

Full Length Research Paper

On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type

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In this paper, the homotopy perturbation method (HPM) is employed to obtain approximate analytical solutions of the time-fractional reaction-diffusion equation of the Fisher type. The method can easily be applied to many problems and is capable of reducing the size of computational work. The fractional derivative is described in the Caputo sense. The analytical/numerical results are compared with existing analytic solutions obtained by Adomian decomposition method (ADM) and differential transformation method (DTM) and the outcomes confirm that the scheme yields accurate and excellent results even when few components are used.

Key words: Fisher equation, diffusion, reaction, fractional partial differential equations.

INTRODUCTION

Reaction-diffusion is a process in which two or more chemicals that diffuse at unequal rates over a surface react with one another in order to form stable patterns. Reaction-diffusion (RD) (Wang and He, 2008; Wilhelmsson et al., 2001; Hundsdorfer et al., 2003) equations are useful in many areas of science and engineering. Recent development of new algorithms for analyzing reaction-diffusion phenomena has led to physically interesting and mathematically challenging problems. The Fisher equation has various applications in the fields of logistic population growth (Fisher, 1937), neurophysiology (Tuckwell, 1988), autocatalytic chemical reaction (Aronson and Weinberg, 1988), branching Brownian motion processes (Bramson, 1978). In the recent years fractional differential equations (Miller and Ross, 1993; Butzer and Westphal, 2000; Diethelm et al., 2005; Zhou and Li 2005) are gaining importance owing to their applications in the field of visco-elasticity (Mahmood et al., 2009), feed-back amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles, neuron modeling. Fractional diffusion equations are used to model problems in finance (Gorenflo et al., 2001; Mainardi et al., 2000; Raberto et al., 2002), and hydrology (Benson et al., 2000). The nature of the

diffusion is characterized by the temporal scaling of the mean-square displacement $\langle r^2(t) \rangle \sim t^\gamma$. For standard diffusion $\gamma = 1$, whereas in anomalous sub-diffusion $\gamma < 1$ and in anomalous super-diffusion $\gamma > 1$. Sub-diffusion typically arises in cases where there are spatial or temporal constraints such as occurring in fractured and porous media and fractal lattices. Super-diffusion may occur in chaotic or turbulent processes through enhanced transport of particles.

Many mathematical and computational methods have been developed over the last century for solving and analyzing differential equations (DEs), which makes DE-based modeling attractive. However, it has serious limitations when applied to physical, chemical and biological systems. The most recent numerical techniques, it is worth mentioning on-standard finite difference methods, hybrid boundary integral procedures. The widely applied techniques (that is, perturbation method) are of great interest to be used in engineering systems (Cole, 1968). To eliminate the limitation of "small parameter", which is assumed in the perturbation method, a new technique based on the homotopy terminology has been proposed. Accordingly, a nonlinear problem is transformed into an infinite number of simple problems without using the perturbation technique. Effectively, letting the small parameter float and

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converge to the unity, the problem will be converted into a special perturbation problem with a small embedding parameter. So the method receives the name, homotopy perturbation method (HPM) (He, 1999, 2000, 2004, 2006, 2008; Khan and Wu, 2011). The scheme has been applied to linear and nonlinear ordinary and partial differential equations. These equations usually describe a dynamic system incorporating the perturbation value (that is, HPM). Recently, it has been applied to a wide class of differential equations, such as Riccati equation (Aminikhah and Hemmatnezhad, 2010), system of ordinary differential equation with time-fractional derivative arising in chemical engineering (Khan et al., 2010), fractional order Riccati equation (Khan et al., 2011), rational approximation solution of the fractional Sharma–Tasso–Oleiver equation (Song et al., 2009). The method was also applied to Cahn-Hilliard equation (Ugurlu and Kaya, 2008), Navier-Stokes equation (Khan et al., 2009) and Biazar discuss the convergence of HPM (Biazar and Ghazvini, 2009) etc.

We present solutions of a more general model of time-fractional Fisher equation:

$$D_t^\alpha u(x,t) = d D_x^2 u(x,t) + F(u,x,t) \tag{1}$$

$0 < \alpha \leq 1, x \in \mathfrak{R}, t > 0$

Where d is the diffusion coefficient and $F(u, x, t)$ is a nonlinear function representing reaction kinetics. It is interesting to observe that Equation 1 reduces to the time-fractional Fisher equation which was originally proposed by Fisher as a model for the spatial and temporal propagation of a virile gene in an infinite medium. If we set:

$$F(u, x, t) = u(1-u)(u-a) \tag{2}$$

It gives rise to the time-fractional Fitzhugh–Nagumo equation (Fitzhugh, 1961; Nagumo et al., 1962; Shih et al., 2005), which is an important nonlinear reaction–diffusion equation and applied to model the transmission of nerve impulses. The balance of this paper is organized as follows: some basic definitions of fractional calculus are given in this study; basic idea of HPM is given and implemented on Equation 1; for a number of special cases of the RD equations including the time-fractional Fitzhugh–Nagumo equation. The approximate solutions are compared with the exact closed-form solutions. Conclusions are given that briefly summarizes the numerical results.

BASIC DEFINITIONS OF FRACTIONAL CALCULUS

Definition 1

Caputo’s definition of the fractional-order derivative is defined as:

$$D^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\mu+1-n}} d\xi, (n-1 < R(\alpha) \leq n, n \in \mathbb{N}), t > 0, \tag{3}$$

Where the parameter μ is the order of the derivative and is allowed to be real or even complex, a is the initial value of function f . In the present work only real and positive μ will be considered. For the Caputo’s derivative we have:

$$D^\mu C = 0, \tag{4}$$

Where C is a constant,

$$D^\alpha t^\gamma = \begin{cases} 0, & (\gamma \leq \alpha - 1) \\ \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} t^{\gamma - \alpha}, & (\gamma > \alpha - 1) \end{cases} \tag{5}$$

Definition 2

For m to be the smallest integer that exceeds α , n to be the smallest integer that exceeds γ , the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D^\mu u(x,t) = \frac{\partial^\mu u(x,t)}{\partial t^\mu} = \begin{cases} \frac{1}{\Gamma(m-\mu)} \int_a^t \frac{f^{(m)}(\xi)}{(t-\xi)^{\mu+1-m}} \frac{\partial^m u(x,t)}{\partial t^m} d\xi, & m-1 < \beta < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \mu = m \in \mathbb{N} \end{cases} \tag{6}$$

Definition 3

The Riemann–Liouville fractional integral operator of order μ , of a function f :

$$J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^x (t-\xi)^{\mu-1} f(\xi) d\xi, \alpha > 0, t > 0 \tag{7}$$

For $\mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$ is defined as:

$$J^0 f(t) = f(t), \quad J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \\ J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} t^{\gamma + \alpha}, \quad J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t) \tag{8}$$

Also, we need two of its basic properties. If:

$$D^\alpha J^\beta f(t) = f(t), \\ J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0 \tag{9}$$

A modification in Riemann–Liouville fractional integral operator of order μ found in the literature (Khan and Faraz, 2011).

