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Full Length Research Paper

Fast Fourier transform technique for the European option pricing with double jumps

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In this paper, we provided a fast algorithm for pricing European options under a double exponential jump-diffusion model based on Fourier transform. We derived a closed-form (CF) representation of the characteristic function of the model. By using fast Fourier transform (FFT) technique, we obtained an approximation numerical solution for the prices of European call options. Our numerical results show that our method is fast, accurate and easy to implement. The proposed option pricing method is useful for empirical analysis of asset returns and managing the corporate credit risks.

Key words: Fast Fourier transforms, characteristic function, double exponential jump diffusion, option pricing.

INTRODUCTION

The Fourier transform is a widely used and a well understood mathematical tool from Physics and Engineering disciplines applicable to numerous tasks, for example signal processing (Allen and Mills, 2004), or as a method for solving partial differential equations (Duffy, 2004). Inside the field of finance, the Fourier inversion method was first proposed in the Stein and Stein (1991) stochastic volatility model that uses the transform method in order to find the distribution of the underlying. Carr and Madan (1999) proposed Fourier transforms with respect to log-strike price; Bakshi and Madan (2000) provided an economic foundation for characteristic functions; Duffie et al. (2000) offered a comprehensive survey that the Fourier methods are applicable to a wide range of stochastic processes; Carr and Wu (2004) applied the transforms to time changed levy processes and the class of generalized affine models. Hurd et al., (2010) expressed the spread option payoff in terms of gamma function and fast Fourier transform (FFT) technique. For an overview of option pricing using Fourier transforms, see Martin (2010).

It has been noticed recently (Barndorff-Nielsen and Shephard 2006; Aigbedion et al., 2008; Maekawa et al., 2008; Akkurt et al., 2009) that sudden changes in the financial time series data are common. Many investors during the current global financial crisis encounter effects of jumps. It has been suggested from extensive empirical studies that markets tend to have both overreaction and under-reaction to various good or bad news. One may interpret the jump part of the model as the market response to outside news. More precisely, in the absence of outside news the asset price simply follows a geometric Brownian motion. Good or bad news arrives according to a Poisson process, and the asset price changes in response according to the jump size distribution. By adding jumps to the archetypal price process with Gaussian innovations, Merton (1976) was able to partly explain the observed deviations from the benchmark model which are characterized by fat tails and excess kurtosis in the returns distribution [for an overview of 'stylized facts' on asset returns see Cont (2001), statistical properties of implied volatilities are summarized in Cont et al., (2002)]. In the sequel also, other authors developed more realistic models based on jump processes (Eberlein and Keller, 1995; Madan et al., 1998; Kou, 2002).

However, they derived option values from an analytical form of the conditional density function, for the value of the underlying on maturity given in the initial state. Many of these original results are quite complicated requiring special functions or infinite summations which are difficult

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to be applied in real option pricing. This paper focus on pricing European options in a double exponential jump diffusion (DEJD) model recently proposed by Kou (2002). The purpose of this paper is to provide a fast, accurate and easy to implement numerical method for European options pricing by FFT technique. The main idea using the transform methods is to take an integral part of the payoff function over the probability distribution obtained by inverting the corresponding Fourier transform. Our contributions are that, we derived the characteristic function of the DEJD process, provided FFT of European call options pricing under the DEJD model and developed applied program codes in Matlab package. These results are important for option valuation and may also have implications in empirical analysis of assets return and managing the corporate credit risks.

PROBLEM DESCRIPTION

We consider an arbitrage-free and frictionless financial market where only riskless asset B and risky asset S are traded continuously up to a fixed horizon date T. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with a filtration satisfying the usual conditions, that is, the filtration is continuous on the right and $\mathcal{F}_0$ contains all p -null sets. The DEJD model assumes that the return process has two components, a continuous part modeled as Brownian motion, and a jump part with jump times driven by a Poisson process. According to Kou (2002), the following dynamic is proposed to model the asset price $S(t)$, under the physical probability measure p

$$\frac{dS(t)}{S(t\_)} = (r - \lambda z)dt + \sigma dW(t) + d\left[\sum_{j=1}^{N(t)} (V_j - 1)\right] \quad (1)$$

