

Full Length Research Paper

Application of iteration perturbation method for nonlinear oscillators with discontinuities

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In this paper, we have applied He's iteration perturbation method for the first time to solve nonlinear oscillators with discontinuities. Three practical examples are explained and introduced. Comparing with exact solutions, just one iteration leads us to high accuracy of solutions which are valid for a wide range of vibration amplitudes as indicated in the following examples.

Key words: Nonlinear oscillators, discontinuities, iteration perturbation method.

INTRODUCTION

Considerable attention has been paid to the study of the nonlinear equations, not only in all areas of physics, but also in applied mathematics, engineering, and other disciplines.

Generally, finding an analytical approximation for nonlinear problems is more difficult than the numerical solution. During the past few decades, several methods have been proposed for obtaining approximate solutions for various types of nonlinear equations (Bayat et al., 2010, 2011a, b, c, d, e; Pakar et al., 2011; Kimiaeifar et al., 2009).

The major concern of this paper is to assess excellent approximations to the exact solutions for the whole range of the oscillation amplitude, reducing the respective error of angular frequency in comparison with the iteration perturbation method. Application of the method to different fields of science and engineering has been discussed by various researchers (He, 2001; Özis and Yildirim, 2009; Rafei et al. 2007).

ITERATION PERTURBATION METHOD

Many researchers have devoted their attention to obtaining approximate solution of nonlinear equations in the form:

$$u'' + u + \mathcal{E}f(u, u') = 0, \quad (1)$$

Subject to the following initial conditions:

$$u(0) = A, \quad u'(0) = 0 \quad (2)$$

We rewrite Equation (1) in the following form:

$$u'' + u + \mathcal{E}u \cdot g(u, u') = 0, \quad (3)$$

Where $g(u, u') = f / u$.

We construct an iteration formula for the aforesaid equation:

$$u''_{n+1} + u_{n+1} + \mathcal{E}u_{n+1} \cdot g(u_n, u'_n) = 0, \quad (4)$$

Where we denote by u_n the n th approximate solution.

For nonlinear oscillation, Equation (4) is of Mathieu type. We will use the perturbation method to find approximately u_{n+1} . The technique is called iteration perturbation method.

APPLICATIONS

In order to assess the advantages and the accuracy of the iteration perturbation method we will consider the following examples.

Here, we will introduce a nonlinear oscillator with

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discontinuity in several different forms:

$$\frac{d^2 u}{dt^2} + h(u) + \beta \operatorname{sgn}(u)u = 0, \tag{5}$$

Or

$$\frac{d^2 u}{dt^2} + h(u) + \beta u |u| = 0, \tag{6}$$

With initial conditions

$$u(0) = A, \frac{du(0)}{dt} = 0 \tag{7}$$

In this work, we assume that $h(u)$ is in a polynomial form. The reason for this assumption is that the discontinuity equations found in the literature belong to this family. Since there are no small parameters in Equation (6) the traditional perturbation methods cannot be applied directly. In the following example, we assume a linear form $h(u)$.

Example 1

Let $h(u) = \alpha u$, in Equation (6)

We can rewrite Equation (6) in the following form:

$$u'' + \alpha u + \beta u |u| = 0 \tag{8}$$

To apply the iteration perturbation method, the solution is expanded and the series of \mathcal{E} is introduced as follows:

$$u = u_0 + \sum_{i=0}^n \mathcal{E}^i u_i \tag{9}$$

$$\alpha = \omega^2 + \sum_{i=0}^n \mathcal{E}^i a_i \tag{10}$$

$$\beta = \sum_{i=0}^n \mathcal{E}^i d_i \tag{11}$$

Substituting Equations (9), (10) and (11) into Equation (8) and equating the terms with the identical powers of \mathcal{E} , a series of linear equations are obtained. Expanding the first two linear terms becomes as follows:

$$\mathcal{E}^0 : \ddot{u}_0 + \omega^2 u_0 = 0, u_0(0) = A, \dot{u}_0(0) = 0 \tag{12}$$

$$\begin{aligned} \mathcal{E}^1 : u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 u_0 |u_0| &= 0, \\ u_1(0) = 0, \dot{u}_1(0) &= 0 \end{aligned} \tag{13}$$