BASIC IDEA AND IMPLEMENTATION OF HPM

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$\Phi(U) = f(r), \quad r \in \Omega, \tag{10}$$

Where Φ represents a general nonlinear differential equation involving both linear and nonlinear parts. Therefore Equation 10 can be rewritten as follows:

$$L(U) + N(U) - f(r) = 0 \tag{11}$$

By the homotopy perturbation technique, we construct a homotopy $v(X, t, q) : \Omega \times [0, 1] \rightarrow \mathfrak{R}$ which satisfies:

$$H(v, q) = (1 - q)[L(v) - L(U_0)] + q[\Phi(v) - f(r)] = 0, \quad q \in [0, 1] \tag{12}$$

or,

$$H(v, q) = L(v) - L(U_0) + qL(U_0) + q[N(v) - f(X, t)] = 0$$

Where $q \in [0, 1]$ is an embedding parameter, U_0 is an initial approximation of (11). Obviously from the definitions we will have:

$$\begin{aligned} H(v, 0) &= L(v) - L(U_0) = 0, \text{ and} \\ H(v, 1) &= \Phi(v) - f(X, t) = 0, \end{aligned} \tag{13}$$

The changing process of q from 0 to 1, is just that of $H(v, q)$ from $L(v) - L(U_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $L(v) - L(U_0)$ and $A(v) - f(r)$ are called homotopic. Applying the perturbation technique, due to the fact that $0 \leq q \leq 1$. can be considered as a small parameter, we can assume that the solution of Equations (10) or (11) can be expressed as a series in q :

$$v = v_0 + q v_1 + q^2 v_2 + q^3 v_3 + \dots \tag{14}$$

When $q \rightarrow 1$, the approximate solution:

$$U = \lim_{q \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots \tag{15}$$

The homotopy perturbation method, which provides an analytical approximate solution, is applied on various nonlinear problems. Here, we implement HPM on Equation 1:

$$(1 - q)D_t^\alpha u(x, t) + q(D_t^\alpha u(x, t) - F(u, x, t)) = 0 \tag{16}$$

Or

$$D_t^\alpha u(x, t) + qF(u, x, t) = 0$$

In view of the homotopy perturbation method, we use the homotopy parameter q to expand the solution:

$$u = u_0 + q u_1 + q^2 u_2 + q^3 u_3 + \dots \tag{17}$$

Substituting Equation (17) into (16), and equating the terms with identical powers of q , we can obtain a series of linear equations. These linear equations are easily obtained by using MATHEMATICA 7 or by writing a computer code to get as many equations as we need in the calculation of the numerical as well as explicit solutions. Here we only write the first few linear equations:

$$D_t^\alpha u_0(x, t) = 0$$

$$D_t^\alpha u_1(x, t) + F_1(u_0, x, t) = 0$$

$$D_t^\alpha u_2(x, t) + F_2(u_0, u_1, x, t) = 0 \tag{18}$$

$$D_t^\alpha u_3(x, t) + F_3(u_0, u_1, u_2, x, t) = 0$$

Firstly, we apply the operator J_t^α , the inverse of the operator D_t^α , on both sides of the first equation of (18) to obtain u_0 . Solving the aforementioned equations, by taking the operator J_t^α both sides of the system of linear fractional differential equation 18.

TIME-FRACTIONAL REACTION-DIFFUSION EQUATIONS OF THE FISHER TYPE

To incorporate our discussion, we consider nonlinear Fisher type reaction-diffusion equations with time-fractional derivative which are arising in engineering sciences and other diverse phenomenon.

Application 1

In this case we will examine the case:

$$D_t^\alpha u = u_{xx} + u(\delta - \epsilon u) \tag{19}$$

Subject to a constant initial condition:

$$u(x, 0) = \lambda \tag{20}$$

Where δ and ϵ , respectively, correspond to the constant intrinsic growth rate and intraspecific competition coefficients. In the 1980s, this model has been extended to heterogeneous environments by Shigesada et al. (1986). Here, we consider δ and ϵ is equal to unity.

According to the HPM, we construct the following simple homotopy:

$$D_t^\alpha u + q(u - u^2) = 0 \tag{21}$$

In view of the homotopy perturbation method, we use the homotopy

parameter q to expand the solution. Substituting Equation (17) into (21), and equating the terms with identical powers of q , we can obtain a series of linear equations.

$$\begin{aligned}
 D_t^\alpha u_0 &= 0 \\
 D_t^\alpha u_1 + u_0 - u_0^2 &= 0 \\
 D_t^\alpha u_2 + u_1 - 2u_0u_1 &= 0 \\
 D_t^\alpha u_3 + u_2 - 2u_0u_2 - u_1^2 &= 0
 \end{aligned}
 \tag{22}$$

Firstly, we apply the operator J_t^α , the inverse of the operator D_t^α , on both sides of the first equation of (22) to obtain u_0 . For avoiding difficult fractional differentiation the few components are:

$$u_0(x, t) = \lambda \tag{23}$$

$$u_1(x, t) = \frac{(\lambda - \lambda^2)t^\alpha}{\Gamma(\alpha + 1)} \tag{24}$$

$$u_2(x, t) = \frac{(\lambda - 3\lambda^2 + 2\lambda^3)t^{2\alpha}}{\Gamma(2\alpha + 1)} \tag{25}$$

$$u_3(x, t) = \left(\frac{\lambda - 5\lambda^2 + 8\lambda^3 - 4\lambda^4}{\Gamma(3\alpha + 1)} - \frac{\Gamma(2\alpha + 1)(\lambda^2 + 2\lambda^3 - \lambda^4)}{(\Gamma(\alpha + 1))^2 \Gamma(3\alpha + 1)} \right) t^{3\alpha} \tag{26}$$

The approximate solution of Equation 19 by the HPM is:

$$u(x, t) = \sum_{k=0}^{10} u_k \tag{27}$$

The exact solution of the Equation 19 as $\alpha \rightarrow 1, \delta = 1, \varepsilon = 1$ is:

$$u(x, t) = \frac{\lambda e^t}{1 - \lambda + \lambda e^t} \tag{28}$$

Application 2

Consider the following Fisher equation:

$$D_t^\alpha u = u_{xx} + Au(1 - u) \tag{29}$$

Subject to initial condition:

$$u(x, 0) = \frac{1}{\left(1 + e^{\sqrt{\frac{A}{6}}x}\right)^2} \tag{30}$$

To solve the problem using the HPM and apply the same procedure in the previous applications, we obtain the following components:

$$u_0(x, t) = \frac{1}{\left(1 + e^{\sqrt{\frac{A}{6}}x}\right)^2} \tag{31}$$

$$u_1(x, t) = \frac{5Ae^{\sqrt{\frac{A}{6}}x}t^\alpha}{3\left(1 + e^{\sqrt{\frac{A}{6}}x}\right)^3\Gamma(\alpha + 1)} \tag{32}$$

$$u_2(x, t) = \frac{25A^2\left(-1 + 2e^{\sqrt{\frac{A}{6}}x}\right)t^{2\alpha}}{18\left(1 + e^{\sqrt{\frac{A}{6}}x}\right)^4\Gamma(2\alpha + 1)} \tag{33}$$

$$u_3(x, t) = \frac{25A^3e^{\sqrt{\frac{A}{6}}x}\left(\left(5 - 15e^{\sqrt{\frac{2A}{3}}x} + 20e^{\sqrt{\frac{2A}{3}}x} - 6e^{\sqrt{\frac{A}{6}}x}\right)(\Gamma(\alpha + 1))^2t^{3\alpha} - 12e^{\sqrt{\frac{A}{6}}x}\Gamma(2\alpha + 1)\right)}{108\left(1 + e^{\sqrt{\frac{A}{6}}x}\right)^6(\Gamma(\alpha + 1))^2\Gamma(3\alpha + 1)} \tag{34}$$

