi)}\right)^2,$$

where $\Omega = \frac{1}{2}\sqrt{p^2 - 4qr}$, $\xi = x - (p^4 - 8p^2 q\mathrm{r} + 16q^2 r^2)t$ and p, q, r are arbitrary constants.

$$u_{14} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{2\Omega^2 \operatorname{csc}h^2(\Omega\xi)}{p + 2\Delta\coth(\Omega\xi)}\right) - \left(\frac{2\Omega^2 \operatorname{csc}h^2(\Omega\xi)}{p + 2\Delta\coth(\Omega\xi)}\right)^2,$$

$$u_{15} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{4\Omega^2 \operatorname{sec}h(2\Omega\xi)\left(1 \mp i\sinh(2\Omega\xi)\right)}{p\cosh(2\Omega\xi) + 2\Delta\sinh(2\Omega\xi) \pm i2\Omega}\right)$$
$$-\left(\frac{4\Omega^2 \operatorname{sec}h(2\Omega\xi)\left(1 \mp i\sinh(2\Omega\xi)\right)}{p\cosh(2\Omega\xi) + 2\Delta\sinh(2\Omega\xi) \pm i2\Omega}\right)^2,$$

$$u_{16} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{4\Omega^2 \csc h(2\Omega\xi)\left(1 \pm \cosh(2\Omega\xi)\right)}{p\sinh(2\Omega\xi) + 2\Omega\cosh(2\Delta\xi) \pm 2\Omega}\right) - \left(\frac{4\Omega^2 \csc h(2\Omega\xi)\left(1 \pm \cosh(2\Omega\xi)\right)}{p\sinh(2\Omega\xi) + 2\Omega\cosh(2\Delta\xi) \pm 2\Omega}\right)^2,$$

$$u_{17} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{\Omega^2 \sec h^2(\Omega\xi/2)}{2\left\{\cosh^2(\Omega\xi/2) - 1\right\}\left\{p + \Omega\left(\tanh(\Omega\xi/2) + \coth(\Omega\xi/2)\right)\right\}}\right) - \left(\frac{\Omega^2 \sec h^2(\Omega\xi/2)}{2\left\{\cosh^2(\Omega\xi/2) - 1\right\}\left\{p + \Omega\left(\tanh(\Omega\xi/2) + \coth(\Omega\xi/2)\right)\right\}}\right)^2,$$

$$u_{18} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) - \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{\left(A\sin(2\Omega\xi) + B\right)\left(pA\sinh(2\Omega\xi) + pB - 2\Omega\sqrt{A^2+B^2} + 2A\Omega\cosh(2\Omega\xi)\right)}\right) - \left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) - \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{\left(A\sin(2\Omega\xi) + B\right)\left(pA\sinh(2\Omega\xi) + pB - 2\Omega\sqrt{A^2+B^2} + 2A\Omega\cosh(2\Omega\xi)\right)}\right)^2,$$

$$u_{19} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) + \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{\left(A\sin(2\Omega\xi) + B\right)\left(pA\sinh(2\Omega\xi) + pB + 2\Omega\sqrt{A^2+B^2} + 2A\Omega\cosh(2\Omega\xi)\right)}\right) - \left(\frac{4A\Omega^2\left(A - B\sinh(2\Omega\xi) + \sqrt{A^2+B^2}\cosh(2\Omega\xi)\right)}{\left(A\sin(2\Omega\xi) + B\right)\left(pA\sinh(2\Omega\xi) + pB + 2\Omega\sqrt{A^2+B^2} + 2A\Omega\cosh(2\Omega\xi)\right)}\right)^2,$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 < 0$.

$$u_{20} = -\frac{p^2}{6} + \frac{2}{3}qr - p\left(\frac{2\Omega^2 \sec h(\Omega\xi)}{2\Omega\sinh(\Omega\xi) - p\cosh(\Omega\xi)}\right) - \left(\frac{2\Omega^2 \sec h(\Omega\xi)}{2\Omega\sinh(\Omega\xi) - p\cosh(\Omega\xi)}\right)^2,$$

$$u_{21} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{2\Omega^2 \csc h(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right) - \left(\frac{2\Omega^2 \csc h(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right)^2,$$

$$u_{22} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{4\Omega^2 \sec h(2\Omega\xi)(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) - 2\Omega\sinh(2\Omega\xi) \mp i2\Omega}\right) - \left(\frac{4\Omega^2 \sec h(2\Omega\xi)(1 \mp i\sinh(2\Omega\xi))}{p\cosh(2\Omega\xi) - 2\Omega\sinh(2\Omega\xi) \mp i2\Omega}\right)^2,$$

$$u_{23} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{4\Omega^2 \csc h(2\Omega\xi)(1 \pm \cosh(2\Omega\xi))}{2\Omega\cosh(2\Omega\xi) - p\sinh(2\Omega\xi) \pm 2\Omega}\right) - \left(\frac{4\Omega^2 \csc h(2\Omega\xi)(1 \pm \cosh(2\Omega\xi))}{2\Omega\cosh(2\Omega\xi) - p\sinh(2\Omega\xi) \pm 2\Omega}\right)^2,$$

$$u_{24} = -\frac{p^2}{6} + \frac{2}{3}qr + p\left(\frac{2\Omega^2 \csc h(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right) - \left(\frac{2\Omega^2 \csc h(\Omega\xi)}{2\Omega\cosh(\Omega\xi) - p\sinh(\Omega\xi)}\right)^2.$$

Family 3: When $r=0$ and $pq\neq 0$, the solutions of Equation (10) are,

$$u_{25}=-\frac{p^2}{6}+\frac{2}{3}qr+p\left(\frac{p(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-\sinh(p\xi)}\right)-\left(\frac{p(\cosh(p\xi)-\sinh(p\xi))}{d+\cosh(p\xi)-\sinh(p\xi)}\right)^2,$$

$$u_{26}=-\frac{p^2}{6}+\frac{2}{3}qr+p\left(\frac{pd}{d+\cosh(p\xi)+\sinh(p\xi)}\right)-\left(\frac{pd}{d+\cosh(p\xi)+\sinh(p\xi)}\right)^2.$$

Family 4: When $q\neq 0$ and $r=p=0$, the solutions of Equation (10) are,

$$u_{27}=-\frac{p^2}{6}+\frac{2}{3}qr-p\left(\frac{q}{q\xi+c_1}\right)-\left(\frac{q}{q\xi+c_1}\right)^2,$$

where c_1 is an arbitrary constant.