where interest rate r and the volatility σ are assumed to be constants, $W(t)$ is a standard Brownian motion which is $\mathcal{F}_t$ -adapted, $N(t)$ is a Poisson process with constant intensity λ and $V = \{V_j\}, j = 1, 2, \cdots$ is a sequence of independent identically distribute non-negative random variables, $z = E(V-1)$ and suppose $Y = \log V$ has an asymmetric double exponential distribution with the density:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{(y\geq 0)} + q\eta_2 e^{\eta_2 y}\mathbf{1}_{(y<0)},\ \eta_1 > 1,\ \eta_2 > 0, \quad (2)$$

where $\mathbf{1}$ denotes the indicator function, so $\mathbf{1}_{(y\geq 0)}$ equals 1 if $y \geq 0$, but 0 otherwise. $p,\ q \geq 0,\ p + q = 1$ is, respectively up-move jump and down-move jump. From (2), we can obtain

$$z = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

In the model, all sources of randomness, $N(t)$, $W(t)$ and Y_j are assumed to be independent.

Solving the stochastic different equation, (1) gives the dynamics of the asset price as follows:

$$S(t) = S(0)\exp\left[(r - \frac{1}{2}\sigma^2 - \lambda z)t + \sigma W(t)\right]\prod_{j=1}^{N(t)} V_j \quad . \quad (3)$$

DERIVING THE CHARACTERISTIC FUNCTION

To obtain FFT of European call options pricing, we need to compute the characteristic function of the DEJD process. The characteristic function of the process (3) is defined as:

$$\phi(u) = E\left[e^{iu\ln S(t)}\right]. \quad (4)$$

where i is the imaginary unit. We now compute the characteristic function of the process (3).

Theorem

Let $X_T = \ln S(T)$, suppose the asset price $S(t)$ follows (3), we have:

$$\phi(u) = e^{iuX_0 + \left[iu\left(r - \frac{1}{2}\sigma^2 - \lambda z\right) - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1\right)\right]T} \quad (5)$$

Proof: From (3), we have:

$$X_T = X_0 + \left(r - \frac{1}{2}\sigma^2 - \lambda z\right)T + \sigma\int_0^T dW(s) + \sum_{j=1}^{N(T)} Y_j,$$

where $W(t)$ is a standard Brownian motion satisfying $W(t) \sim N(0,t)$. The characteristic function of (3) can be transformed into the following equal.

$$\phi(u) = e^{iuX_0 + iu\left(r - \frac{1}{2}\sigma^2 - \lambda z\right)T} \cdot E[e^{i\sigma uW(T)}] \cdot E\left[\exp\left(iu\sum_{j=1}^{N(T)} Y_j\right)\right] \quad (6)$$

Because of $W(t) \sim N(0,t)$, we can obtain

$$E\left(e^{\sigma iuW(T)}\right) = e^{-\frac{1}{2}\sigma^2 u^2 T} \quad (7)$$

From (2), we can obtain

$$Ee^{iuY_i} = \int_{-\infty}^0 q\eta_2 e^{-\eta_2 y} e^{iuY_i} dy + \int_0^\infty p\eta_1 e^{-\eta_1 y} e^{iuY_i} dy = \frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1. \quad (8)$$

Because Y_j is a sequence of independent identically distribute random variables, we can obtain:

$$E\left[\exp\left(iu\sum_{j=1}^{N(T)} Y_j\right)\right] = E\left(e^{iuY_1} \cdot e^{iuY_2} \cdots e^{iuY_{N_T}}\right)$$
$$= Ee^{iuY_1} \cdot Ee^{iuY_2} \cdots Ee^{iuY_{N_T}}$$
$$= \lambda T\left(\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1\right). \quad (9)$$

Put (7) and (9) into (6), we can obtain our theorem.