The approximate solution of Equation 29 is given by:

$$u(x, t) = \sum_{k=0}^{10} u_k \tag{35}$$

The exact solution of the Equation 29 with condition (30) as $\alpha \rightarrow 1$ is:

$$u(x, t) = \frac{1}{\left(1 + e^{\sqrt{\frac{A}{6}}x - \frac{5}{6}At}\right)^2} \tag{36}$$

Application 3

In this case we will examine:

$$D_t^\alpha u = u_{xx} + u(1 - u^\delta) \tag{37}$$

Subject to an initial condition:

$$u(x, 0) = \frac{1}{\left(1 + e^{\frac{3x}{2}}\right)^3} \tag{38}$$

Similarly, we construct the following homotopy:

$$D_t^\alpha u + q(u - u^7) = 0 \tag{39}$$

Solving the systems accordingly, thus we obtain:

$$u_0(x,t) = \frac{I}{\left(1 + e^{\frac{3x}{2}}\right)^{\frac{1}{3}}} \tag{40}$$

$$u_1(x,t) = \frac{10e^{\frac{3x}{2}} t^\alpha}{\left(1 + e^{\frac{3x}{2}}\right)^{\frac{3}{4}} \Gamma(\alpha + 1)} \tag{41}$$

$$u_2(x,t) = \frac{25e^{\frac{3x}{2}} \left(-3 + e^{\frac{3x}{2}}\right) t^{2\alpha}}{32 \left(1 + e^{\frac{3x}{2}}\right)^{\frac{7}{3}} \Gamma(2\alpha + 1)} \tag{42}$$

$$u_3(x,t) = \frac{25e^{\frac{3x}{2}} t^{3\alpha} \left(45 + 123e^{\frac{3x}{2}} - 85e^{3x} + 5e^{\frac{9x}{2}}\right) \left(\Gamma(\alpha + 1)\right)^2 - 84e^{\frac{3x}{2}} \Gamma(2\alpha + 1)}{64 \left(1 + e^{\frac{3x}{2}}\right)^{\frac{13}{3}} \left(\Gamma(2\alpha + 1)\right)^2 \Gamma(\alpha + 1)} \tag{43}$$

The approximate solution of Equation 37 by the HPM is:

$$u(x,t) = \sum_{k=0}^9 u_k \tag{44}$$

The closed form solution of the problem is given by:

$$u(x,t) = \left(\frac{1}{2} \tan\left(\frac{-3}{4} \left(x - \frac{5t}{2}\right)\right) + \frac{1}{2}\right)^{\frac{1}{3}} \tag{45}$$

Application 4

We consider the time-fractional Fitzhugh–Nagumo equation. This equation models the transmission of nerve impulses, and in the area of population genetics, in circuit theory, also this equation is an important nonlinear reaction–diffusion equation. This equation has three constant solutions; $u = 0, I, a$. The case with $0 < a < I$ is what the genetics (Kawahara et al., 1983) refer to as the heterozygote inferiority.

$$D_t^\alpha u = u_{xx} + u(1 - u)(u - a), \quad 0 < a < I \tag{46}$$

Subject to initial condition:

$$u(x,0) = \frac{I}{1 + e^{\frac{-x}{\sqrt{2}}}} \tag{47}$$

Similarly, we construct the following homotopy:

$$D_t^\alpha u + q((1 + a)u^2 - au - u^3) = 0 \tag{48}$$

Solving the systems accordingly, thus we obtain:

$$u_0(x,t) = \frac{I}{1 + e^{\frac{-x}{\sqrt{2}}}} \tag{49}$$

$$u_1(x,t) = \frac{(1 - 2a)e^{\frac{x}{\sqrt{2}}} t^\alpha}{2 \left(1 + e^{\frac{x}{\sqrt{2}}}\right)^2 \Gamma(\alpha + 1)} \tag{50}$$

$$u_2(x,t) = \frac{(1 - 2a)e^{\frac{x}{\sqrt{2}}} \left(-1 + e^{\frac{x}{\sqrt{2}}}\right) t^{2\alpha}}{4 \left(-1 + e^{\frac{x}{\sqrt{2}}}\right)^4 \Gamma(2\alpha + 1)} \tag{51}$$

$$u_3(x,t) = \frac{\left[\begin{aligned} & \left((1 - 2a)^2 e^{\frac{x}{\sqrt{2}}} t^{3\alpha} \left(-1 + e^{\frac{x}{\sqrt{2}}}\right) \left(1 - 6e^{\frac{x}{\sqrt{2}}} - e^{\sqrt{2}x} + 2\left(-1 + e^{\frac{x}{\sqrt{2}}}\right)\right) \left(\Gamma(\alpha + 1)\right)^2 \right) \\ & - 2e^{\frac{x}{\sqrt{2}}} \left(1 + a - 2e^{\frac{x}{\sqrt{2}}} + ae^{\frac{x}{\sqrt{2}}}\right) \Gamma(2\alpha + 1) \end{aligned} \right]}{\left(-1 + e^{\frac{x}{\sqrt{2}}}\right) \left(\Gamma(\alpha + 1)\right)^2 \Gamma(3\alpha + 1)} \tag{52}$$

The approximate solution of Equation 46 by the HPM is:

$$u(x,t) = \sum_{k=0}^8 u_k \tag{53}$$

As $\alpha \rightarrow I$ the close form solution is given by:

$$u(x,t) = \frac{I}{1 + e^{x+ct}}$$

Where,

$$c = \frac{1}{\sqrt{2}} - \sqrt{2}a.$$

Which is a good agreement with Wazwaz and Gorguis (2004) solution.