Because of the arbitrariness of the parameters p, q and r in the above families of solution, the physical quantities u and v may possess rich structures.

Graph is a powerful tool for communication and describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given below (Graph 1a to h). The graphs readily have shown the solitary wave form of the solutions.

New approach of (*G`/G*)-expansion method

To convert Equation (10) into ODE we used the following transformation

$$u(x,t)=u(\xi),\ \xi=kx+\omega t, \qquad (17)$$

where k and ω are arbitrary constant. Substituting Equation (17) into (10) and using the chain rule and $\xi_x=k, \xi_t=\omega$, we obtain

$$\omega u'+30k^3uu'''+30k^3u'u''+180ku^2u'+k^5u'''''=0. \qquad (18)$$

Integrating the above equation once, ignoring the constant of integration equal to zero we have the following equation

$$\omega u+60ku^3+30k^3uu''+k^5u''''=0.$$

For *m* = 2, we obtained the trail solution

$$u=a_0+a_1\left(\frac{G(\xi)'}{G(\xi)}\right)+a_2\left(\frac{G(\xi)'}{G(\xi)}\right)^2. \qquad (19)$$

where $G(\xi)$ satisfying the following Riccati equation

$$G(\xi)G''(\xi)-\delta_1G^2(\xi)+\delta_2\left(G'(\xi)\right)^2=0. \qquad (20)$$

Putting Equation (20) into (18) coupled with auxiliary equation; the Equation (18) yields an algebraic equation involving power of $\left(\frac{G(\xi)'}{G(\xi)}\right)$ as

$$C_0\left(\frac{G(\xi)'}{G(\xi)}\right)^0+C_1\left(\frac{G(\xi)'}{G(\xi)}\right)^1+C_2\left(\frac{G(\xi)'}{G(\xi)}\right)^2+C_3\left(\frac{G(\xi)'}{G(\xi)}\right)^3+\cdots+C_6\left(\frac{G(\xi)'}{G(\xi)}\right)^6=0.$$

Compare the like powers of $\left(\frac{G(\xi)'}{G(\xi)}\right)$ we have system of equations

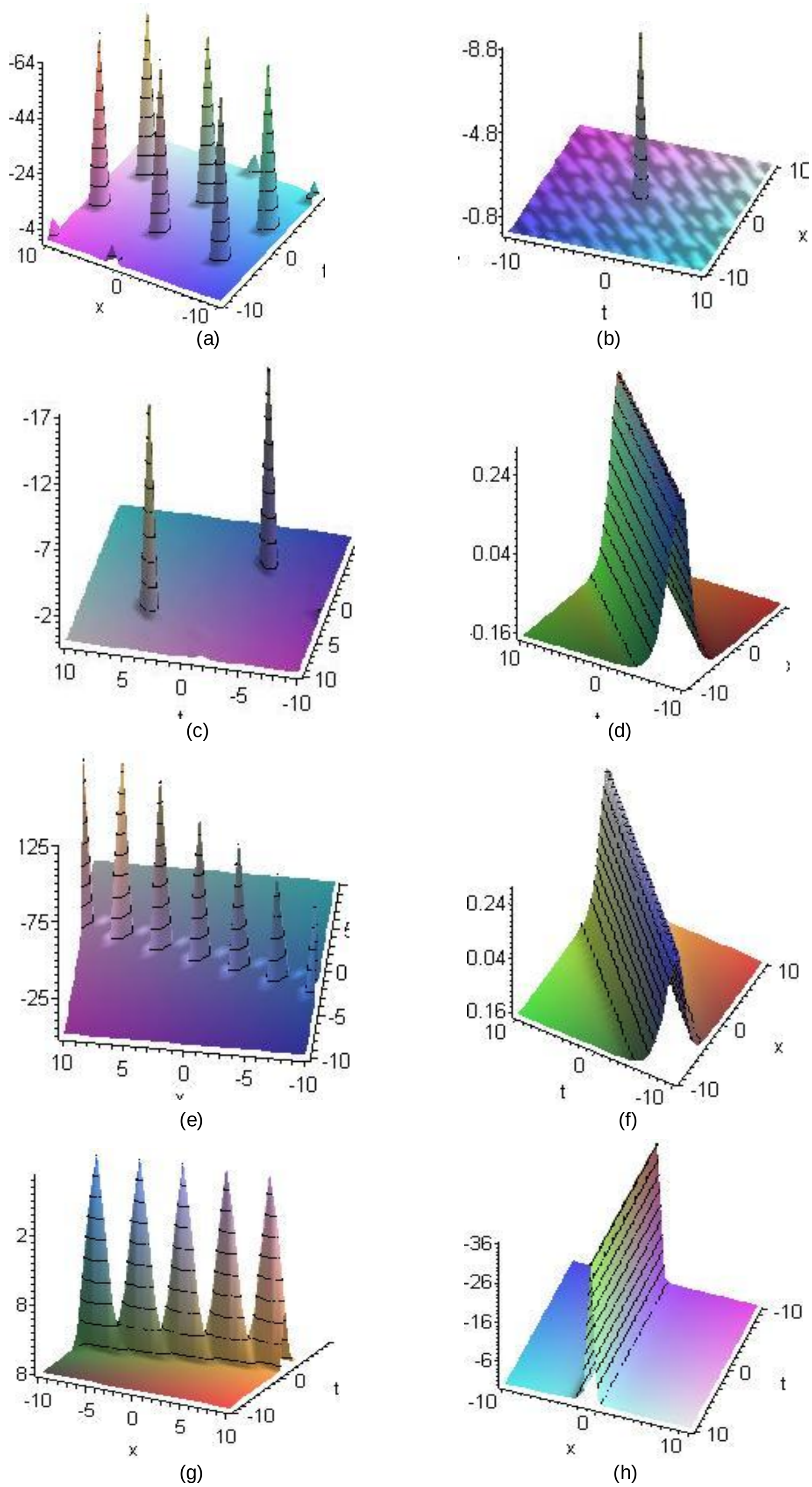
$\left(\frac{G(\xi)'}{G(\xi)}\right)^0$: $\quad -16k^5a_2\delta_1^3\delta_2-16k^5a_2\delta_1^3+\omega a_0+60ka_0^3+60k^3a_0a_2\delta_1^2=0,$