A FAST ALGORITHM FOR EUROPEAN OPTION PRICING BASED ON FAST FOURIER TRANSFORM (FFT)

Let K be the strike price and T the expiration of a European call option with terminal asset price $S(T)$, where $S(T)$ is governed by the dynamics (3). By using the rational expectations argument with a hyperbolic absolute risk aversion (HARA) type utility function for the representative agent, one can choose a particular risk-neutral measure *Q*, under which the price of a European call option is computed as the discounted risk-neutral expectation of the terminal payoff $(S(T)-K)^+ = \max(S(T)-K,0)$:

$$C(t,S(T)) = e^{-r(T-t)}E(S(T)-K)^+, \quad (10)$$

Assume that for simplicity (without loss of "generality") we let $t=0$ and define $k=\ln K$. Furthermore, we express the call option pricing function (10) as a function of the log strike K rather than the terminal log asset price X_T :

$$C_T(k) = \int_k^{\infty} e^{-rT}\left(e^{X_T} - e^k\right) q_T(X_T)\, dX_T, \quad (11)$$

where $q_T(X_T)$ is the density of the process X_T.

Unfortunately, the call price function (11) is not square-integrable because $C_T(k)$ converges to $S(0)$ for $k \to -\infty$. Hence, Carr and Madan (1999) introduced a new technique with the key idea to calculate the Fourier transform of a modified call option price with respect to the logarithmic strike price. With this specification and a FFT routine, a whole range of option prices can be obtained within a single Fourier inversion. In this section, we develop the numerical solutions of the prices by using the idea of Carr and Madan (1999).

Fourier transforms of in-the-money option prices

By introducing an exponential damping factor $e^{\alpha k}$ with $\alpha > 0$, it is possible to make the integrand in (11) square-integrable; we modified the pricing function (11) by $c_T(k) = e^{\alpha k} C_T(k)$. The European call price can be easily recovered by applying the FFT:

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi}\int_0^{\infty} \mathrm{Re}[e^{-ivk}\psi_T(v)]dv, \quad (12)$$

where
$$\psi_T(v) = \frac{e^{-rT}\phi_T\left(v-(\alpha+1)i\right)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}.$$
In this instance, an efficient implementation of the FFT requires a closed form representation of the characteristic function $\phi(u)$. We have seen that the asset price dynamics (3) does indeed have an analytical characteristic function (5).

This method is viable when α is chosen in a way that the damped option price is well behaved. Damping the option price with $e^{\alpha k}$ makes it integrable for the negative axis $k<0$. On the other hand, for $k>0$, the option prices increase by the exponential $e^{\alpha k}$ which influences the integrability for the positive axis. A sufficient condition of (11) to be integrable for both sides (square integrability) is given by $\psi_T(0)$ being finite, that is, $\phi_T(-(\alpha+1)i) < \infty.$ which is equivalent to:

$$\phi_T(-(\alpha+1)i) = \mathrm{E}[S(T)^{1+\alpha}] < \infty. \quad (13)$$

Therefore, $c_T(k)$ is well behaved when the moments of order $1+\alpha$ of the underlying exist and are finite. If not all moments of $S(T)$ exist, this will impose an upper bound on α.

Using the trapezoid rule for the integral on the right-hand side of (12) and setting $v_j = \eta(j-1),$ an approximation for $C_T(k)$ is:

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi}\sum_{j=1}^{N} e^{-iv_j k}\psi_T(v_j)\eta. \quad (14)$$

The FFT returns N values of k and we employ a regular spacing of size h, so that our values for k are

$$k_u = -b + h(u-1) \quad \text{for } u = 1,2,\cdots,N. \quad (15)$$

This gives us log strike levels ranging from $-b$ to b, where,

$$b = \frac{1}{2}Nh. \quad (16)$$

In order to apply FFT we define:

$$\eta h = \frac{2\pi}{N}. \quad (17)$$

To obtain an accurate integration with larger values of η, we incorporate Simpson's rule weightings into our summation. From (12) to (17) and Simpson's rule weightings, we obtain European call option value as:

$$C(k_u) = \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(i-1)} e^{ibv_j} \psi(v_j)$$
$$\frac{\eta}{3}[3 + (-1)^j - \omega_{j-1}], \quad (18)$$

where ω_j is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The summation in (18) is an exact application of the FFT.