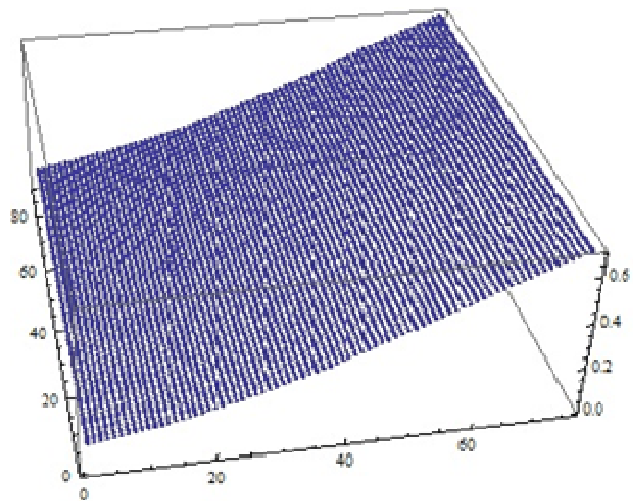


Figure 1. Exact solution $u(x,t)$ for $\alpha = 1, \lambda = 0.1$.

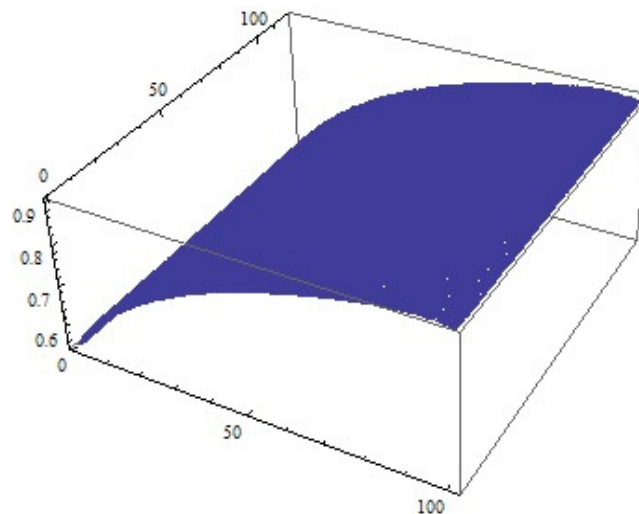


Figure 3. Approximate solution $u(x,t)$ for $\alpha = 0.8, \lambda = 0.55$.

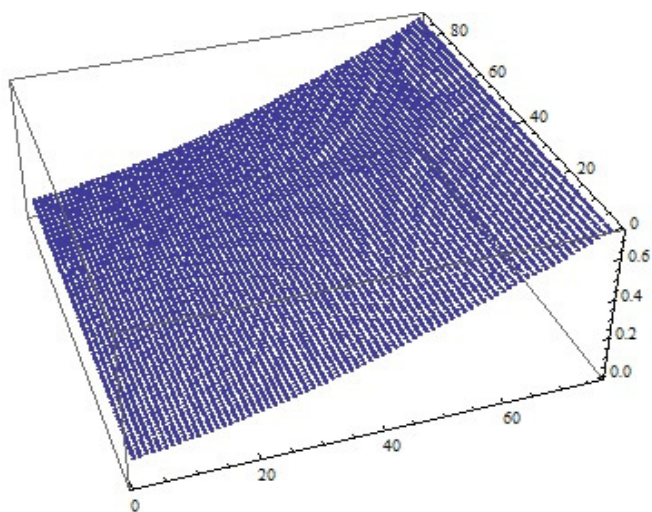


Figure 2. Approximate solution $u(x,t)$ for $\alpha = 1, \lambda = 0.1$.

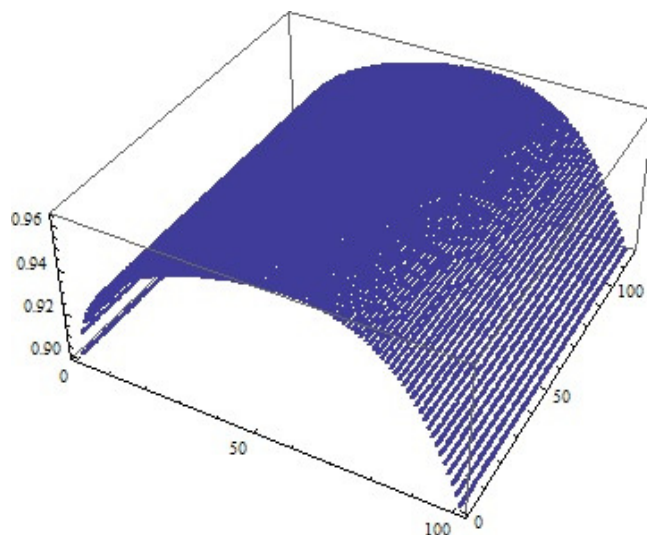


Figure 4. Approximate solution $u(x,t)$ for $\alpha = 0.6, \lambda = 0.9$.

DISCUSSION AND CONCLUSION

In this paper we obtain the analytical solutions of nonlinear time-fractional reaction-diffusion equations of the Fisher type using He's homotopy perturbation method. Figures 1 and 2 shows a very good agreement to the analytical solution of time-fractional RD-equation with constant initial condition in the time interval $(0, 3)$ by using 10th order of the series, which indicates that the speed of convergence of HPM is very fast. Figures 2, 3 and 4 shows the solution surfaces of the fractional Brownian motion for different values of α and λ . Figures 5 shows that a decrease in the fractional order α corresponds to an increase in the function $u(x,t)$.

Similar effects are for the function $u(x,t)$ (Figure 6) for fixed value of x and λ . It is seen from Figure 6 that four consecutive values of $\alpha = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ occur where first three are slow (slow diffusion) and in the positive direction but the fourth one is faster (fast diffusion). In Figures 7 to 8, the solution surfaces, respectively are depicted for different values of α . Figures 9 and 10 are prepared to show the influence of α on the function $u(x,t)$. It is clearly seen that a $u(x,t)$ increase with the increases in t for $\alpha = 1, \frac{9}{10}, \frac{8}{10}, \frac{7}{10}$.