$\left(\frac{G(\xi)'}{G(\xi)}\right)^1$: $\quad -60k^3a_0a_1\delta_1\delta_2-60k^3a_0a_1\delta_1+60k^3a_1a_2\delta_1^2+\cdots+16k^5a_1\delta_1^2=0,$

$\left(\frac{G(\xi)'}{G(\xi)}\right)^2$: $\quad -60\delta_1a_1^2k^2+180ka_0a_1^2-240k^3a_0a_2\delta_1\delta_2+\cdots+60k^3a_2^2\delta_1^2=0,$

$\left(\frac{G(\xi)'}{G(\xi)}\right)^3$: $\quad -300k^3a_1a_2\delta_1\delta_2+360ka_0a_1a_2-300k^3a_1a_2\delta_1+\cdots-40k^5a_1\delta_1=0,$

$\left(\frac{G(\xi)'}{G(\xi)}\right)^4$: $\quad 180ka_2^2a_0+360k^3a_0a_2\delta_2+180k^3a_0a_2\delta_2+\cdots-240k^5a_2\delta_1=0,$



Graph 1. Solitons corresponding to solutions (a) u_1 for $p=q=r=1$ (b) u_2 for $p=q=1, r=2$ (c) u_8 for $p=q=r=2$ (d) u_{13} for $p=3, q=2, r=1$ (e) u_{14} for $p=2, q=1, r=0.5$ (f) u_{20} for $p=3, q=1, r=2$ (g) u_{26} for $p=1.5, q=1, r=0$ (h) u_{27} for $p=0, q=1, r=0$.

$$\left(\frac{G(\xi)'}{G(\xi)}\right)^5: \quad 240k^3a_1a_2\delta_2^2 + 480k^3a_1a_2\delta_2 + 24k^5a_1 + \cdots + 96k^5a_1\delta_2 = 0,$$

$$\left(\frac{G(\xi)'}{G(\xi)}\right)^6: \quad 60ka_2^3 + 180k^3a_2^2 + 120k^5a_2\delta_2^4 + 120k^5a_2 + \cdots + 480k^5a_2\delta_2 = 0.$$

Solving the above system for unknown parameters, we have the following solution sets.

1st solution set

$$k = k, \omega = -16k^5(\delta_2^2 + 2\delta_2 + 1)\delta_1^2, a_0 = k^2\delta_1 + k^2\delta_1\delta_2, a_1 = 0, a_2 = -k^2(\delta_2^2 + 2\delta_2 + 1).$$

Family 1: When $\delta_1, \delta_2 \neq 0$,

$$u(\xi) = k^2\delta_1 + k^2\delta_1\delta_2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}$.

Family 2: When $\delta_1 < 0$, and $(1 + \delta_2) > 0$, or $\delta_1 > 0$, and $(1 + \delta_2) < 0$

$$u(\xi) = k^2\delta_1 + k^2\delta_1\delta_2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)-\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{-\delta_1}+\sqrt{-\delta_1}}{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)+\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}$.

Family 3: When $\delta_1 \neq 0$, and $\delta_2 = 0$,

$$u(\xi) = k^2\delta_1 + k^2\delta_1\delta_2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]-1}$.

Family 4: When $\delta_1 = 0$, and $\delta_2 \neq 0$,

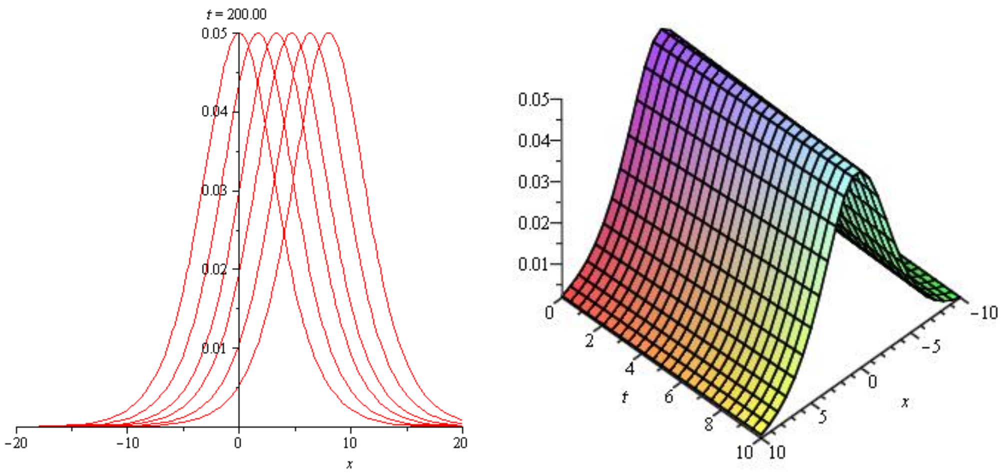
$$u(\xi) = k^2\delta_1 + k^2\delta_1\delta_2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}\frac{1}{1+\delta_2}$.

Family 5: When $\delta_1 = 0$, and $\delta_2 = 0$.

$$u(\xi) = k^2\delta_1 + k^2\delta_1\delta_2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}$.



(a) (b)

Graph 2. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.

In all cases $\xi = kx - 16k^5(\delta_2^2 + 2\delta_2 + 1)\delta_1^2\, t.$

Graph 2a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

2nd solution set

$$k = k, \omega = 60k^5\delta_1^2\left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)\delta_2^2 + 120k^5\delta_1^2\left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)\delta_2 + 60k^5\delta_1^2\left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right) - 52k^5\delta_1^2\delta_2^2 - 104k^5\delta_2\delta_1^2 - 52k^5\delta_1^2, a_0 = \left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2, a_1 = 0, a_2 = -k^2(\delta_2^2 + 2\delta_2 + 1).$$

Family 1: When $\delta_1, \delta_2 \neq 0,$

$$u(\xi) = \left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right) + \sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{\delta_1} + \sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right) + \sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2} - \sqrt{1+\delta_2}}.$