Fourier transforms of out-of-the-money option prices

For very short maturities, the call value approaches to its non-analytic intrinsic value causes the integrand in the Fourier inversion oscillate high, and therefore difficult to numerically integrate. We introduce an alternative approach that works with time values only, which is quite similar to the previous approach. But in this case, the call price is obtained via the Fourier transform of a modified time value, where the modification involves a hyperbolic sine function instead of an exponential function.

Let $z_T(k)$ denote the time value of an out-of-the-money option, that is, for $k < X_T$ we have the put price for $z_T(k)$ and for $k > X_T$ we have the call price. Scaling $S(0) = 1$ for simplicity, $z_T(k)$ is defined by:

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} [(e^k - e^{X_T}) \mathbf{1}_{X_T < k, k < 0}$$
$$+ (e^{X_T} - e^k) \mathbf{1}_{X_T > k, k > 0}] q_T(X_T) dX_T \quad (19)$$

Let $\zeta_T(u)$ be the Fourier transform of $z_T(k)$

$$\zeta_T(u) = \int_{-\infty}^{\infty} e^{iuk} z_T(k) dk, \quad (20)$$

Considering a damping function $\sinh(\alpha k)$, the time value of an option follows a Fourier inversion,

$$z_T(k) = \frac{1}{\pi \sinh(\alpha k)} \int_0^{\infty} e^{-ivk} \Upsilon_T(v) dv, \quad (21)$$

Where,

$$\Upsilon_T(v) = \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2}$$

The use of the FFT for calculating out-of-the-money option prices is similar to (18). The only differences are that they replace the multiplication by $e^{-\alpha k_u}$ with a division by $\sinh(\alpha k)$ and the function call to $\psi_T(v)$ is replaced by a function call to $\Upsilon_T(v)$.

NUMERICAL RESULTS AND DISCUSSION

In this section, we used 128 European call option levels at four combinations of parameter settings and compared the accuracy and speed with that of the FFT technique and closed-form (CF) formulae derived by Kou (2002). Throughout this section, we shall use the default parameters:

$S = 100, r = 0.05, p = 0.6, \sigma = 0.3, T = 1.$ For our FFT methods, we used $N = 4096$ points in our quadrature, implying a log strike spacing of $h = \frac{\pi}{300} = 0.01047$ which is adequate for practice. For the choice of the dampening coefficient in the transform of the modified call price, we used a value of $\alpha = 2.74$. Numerical outcomes are listed in Table 1 and applied program codes in Matlab package are presented in the Appendix.

Our numerical experiment shows that the FFT approach is considerably faster than the CF method of Kou (2002). The FFT takes about 2 s to produce 128 option prices at four combinations of parameter settings, while the CF method of Kou takes about 100 s.

Furthermore, Table 1 compared the pricing accuracy between the two methods across a range of strike prices. If we consider the CF method to be the benchmark, the relative percentage differences are all less than 0.17%.

In addition, the program of the FFT is very simple. It requests only a dozen lines of code in Matlab, while the one of the CF method is more complicated and requests about seventy lines. Our numerical example confirms that our FFT numerical solution is correct. It also illustrates how much more efficient this technique is.

CONCLUSIONS

In the present paper, we derived a CF representation of the characteristic function of the DEJD process. By using FFT, we obtained fast, accurate and easy to implement numerical solution to European call options under the DEJD model. It should be noted that this study has examined only European call options pricing. As for put options pricing, one can obtain easily corresponding result by the put-call parity (Black and Scholes, 1973).

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Table 1. Accuracy check and the FFT method versus CF solution.