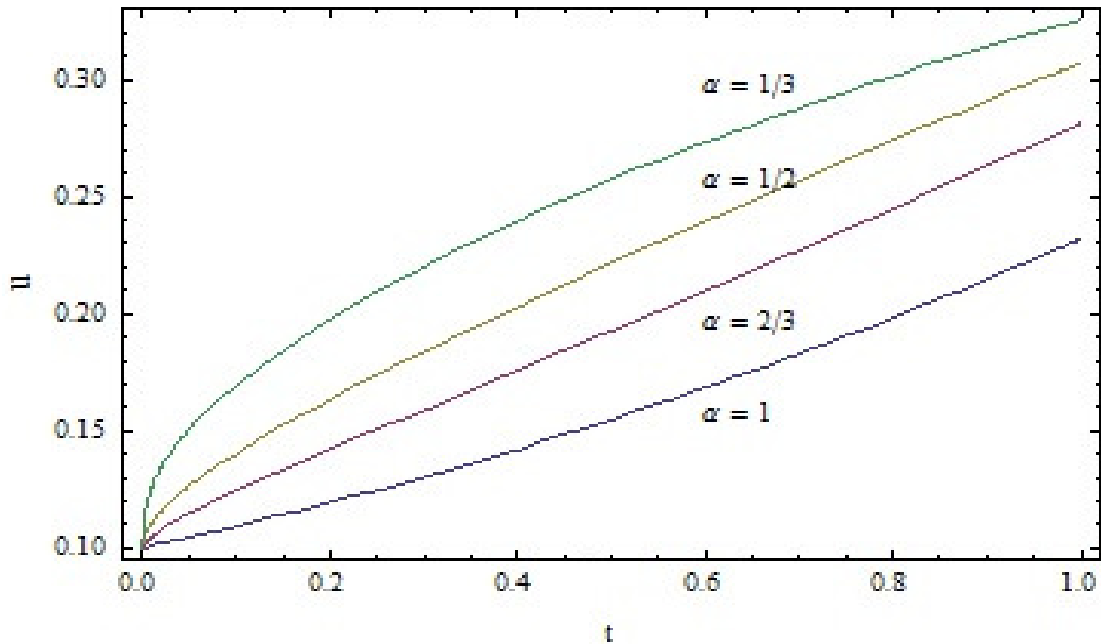


Figure 5. Approx. solution $u(x,t)$ for $x = 0.5, \lambda = 0.1$. (color figure can be viewed in the online issue)

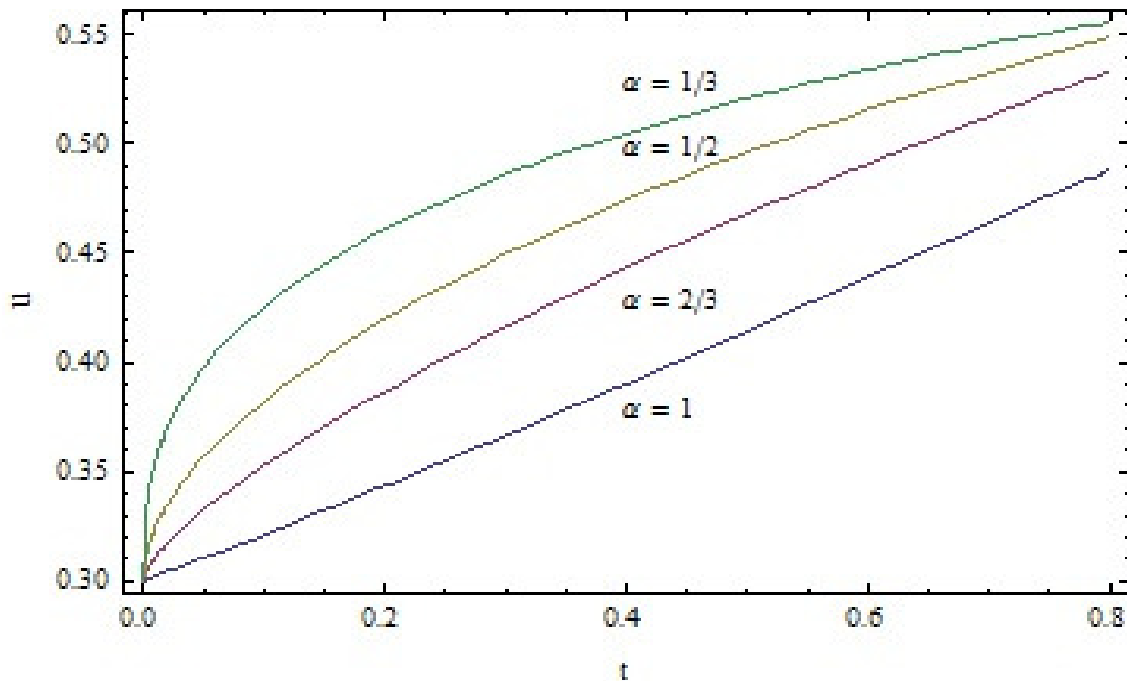


Figure 6. Approx. solution $u(x,t)$ for $x = 0.5, \lambda = 0.3$. (color figure can be viewed in the online issue)

Figures 11 and 12 are plotted for approximate solution of generalized time-fractional Fisher equation found in Application 3. Finally, the solution surfaces of

time-fractional Fitzhugh–Nagumo equation are depicted in Figures 13 to 16. In Figures 17 and 18 the function $u(x,t)$ vs. x are plotted for different a and α .

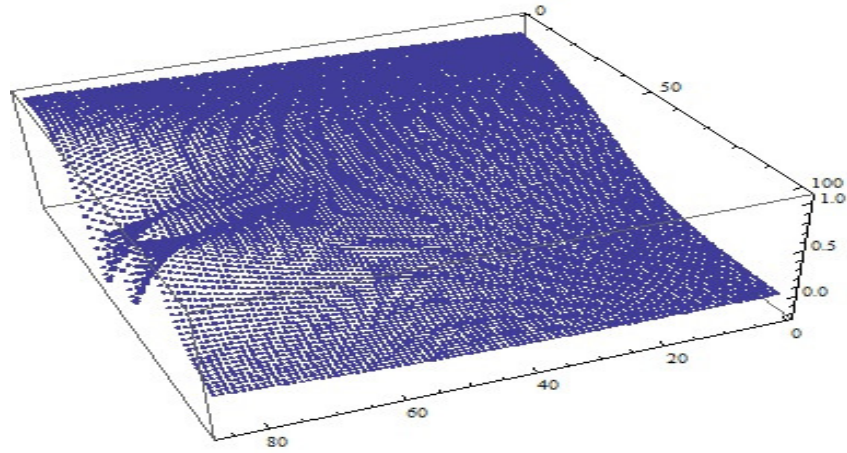


Figure 7. Approx. solution $u(x,t)$ for $\alpha = 1, A = 6$.

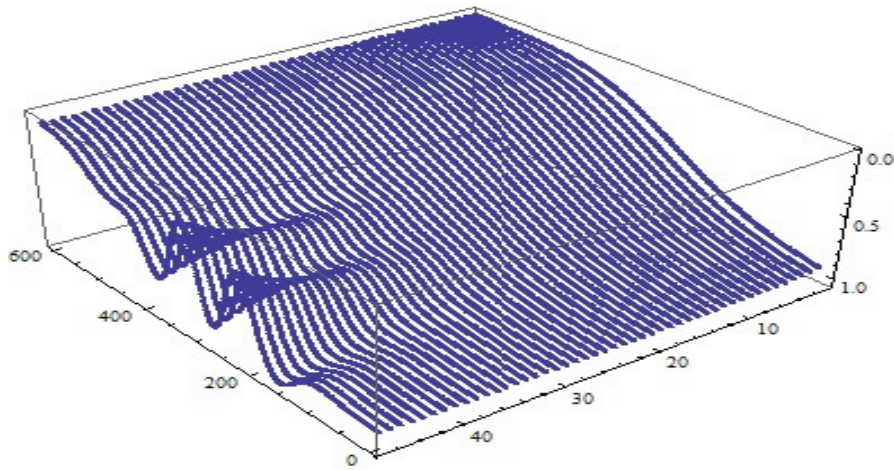


Figure 8. Approx. solution $u(x,t)$ for $\alpha = 0.6, A = 6$.