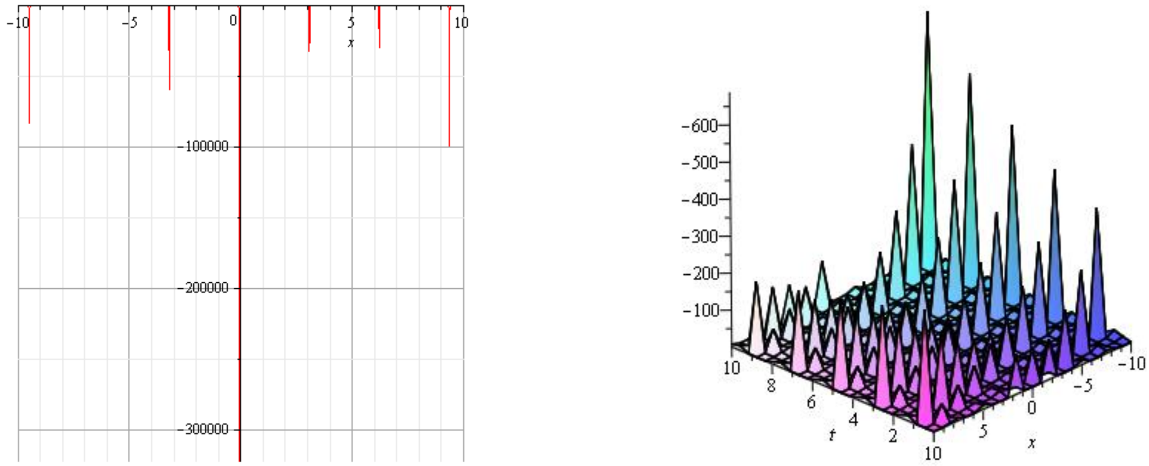
Family 2: When $\delta_1 < 0$, and $(1 + \delta_2) > 0$, or $\delta_1 > 0$, and $(1 + \delta_2) < 0$

$$u(\xi) = \left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right) - \sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{-\delta_1} + \sqrt{-\delta_1}}{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right) + \sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2} - \sqrt{1+\delta_2}}.$

Family 3: When $\delta_1 \neq 0$, and $\delta_2 = 0,$

$$u(\xi) = \left(\frac{1}{2} + \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$



(a) (b)

Graph 3. (a) 2D and (b) 3D periodic wave solutions of Equation (10) for different values of parameters.

where $\left(\frac{G'(\xi)}{G(\xi)}\right)=\frac{\left[\cosh\left(2\sqrt{\delta_2}\xi\right)+\sinh\left(2\sqrt{\delta_2}\xi\right)\right]\sqrt{\delta_2}+\sqrt{\delta_2}}{\left[\cosh\left(2\sqrt{\delta_2}\xi\right)+\sinh\left(2\sqrt{\delta_2}\xi\right)\right]-1}$.

Family 4: When $\delta_1=0$, and $\delta_2\neq 0$,

$$u(\xi)=\left(\frac{1}{2}+\frac{1}{30}\sqrt{105}\right)(\delta_2+1)\delta_1k^2-k^2(\delta_2^2+2\delta_2+1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right)=\frac{1}{1+\xi}\frac{1}{1+\delta_2}$.

Family 5: When $\delta_1=0$, and $\delta_2=0$.

$$u(\xi)=\left(\frac{1}{2}+\frac{1}{30}\sqrt{105}\right)(\delta_2+1)\delta_1k^2-k^2(\delta_2^2+2\delta_2+1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right)=\frac{1}{1+\xi}$.

In all cases $\xi=kx+\begin{bmatrix}60k^5\delta_1^2\left(\frac{1}{2}+\frac{1}{30}\sqrt{105}\right)\delta_2^2+120k^5\delta_1^2\left(\frac{1}{2}+\frac{1}{30}\sqrt{105}\right)\delta_2+\\60k^5\delta_1^2\left(\frac{1}{2}+\frac{1}{30}\sqrt{105}\right)-52k^5\delta_1^2\delta_2^2-104k^5\delta_2\delta_1^2-52k^5\delta_1^2\end{bmatrix}t.$

Graph 3a and b show 2D and 3D periodic wave solutions of Equation (10) for different values of parameters.

3rd solution set

$$\begin{aligned}k=k,\omega&=60k^5\delta_1^2\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)\delta_2^2+120k^5\delta_1^2\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)\delta_2\\&\quad+60k^5\delta_1^2\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)-52k^5\delta_1^2\delta_2^2-104k^5\delta_2\delta_1^2-52k^5\delta_1^2,a_0\\&=\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)(\delta_2+1)\delta_1k^2,a_1=0,a_2=-k^2(\delta_2^2+2\delta_2+1).\end{aligned}$$

Family 1: When $\delta_1, \delta_2 \neq 0$,

$$u(\xi) = \left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)+\sinh\left(2\sqrt{\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}$.

Family 2: When $\delta_1 < 0$, and $(1 + \delta_2) > 0$, or $\delta_1 > 0$, and $(1 + \delta_2) < 0$

$$u(\xi) = \left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)-\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{-\delta_1}+\sqrt{-\delta_1}}{\left[\cos\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)+\sin\left(2\sqrt{-\delta_1(1+\delta_2)}\xi\right)\right]\sqrt{1+\delta_2}-\sqrt{1+\delta_2}}$.

Family 3: When $\delta_1 \neq 0$, and $\delta_2 = 0$,

$$u(\xi) = \left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]\sqrt{\delta_1}+\sqrt{\delta_1}}{\left[\cosh\left(2\sqrt{\delta_1}\xi\right)+\sinh\left(2\sqrt{\delta_1}\xi\right)\right]-1}$.

Family 4: When $\delta_1 = 0$, and $\delta_2 \neq 0$,

$$u(\xi) = \left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}\frac{1}{1+\delta_2}$.

Family 5: When $\delta_1 = 0$, and $\delta_2 = 0$.

$$u(\xi) = \left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)(\delta_2 + 1)\delta_1 k^2 - k^2(\delta_2^2 + 2\delta_2 + 1)\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$

where $\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{1}{1+\xi}$.