		$\eta_1=\eta_2=20.0$		$\eta_1=\eta_2=40.0$	
		Price FFT	**Price CF**	**Price FFT**	**Price CF**
$K=90$	$\lambda=1.0$	19.9548	19.9548	19.7603	19.7633
	$\lambda=3.0$	20.4646	20.4569	19.8932	19.8941
	$\lambda=5.0$	20.9581	20.9431	20.0247	20.0237
$K=100$	$\lambda=5.0$	14.5191	14.5393	14.2874	14.3099
	$\lambda=3.0$	15.1203	15.1348	14.4447	14.4657
	$\lambda=5.0$	15.6960	15.7051	14.6000	14.6196
$K=110$	$\lambda=1.0$	10.3318	10.3485	10.0857	10.1033
	$\lambda=3.0$	10.9677	10.9817	10.2511	10.2681
	$\lambda=5.0$	11.5753	11.5867	10.4144	10.4307

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APPENDIX

Matlab codes for in-the-money the options pricing by FFT

```
function CallValue
=inDexpJ(ata1,ata2,lamta,sigma,r,p,s0,strike,T)
%sigma = volatility
%T = maturity
%r = interest rate
%s0 = initial asset price
x0 = log(s0);
alpha = 2.74;
N= 4096;
c = 600;
eta = c/N;
b =pi/eta;
u = [0:N-1]*eta;
lamda = 2*b/N;
position = (log(strike) + b)/lamda + 1;  %position of call
v = u - (alpha+1)*i;
A=i*v*x0;
k=p*ata1/(ata1-1)+(1-p)*ata2/(ata2+1)-1;
l=p*ata1./(ata1-i*v)+(1-p)*ata2./(ata2+i*v)-1;
B=T*(i*v*(r-0.5*sigma^2-lamta*k)-
0.5*sigma^2*(v.^2)+lamta*l);
charFunc=exp(A+B);
ModifiedCharFunc = charFunc*exp(-r*T)./(alpha^2 ...
+ alpha - u.^2 + i*(2*alpha +1)*u);
SimpsonW = 1/3*(3 + (-1).^[1:N] - [1, zeros(1,N-1)]);
FftFunc = exp(i*b*u).*ModifiedCharFunc*eta.*SimpsonW;
payoff = real(fft(FftFunc));
CallValueM = exp(-log(strike)*alpha)*payoff/pi;
format short;
CallValue = CallValueM(round(position));
```

Matlab codes for out-of-the-money the options pricing by FFT

```
function CValue =outDexpJ(ata1,ata2,lamta,sigma,r
,p,s0,strike,T)
x0 = log(s0);
alpha = 2.74;
N= 4096;
c = 600;
eta = c/N;
b =pi/eta;
u = [0:N-1]*eta;
lamda = 2*b/N;
position = (log(strike) + b)/lamda + 1; %position of call
w1 = u-i*alpha;
w2 = u+i*alpha;
v1 = u-i*alpha -i;
v2 = u+i*alpha -i;
k=p*ata1/(ata1-1)+(1-p)*ata2/(ata2+1)-1;
A1=i*v1*x0;
l1=p*ata1./(ata1-i*v1)+(1-p)*ata2./(ata2+i*v1)-1;
B1=T*(i*v1*(r-0.5*sigma^2-lamta*k)-
0.5*sigma^2*(v1.^2)+lamta*l1);
charFunc1=exp(A1+B1);
ModifiedCharFunc1 = exp(-r*T)*(1./(1+i*w1) - ...
exp(r*T)./(i*w1) - charFunc1./(w1.^2 - i*w1));
A2=i*v2*x0;
l2=p*ata1./(ata1-i*v2)+(1-p)*ata2./(ata2+i*v2)-1;
B2=T*(i*v2*(r-0.5*sigma^2-lamta*k)-
0.5*sigma^2*(v2.^2)+lamta*l2);
charFunc2=exp(A2+B2);
ModifiedCharFunc2 = exp(-r*T)*(1./(1+i*w2) - ...
exp(r*T)./(i*w2) - charFunc2./(w2.^2 - i*w2));
ModifiedCharFuncCombo = (ModifiedCharFunc1 - ...
ModifiedCharFunc2)/2 ;
SimpsonW = 1/3*(3 + (-1).^[1:N] - [1, zeros(1,N-1)]);
FftFunc = exp(i*b*u).*ModifiedCharFuncCombo*eta.*...
SimpsonW;
payoff = real(fft(FftFunc));
CallValueM = payoff/pi/sinh(alpha*log(strike));
format short;
CValue = CallValueM(round(position));
```