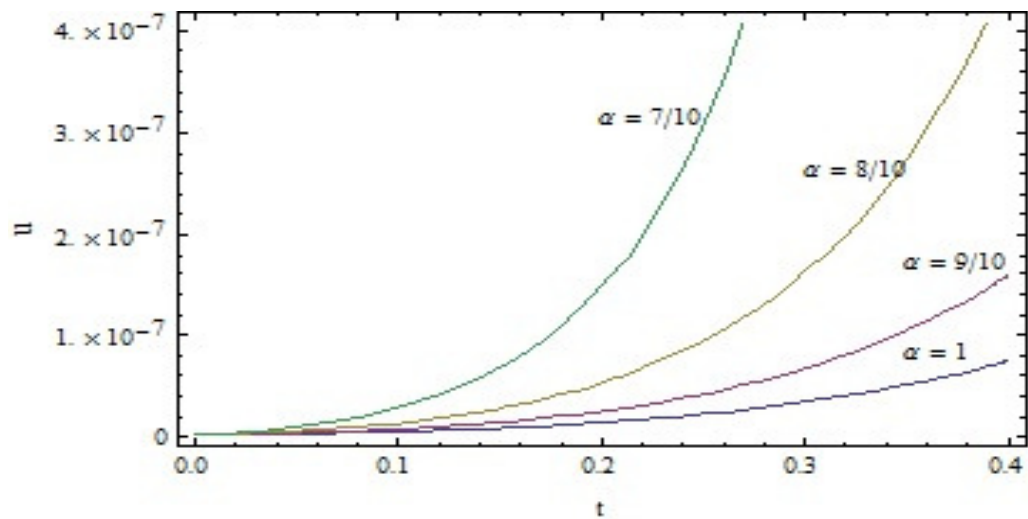


Figure 9. Approx. solution $u(x,t)$ for $x = 10, A = 6$. (Colored figure can be viewed in the online issue).

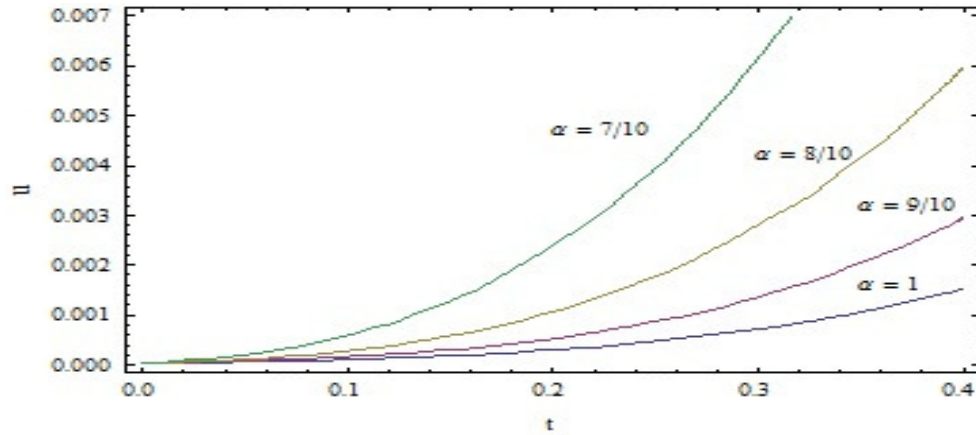


Figure 10. Approx. solution $u(x,t)$ for $x = 5, A = 6$. (color figure can be viewed in the online issue)

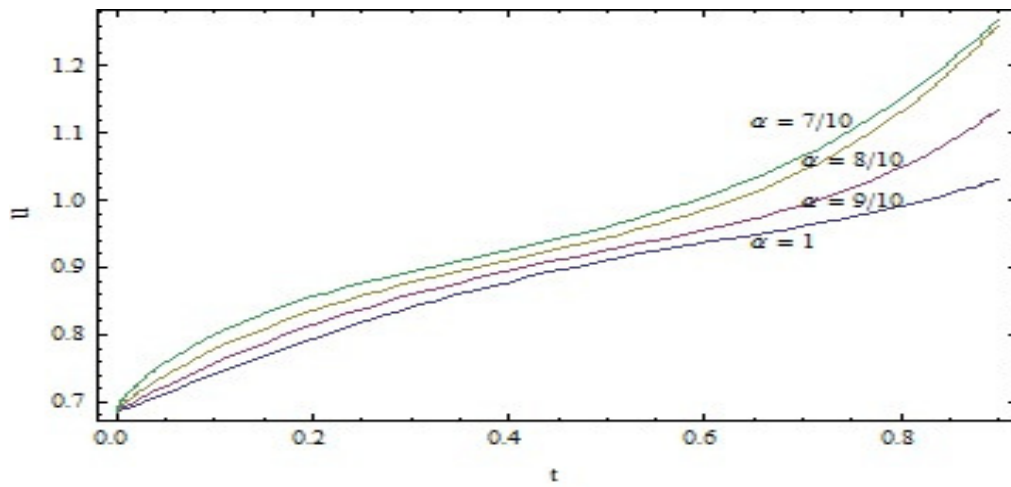


Figure 11. Approximate solution $u(x,t)$ for $x = 0.5$. (color figure can be viewed in the online issue)

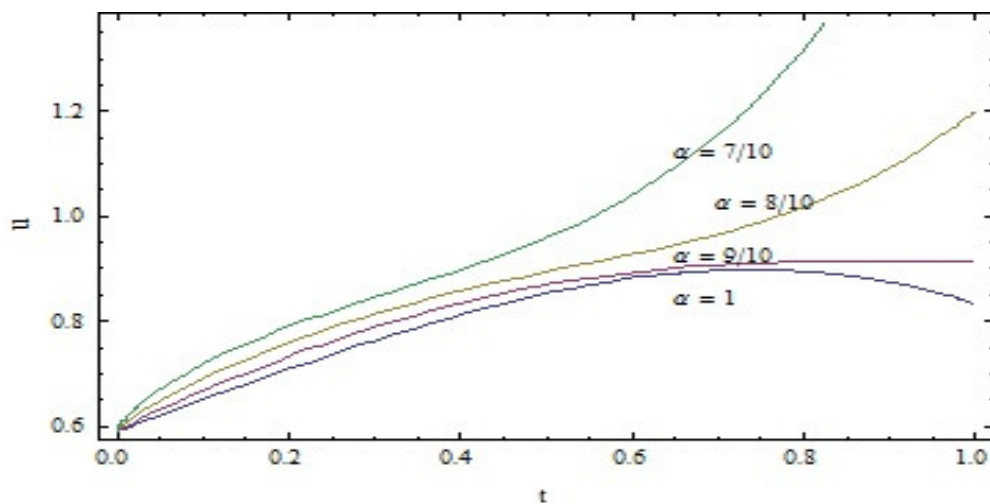


Figure 12. Approximate solution $u(x,t)$ for $x = 0.9$. (Colored figure can be viewed in the online issue).