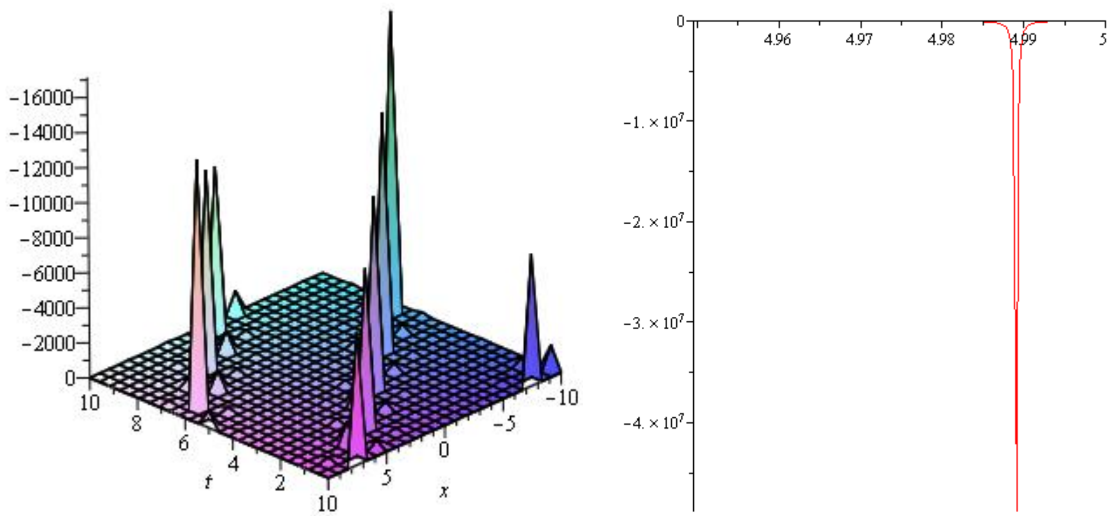
In all cases $\xi = kx + \begin{bmatrix} 60k^5\delta_1^2\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)\delta_2^2 + 120k^5\delta_1^2\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)\delta_2 + \\ 60k^5\delta_1^2\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right) - 52k^5\delta_1^2\delta_2^2 - 104k^5\delta_2\delta_1^2 - 52k^5\delta_1^2 \end{bmatrix} t.$

Graph 4a and b show 2D and 3D periodic wave solutions of Equation (10) for different values of parameters.

(*U`/U*)-expansion method

For $m = 2$, we obtained the trail solution

$$u = a_0 + a_1\left(\frac{U(\xi)^{`}}{U(\xi)}\right) + a_2\left(\frac{U(\xi)^{`}}{U(\xi)}\right)^2. \qquad (21)$$



(a) (b)

Graph 4. (a) 2D and (b) 3D periodic wave solutions of Equation (10) for different values of parameters.

where $G(\xi)$ satisfying the following Riccati equation

$$U'(\xi) = AU + B. \qquad (22)$$

Putting Equation (22) into (18) coupled with auxiliary equation; the Equation (18) yields an algebraic equation involving power of $\left(\frac{G(\xi)'}{G(\xi)}\right)$ as

$$\frac{1}{U^6}\left(C_0U^0 + C_1U^1 + C_2U^2 + C_3U^3 + \cdots + C_6U^6\right) = 0.$$

Compare the like powers of U we have system of equations

$$U^0: \quad 120B^6k^5a_2 + 180k^3B^6a_2^2 + 60ka_2^3B^6 = 0,$$

$$U^1: \quad 180ka_1B^5a_2^2 + 360ka_2^3AB^5 + 240k^3B^5a_1a_2 + \cdots + 24B^5k^5a_1 = 0,$$

$$U^2: \quad 180ka_0a_2^2B^4 + 180ka_1^2B^4a_2 + 900ka_2^3A^2B^4 + \cdots + 60k^3a_1^2B^4 = 0,$$

$$U^3: \quad 1200ka_2^3A^3B^3 + 60k^3B^3a_0a_1 + 150k^3B^3a_1^2A + \cdots + 360ka_0a_1B^3a_2 = 0,$$

$$U^4: \quad 1080ka_0a_1A^2a_2B + 180ka_0^2a_1B + 180ka_1^3A^2B + \cdots + 1080ka_2^2a_0A^2B^2 = 0,$$

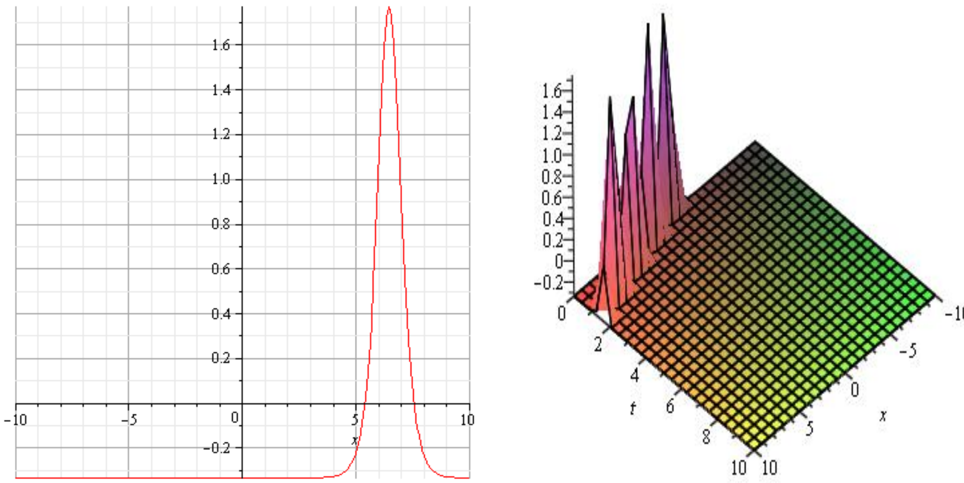
$$U^5: \quad 1080ka_0a_1Aa_2B^2 + 180ka_1a_0^2B + 180ka_1^3A^2B + \cdots + 2\omega Aa_2B = 0,$$

$$U^6: \quad 180ka_0a_2^2A^4 + 180ka_0a_1^2A^2 + 180ka_0^2a_1A + \cdots + 180ka_0^2a_1A = 0.$$

Solving the above system for unknown parameters, we have the following solution sets

1st solution set

$$k = k, \omega = -k^5A^4, a_0 = 0, a_1 = k^2A, a_2 = -k^2,$$



(a) (b)

Graph 5. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.