Substituting the solution into Equation (12), for example, $u_0 = A \cos(\omega t)$, the differential equation for u_1 becomes:

$$\begin{aligned} u_1'' + \omega^2 u_1 + a_1 A \cos(\omega t) \\ + d_1 A \cos(\omega t) |A \cos(\omega t)| &= 0, \\ u_1(0) = 0, u_1'(0) &= 0 \end{aligned} \tag{14}$$

Note that the following Fourier series expansion is valid.

$$\begin{aligned} |A \cos(\omega t)| \cos(\omega t)^{2n-1} &= \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) \\ &= c_1 \cos(\omega t) + c_3 \cos(3\omega t) + \dots \end{aligned} \tag{15}$$

Where c_i can be determined by Fourier series, for example, Equation (16) in Equation (14) gives:

$$\begin{aligned} c_1 &= \frac{2}{\pi} \int_0^{\pi} |\cos(\omega t)|^{2n} \cos(\omega t) d(\omega t) \\ &= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} \cos(\omega t)^{2n+1} d(\omega t) \right. \\ &\quad \left. - \int_0^{\pi} \cos(\omega t)^{2n+1} d(\omega t) \right) \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \end{aligned} \tag{16}$$

$$\begin{aligned} u_1'' + \omega^2 u_1 + a_1 A \cos(\omega t) \\ + d_1 A \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) &= 0 \end{aligned} \tag{17}$$

Avoiding the presence of a secular term requires that

$$a_1 + d_1 c_1 A^2 = 0 \tag{18}$$

Also, substituting $\mathcal{E} = 1$, into Equations (9) and (10) gives:

$$\alpha = \omega^2 + a_1 \tag{19}$$

$$\beta = d_1 \tag{20}$$

From Equations (18), (19) and (20), the first-order approximation to the angular frequency is:

$$\omega = \sqrt{\alpha + \frac{8\mathcal{E}A}{3\pi}} \tag{21}$$

Case 1

If $\alpha = 1$, we have:

$$\omega = \sqrt{1 + \frac{8\varepsilon A}{3\pi}} \tag{22}$$

It is the same as that obtained by the Homotopy perturbation method and the Variational method (He, 2004; Tao, 2008).

Case 2

If $\alpha = 0$, we have

$$\omega = \sqrt{\frac{8\varepsilon A}{3\pi}} \tag{23}$$

The obtained frequency in Equation (23) is valid for the whole solution domain $0 < A < \infty$.

Example 2

If $h(u) = \alpha u^3$, in Equation (6).

Then we have

$$\frac{d^2 u}{dt^2} + \alpha u^3 + \beta u |u| = 0 \tag{24}$$

To apply the iteration perturbation method, the solution is expanded and the series of ε is introduced as follows:

$$u = u_0 + \sum_{i=0}^n \varepsilon^i u_i \tag{25}$$

$$0 = \omega^2 + \sum_{i=0}^n \varepsilon^i a_i \tag{26}$$

$$1 = \sum_{i=0}^n \varepsilon^i d_i \tag{27}$$

Substituting Equations (25), (26) and (27) into Equation (24) and equating the terms with the identical powers of ε , a series of linear equations are obtained. Expanding the first two linear terms becomes as follows:

$$\begin{aligned} \varepsilon^0 : \ddot{u}_0 + \omega^2 u_0 &= 0, \\ u_0(0) = A, \dot{u}_0(0) &= 0 \end{aligned} \tag{28}$$

$$\begin{aligned} \varepsilon^1 : u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 \alpha u_0^3 \\ + \beta u_0 |A \cos(\omega t)| = 0, \\ u_1(0) = 0, \dot{u}_1(0) = 0 \end{aligned} \tag{29}$$

Substituting the solution into Equation (28), for example, $u_0 = A \cos(\omega t)$, the differential equation for u_1

becomes:

$$\begin{aligned} u_1'' + \omega^2 u_1 + a_1 A \cos(\omega t) + d_1 \alpha A^3 \cos^3(\omega t) \\ + \beta A \cos(\omega t) |A \cos(\omega t)| = 0 \end{aligned} \tag{30}$$