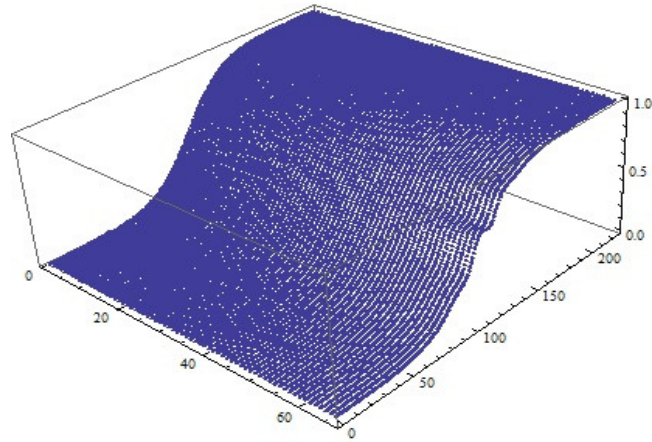


Figure 13. Approximate solution $a = 0.3, \alpha = 0.8$

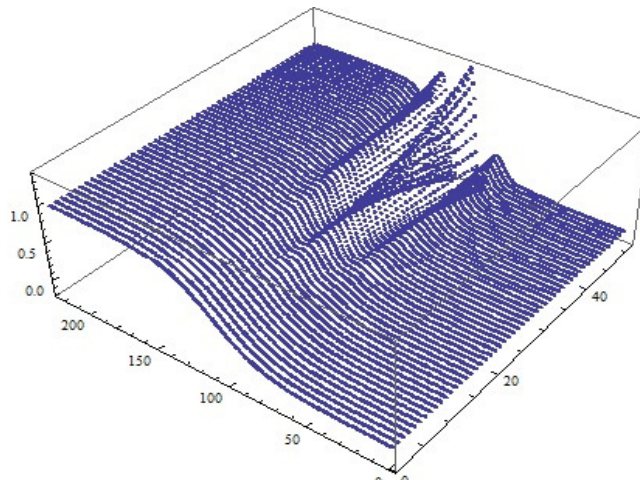


Figure 14. Approximate solution $a = 0.3, \alpha = 0.3$

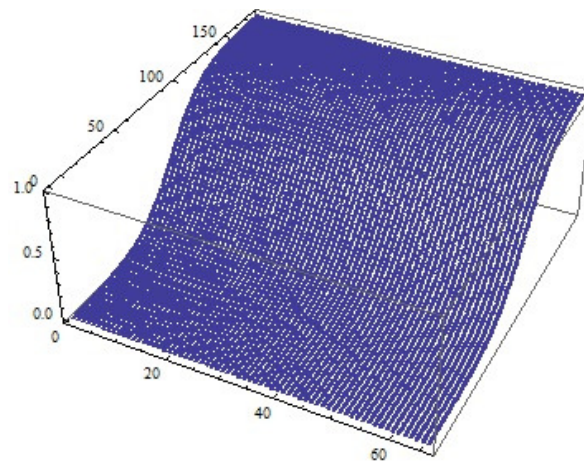


Figure 15. Approximate solution $a = 0.9, \alpha = 1$

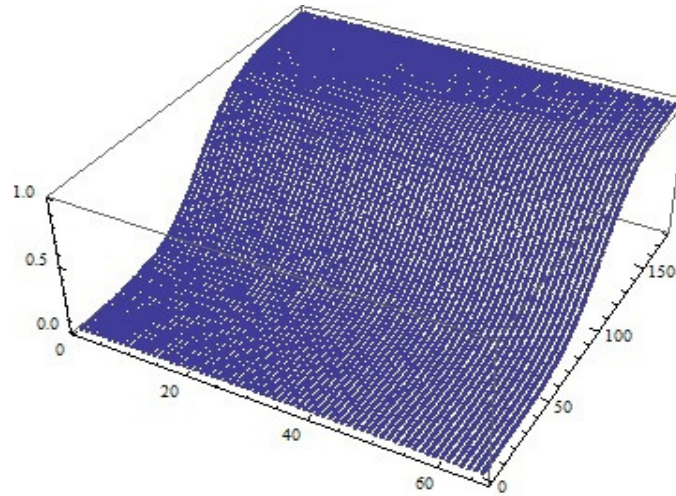


Figure 16. Exact solution at $a = 0.9, \alpha = 1$.

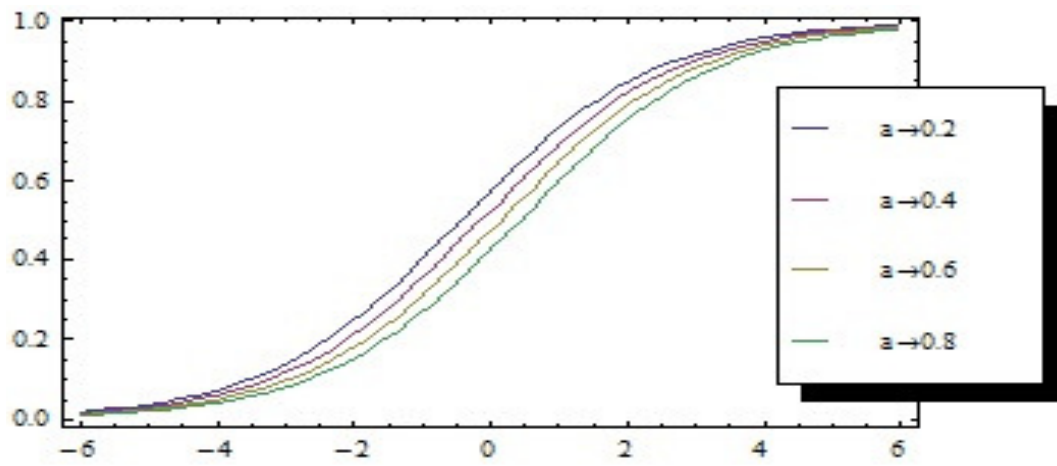


Figure 17. Approximate solution $u(x,t)$ for $t = 0.5, \alpha = 1$ (colored figure can be viewed in the online issue).