Substituting the solution set into trial solution

$$u(\xi)=k^2A\left(\frac{A\mathrm{e}^{A(kx-k^5A^4t)}}{-\frac{B}{A}+\mathrm{e}^{A(kx-k^5A^4t)}}\right)-k^2\left(\frac{A\mathrm{e}^{A(kx-k^5A^4t)}}{-\frac{B}{A}+\mathrm{e}^{A(kx-k^5A^4t)}}\right)^2,$$

Graph 5a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

2nd solution set

$$k=\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A},\ \omega=-\frac{60\sqrt{-15a_0+\sqrt{105}\,a_0}\,a_0a_0^2}{A},\ a_0=a_0,\ a_1=\frac{-15a_0+\sqrt{105}\,a_0}{A},\ a_2=-\frac{-15a_0+\sqrt{105}\,a_0}{A^2},$$

Substituting the solution set into trial solution

$$u(\xi)=a_0+\frac{-15a_0+\sqrt{105}\,a_0}{A}\left(\frac{A\mathrm{e}^{A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}{-\frac{B}{A}+\mathrm{e}^{A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}\right)-\frac{-15a_0+\sqrt{105}\,a_0}{A^2}\left(\frac{A\mathrm{e}^{A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}{-\frac{B}{A}+\mathrm{e}^{A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}\right)^2,$$

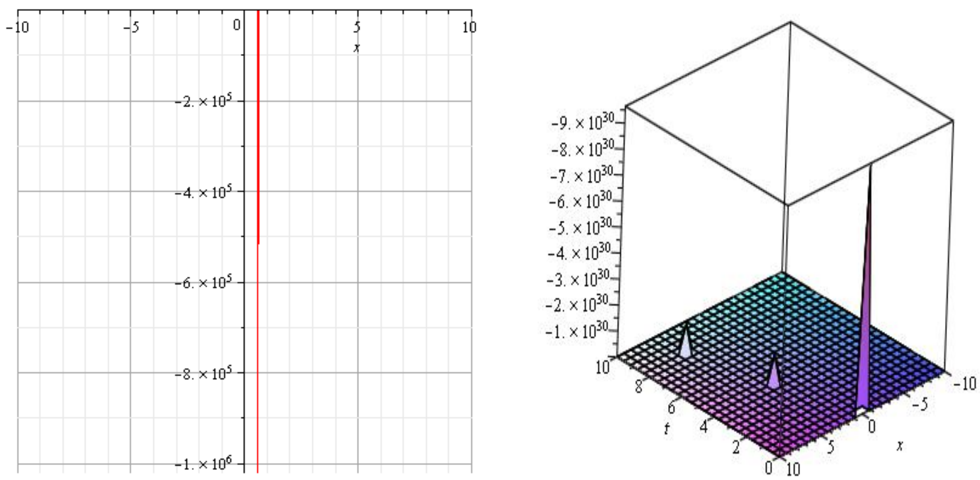
Graph 6a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

3rd solution set

$$k=-\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A},\ \omega=\frac{60\sqrt{-15a_0+\sqrt{105}\,a_0}\,a_0a_0^2}{A},\ a_0=a_0,\ a_1=\frac{-15a_0+\sqrt{105}\,a_0}{A},\ a_2=-\frac{-15a_0+\sqrt{105}\,a_0}{A^2},$$

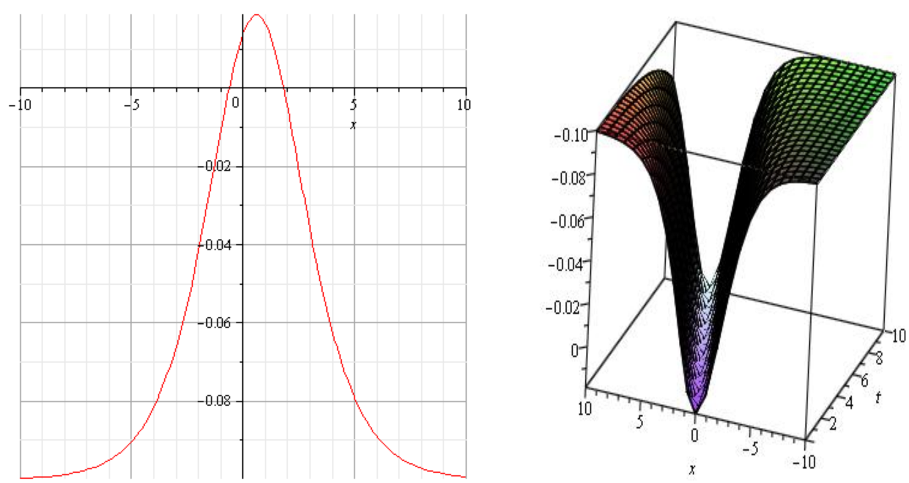
Substituting the solution set into trial solution

$$u(\xi)=a_0+\frac{-15a_0+\sqrt{105}\,a_0}{A}\left(\frac{A\mathrm{e}^{-A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}{-\frac{B}{A}+\mathrm{e}^{-A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}\right)-\frac{-15a_0+\sqrt{105}\,a_0}{A^2}\left(\frac{A\mathrm{e}^{-A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}{-\frac{B}{A}+\mathrm{e}^{-A\frac{\sqrt{-15a_0+\sqrt{105}\,a_0}}{A}(x-60a_0^2t)}}\right)^2,$$



(a) (b)

Graph 6. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.



(a) (b)

Graph 7. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.

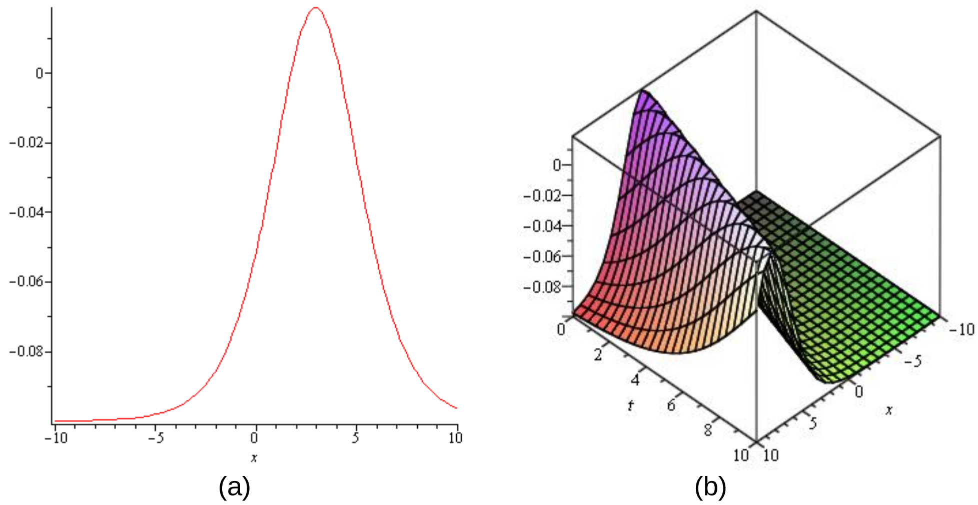
Graph 7a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

4th solution set

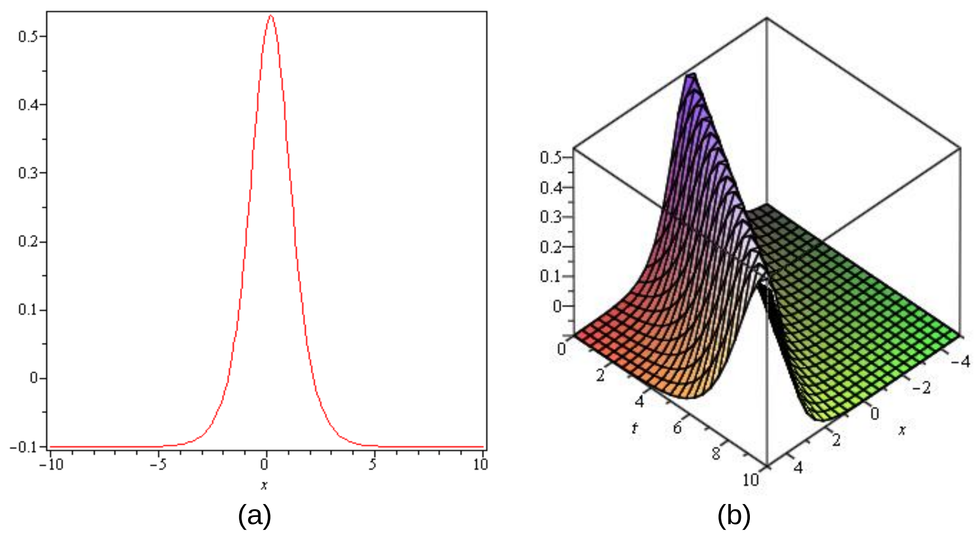
$$k=\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A},\ \omega=-\frac{60\sqrt{-15a_0-\sqrt{105}\,a_0}\,a_0^2}{A},\ a_0=a_0,\ a_1=\frac{-15a_0-\sqrt{105}\,a_0}{A},\ a_2=-\frac{-15a_0-\sqrt{105}\,a_0}{A^2},$$

Substituting the solution set into trial solution

$$u(\xi)=a_0+\frac{-15a_0-\sqrt{105}\,a_0}{A}\left(\frac{A\,e^{\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}\left(x-60a_0^2t\right)}}{-\frac{B}{A}+e^{\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}\left(x-60a_0^2t\right)}}\right)-\frac{-15a_0-\sqrt{105}\,a_0}{A^2}\left(\frac{A\,e^{\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}\left(x-60a_0^2t\right)}}{-\frac{B}{A}+e^{\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}\left(x-60a_0^2t\right)}}\right)^2,$$



Graph 8. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.



Graph 9. (a) 2D and (b) 3D travelling wave solutions of Equation (10) for different values of parameters.

Graph 8a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

5th solution set

$$k = -\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A},\ \omega = \frac{60\sqrt{-15a_0-\sqrt{105}\,a_0}\,a_0^2}{A},\ a_0 = a_0,\ a_1 = \frac{-15a_0-\sqrt{105}\,a_0}{A},\ a_2 = -\frac{-15a_0-\sqrt{105}\,a_0}{A^2},$$

Substituting the solution set into trial solution

$$u(\xi) = a_0 + \frac{-15a_0-\sqrt{105}\,a_0}{A}\left(\frac{A e^{-\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}(x-60a_0^2 t)}}{-\frac{B}{A}+e^{-\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}(x-60a_0^2 t)}}\right) - \frac{-15a_0-\sqrt{105}\,a_0}{A^2}\left(\frac{A e^{-\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}(x-60a_0^2 t)}}{-\frac{B}{A}+e^{-\frac{\sqrt{-15a_0-\sqrt{105}\,a_0}}{A}(x-60a_0^2 t)}}\right)^2,$$

Graph 9a and b show 2D and 3D travelling wave solutions of Equation (10) for different values of parameters.

Conclusion

Alternative (G'/G)-expansion along with the generalized Riccati equation and ($\grave{U}/U$)-expansion methods are successfully used for searching abundant exact travelling wave solutions to the (1+1)-dimensional CDG equation with the help of symbolic computation. Numerical results re-confirm the efficiency of the proposed algorithms. It is concluded that suggested schemes can be extended for other kinds of NLEEs in mathematical physics.

REFERENCES

Abazari R (2010). The (G'/G)-expansion method for Tziteica type nonlinear evolution equations. Math. Comput. Model. 52:1834-1845.

Abbasbandy S (2007a). A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. J. Comput. Appl. Math. 207:59-63.

Abbasbandy S (2007b). Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Int. J. Numer. Meth. Engr. 70:876-881.

Abdollahzadeh M, Hosseini M, Ghanbarpour M, Shirvani H (2010). Exact traveling solutions for fifth order Caudrey-Dodd-Gibbon equation. Int. J. Appl. Math. Comput. 2(4):81-90.

Abdou MA, Soliman AA, Basyony ST (2007). New application of exp-function method for improved Boussinesq equation. Phys. Lett. A 369:469-475.

Adomian G (1994). Solving frontier problems of physics: The decomposition method, Boston, MA: Kluwer Academic.

Akbar MA, Ali NHM (2011a). The modified alternative (G'/G)-expansion method for finding the exact solutions of nonlinear PDEs in mathematical physics. Int. J. Phys. Sci. 6(35):7910-7920.

Akbar MA, Ali NHM (2011b). Exp-function method for Duffing equation and new solutions of (2+1) dimensional dispersive long wave equations. Prog. Appl. Math. 1(2):30-42.

Ali AT (2011). New generalized Jacobi elliptic function rational expansion method. J. Comput. Appl. Math. 235:4117-4127.

Bekir A (2008). Application of the (G'/G)-expansion method for nonlinear evolution equations. Phys. Lett. A 372:3400-3406.