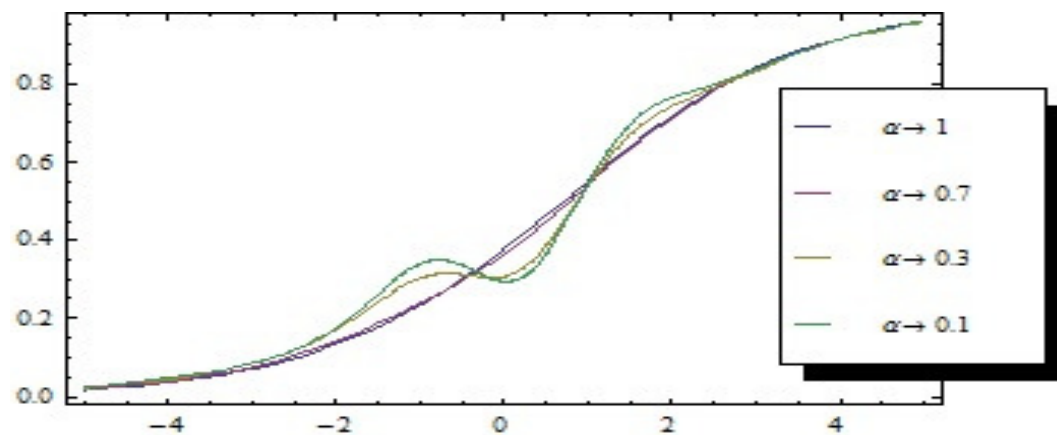


Figure 18. Approximate solution $u(x,t)$ for $t = 0.9, a = 0.9$ (colored figure can be viewed in the online issue).

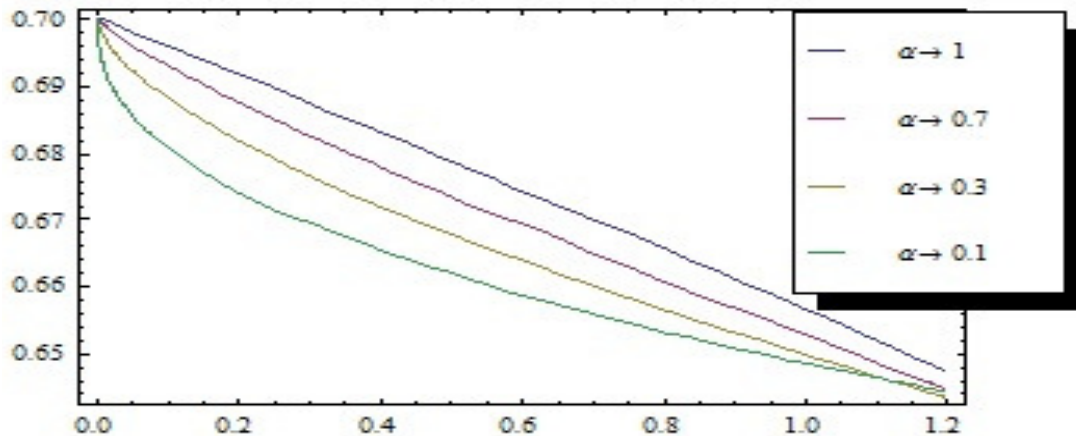


Figure 19. Approximate solution $u(x,t)$ for $x = 0.5$. $a=0.7$ (colored figure can be viewed in the online issue).

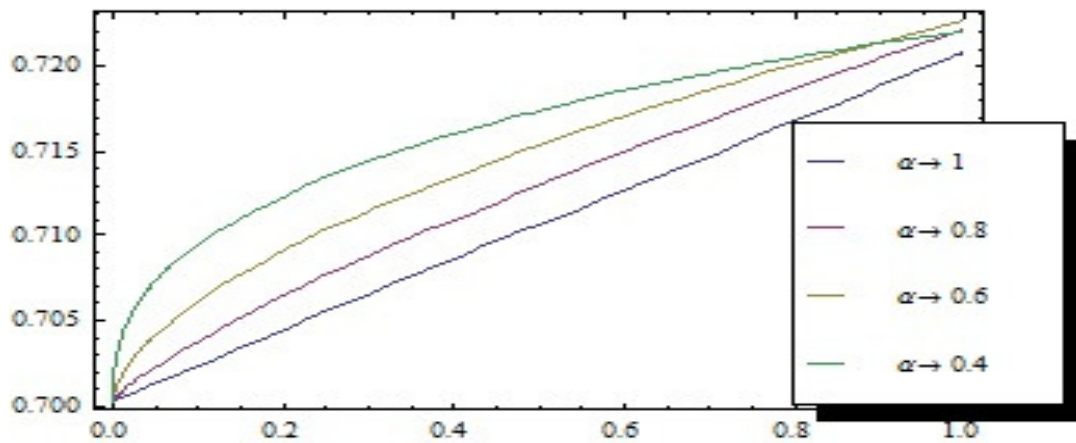


Figure 20. Approximate solution $u(x,t)$ for $x = 0.9$. $a=0.4$ (colored figure can be viewed in the online issue).

Table 1. $\lambda = 0..3$

t	$\alpha = .75$		$\alpha = 1$		Exact
	HPM	GDTM	HPM	GDTM	
0.1	0.548156474808	0.548156465921	0.5249791874789	0.5249791875000	0.5249791874789
0.2	0.580337900826	0.580337566847	0.5498340000000	0.5498339973125	0.5498339973124
0.3	0.607777641025	0.607774880331	0.5744425625000	0.5744425168154	0.5744425168116
0.4	0.632139529406	0.632139529406	0.5986880000000	0.59868766017636	0.5986876601124

Figure 19 and 20 are depicted for $a=0.7$ and $a=0.4$ respectively. It is interesting to observe that the function $u(x,t)$ of ordinary and generalized Fitzhugh–Nagumo equation shows a nonlinear behavior with respect to fractional parameter α in the sense that do not remain in phase for smaller values of time. However, the figures

are not plotted for different values of times.

Numerical comparison between GDTM (Rida et al., 2010) and HPM are found in Tables 1 to 3 which shows that HPM is more promising. It is also found that the results is in complete agreement with the result of HPM (Ağırseven and Ozis, 2010; Dehghan et al., 2010) and

Table 2. $A = 6, x = 2$

t	$\alpha = .75$		$\alpha = 1$		Exact
	GDTM	HPM	GDTM	HPM	
0.02	0.164993813210	0.024139537033	0.01726180555555	0.0169279385146	0.0169282151010
0.04	0.138147422907	0.035155268075	0.01734444444444	0.0201117284508	0.0201217246134
0.06	0.109336625176	0.048947313064	0.0168604166666	0.0238370103628	0.0238595181173
0.08	0.093578766505	0.066092858687	0.01575555555555	0.0281487518402	0.0282178229330

Table 3. $x = 0.5$

t	$\alpha = .75$		$\alpha = 1$		Exact
	GDTM	HPM	GDTM	HPM	
0.10	0.788135123734	0.788773986627	0.7410695562265	0.7412818056634	0.7412818057112
0.15	0.819608638370	0.822391378425	0.7675559304602	0.7681565182997	0.7681565210222
0.20	0.845635115405	0.863770823666	0.7931210778654	0.7937004785344	0.7937005259840
0.25	0.867856851346	0.951573461580	0.8171479502783	0.8176979303098	0.8176983602097

ADM (Gorguis and Wazwaz, 2004) for ordinary cases. A considerable advantage of the HPM is that the solutions are found very easily by using MATHEMATICA 7 without any transformation.

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