Gardner CS, Greene JM, Kruskal MD, Miura RM (1967). A method for solving the Korteweg–de Vries equation. Phys. Rev. Lett. 19:1095-1099.

He JH, Wu XH (2006). Exp-function method for nonlinear wave equations. Chaos Solit. Fract. 30:700-708.

Hirota R (1971). Exact solution of the KdV equation for multiple collisions of Solitons. Phys. Rev. Lett. 27:1192-1194.

Jiang B, Bi Q (2010). A study on the bilinear Caudrey-Dodd-Gibbon equation. Nonlin. Anal. 72:4530-4533.

Jin L (2010). Application of the variational iteration method for solving the fifth order Caudrey-Dodd-Gibbon equation. Int. Math. Forum 5(66):3259-3265.

Liao SJ (1992a). The Homotopy Analysis Method and its applications in mechanics. Ph.D. Dissertation (in English). Shanghai Jiao Tong University.

Liao SJ (1992b). A kind of linear invariance under homotopy and some simple applications of it in mechanics. Bericht Nr. 520, Institut fuer Schifl'bau der Universitaet Hamburg.

Liu X, Tian L, Wu Y (2010). Exact solutions of the generalized Benjamin-Bona-Mahony equation. Math. Prob. Engr. Article ID 796398, 5 pages, doi:10.1155/2010/796398.

Malfliet M (1992). Solitary wave solutions of nonlinear wave equations. Am. J. Phys. 60:650–654.

Mohyud-Din ST (2007). Homotopy perturbation method for solving fourth-order boundary value problems. Math. Prob. Engr. Article ID 98602, doi:10.1155/2007/98602.

Mohyud-Din ST (2008). Variational iteration method for solving fifth-order boundary value problems using He's polynomials. Math. Prob. Engr. Article ID 954794, doi: 10:1155/2008/954794.

Mohyud-Din ST, Noor MA, Noor KI (2009). Some relatively new techniques for nonlinear problems. Math. Prob. Engr. Article ID 234849, doi:10.1155/2009/234849.

Mohyud-Din ST, Noor MA, Waheed A (2010). Exp-function method for generalized travelling solutions of Calogero-Degasperis-Fokas equation. Zeitschrift für Naturforschung A- A J. Phys. Sci. 65a:78-84.

Mohyud-Din ST, Yildirim A, Demirli G (2011a). Analytical solution of wave system in R^n with coupling controllers. Int. J. Numer. Meth. Heat Fluid Flow, Emerald 21:198-205.

Mohyud-Din ST, Yildirim A, Sariaydin S (2011b). Numerical soliton solution of the Kaup–Kuper shmidt equation. Int. J. Numer. Meth. Heat Fluid Flow, Emerald 21(3):272-281.

Naher H, Abdullah FA, Akbar MA (2011a). The (G'/G)-expansion method for abundant travelling wave solutions of Caudrey-Dodd-Gibbon equation. Math. Prob. Engr. Article ID 218216, 11 pp. doi:10.1155/2011/218216.

Naher H, Abdullah FA, Akbar MA (2011b). The exp-function method for new exact solutions of the nonlinear partial differential equations. Int. J. Phys. Sci. 6(29):6706-6716.

Rogers C, Shadwick WF (1982). Backlund Transformations. Academic Press, New York.

Shi Y, Dai Z, Han S, Huang L (2010). The multi-wave method for nonlinear evolution equations. Math. Comput. Appl. 15(5):776-783.

Taghizadeh N, Mirzazadeh M (2011). The first integral method to some complex nonlinear partial differential equations. J. Comput. Appl. Math. 235:4871-4877.

Usman M, Yildirim A, Mohyud-Din ST (2011). A Reliable algorithm for physical problems, Int. J. Phys. Sci. 6(1):146-153.

Wang ML (1996). Exact solutions for a compound KdV-Burgers equation. Phys. Lett. A 213:279-287.

Wang ML, Li X, Zhang J (2008). The (G'/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A 372:417-423.

Wazwaz MA (2009). Partial Differential Equations and Solitary Waves Theory. Springer Dordrecht Heidelberg, London, New York.

Xu Y, Zhou X, Yao L (2008). Solving the fifth order Caudrey-Dodd-Gibbon (CDG) equation using the Exp-function method. Appl. Math. Comput. 206:70-73.

Zayed EME (2009a). The (G'/G)-expansion method and its applications to some nonlinear evolution equations in the mathematical physics. J. Appl. Math. Comput. 30:89-103.

Zayed EME (2009b). New traveling wave solutions for higher dimensional nonlinear evolution equations using a generalized (G'/G)-expansion method. J. Phys. A: Math. Theor. 42:195202-195214.

Zayed EME, Gepreel KA (2009). The (G'/G)-expansion method for finding traveling wave solutions of nonlinear PDEs in mathematical physics. J. Math. Phys. 50:013502-013513.

Zayed EME, Al-Joudi S (2010). Applications of an extended (G'/G)-expansion method to find exact solutions of nonlinear PDEs in mathematical physics. Math. Prob. Engr. Article ID 768573, doi: 10.1155/2010/768573.

Zayed EME (2011). The (G'/G)-expansion method combined with the Riccati equation for finding exact solutions of nonlinear PDEs. J. Appl. Math. Inf. 29(1-2):351-367.

Zhang S, Tong J, Wang W (2008a). A generalized (G'/G)-expansion method for the mKdV equation with variable coefficients. Phys. Lett. A 372:2254-2257.

Zhang J, Wei X, Lu Y (2008b). A generalized (G'/G)-expansion method and its applications. Phys. Lett. A 372:3653-3658.

Zhang J, Jiang F, Zhao X (2010). An improved (G'/G)-expansion method for solving nonlinear evolution equations. Int. J. Comput. Math. 87:1716-1725.

Zheng C, Qiang J, Wang S (2010). Standing, periodic and solitary wave in (1+1)-dimensional Caudrey-Dodd-Gibbon-sawada-Kotera system.

Commun. Theor. Phys. 54:1054-1058.
Zhou YB, Wang ML, Wang YM (2003). Periodic wave solutions to coupled KdV equations with variable coefficients. Phys. Lett. A 308:31–36.
Zhu S (2008). The generalized Riccati equation mapping method in non-linear evolution equation: Application to (2+1)-dimensional Boiti-Leon-Pempinelle equation. Chaos Solit. Fract. 37:1335-1342.