We have the following identity:

$$\cos^3(\omega t) = \frac{3}{4} \cos(\omega t) + \frac{1}{4} \cos(3\omega t) \tag{31}$$

Note that the following Fourier series expansion is valid.

$$\begin{aligned} |A \cos(\omega t)|^{2n-1} \cos(\omega t) &= \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) \\ &= c_1 \cos(\omega t) + c_3 \cos(3\omega t) + \dots \end{aligned} \tag{32}$$

c_i can be determined by Fourier series, for example :

$$\begin{aligned} c_1 &= \frac{2}{\pi} \int_0^{\pi} |\cos(\omega t)|^{2n-1} \cos(\omega t) d(\omega t) \\ &= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} \cos(\omega t)^{2n+1} d(\omega t) \right. \\ &\quad \left. - \int_0^{\pi} \cos(\omega t)^{2n+1} d(\omega t) \right) \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \end{aligned} \tag{33}$$

By means of Equations (31), (32) and (33) we find that:

$$\begin{aligned} u_1'' + \omega^2 u_1 + (a_1 + d_1 A^2 \frac{3}{4}) A \cos(\omega t) \\ + d_1 A^3 \frac{1}{4} \cos(3\omega t) \\ + A^2 \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) = 0 \end{aligned} \tag{34}$$

No secular term in u_1 requires that

$$a_1 + d_1 \alpha A^2 \frac{3}{4} + \beta A \frac{8}{3\pi} = 0 \tag{35}$$

Also, substituting $\varepsilon = 1$, into Equations (26) and (27) gives:

$$0 = \omega^2 + a_1 + \dots \tag{36}$$

$$1 = d_1 \tag{37}$$

From Equations (35), (36) and (37), the first-order approximation to the angular frequency is:

$$\omega = \sqrt{\frac{3\alpha A^2}{4} + \frac{8\beta A}{3\pi}} \tag{38}$$

Case 1

If $\alpha = \beta, \beta = \varepsilon$ we have:

$$\omega = \sqrt{\frac{3\beta A^2}{4} + \frac{8\varepsilon A}{3\pi}} \tag{39}$$

This agrees well with that obtained by the Homotopy perturbation method and the variational method (He, 2004; Tao, 2008). And its period is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{3\beta A^2}{4} + \frac{8\varepsilon A}{3\pi}}} \tag{40}$$

Case 2

If $\varepsilon = 0$, its period can be written as:

$$T = \frac{4\pi}{\sqrt{3}} \beta^{-\frac{1}{2}} A^{-1} \tag{41}$$

The exact period was obtained by Acton and Squire (1985).

$$T_{ex} = 7.4164 \beta^{-\frac{1}{2}} A^{-1} \tag{42}$$

The maximal relative error is less than 2.2% for all $\beta > 0!$

Example 3

Let $h(u) = 0$, in Equation (6)

By setting $\alpha = 0, \beta = 1$, Equation (5) becomes:

$$u'' + \text{sgn}(u) = 0 \tag{43}$$

or

$$u'' + |u| = 0 \tag{44}$$

We can rewrite Equation (44) to the following form:

$$u'' + 0.u + 1.|u| = 0 \tag{45}$$

To apply the iteration perturbation method, the solution is expanded and the series of ε is introduced as follows:

$$u = u_0 + \sum_{i=0}^n \varepsilon^i u_i \tag{46}$$

$$0 = \omega^2 + \sum_{i=0}^n \varepsilon^i a_i \tag{47}$$

$$1 = \sum_{i=0}^n \varepsilon^i d_i \tag{48}$$

Substituting Equations (46), (47) and (48) into Equation (45) and equating the terms with the identical powers of ε , a series of linear equations are obtained. Expanding the first two linear terms becomes as follows:

$$\varepsilon^0 : \ddot{u}_0 + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad \dot{u}_0(0) = 0 \tag{49}$$

$$\varepsilon^1 : u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 u_0 |u_0| (u_0)^{-1} = 0, \tag{50}$$

$$u_1(0) = 0, \quad \dot{u}_1(0) = 0$$

Substituting the solution into Equation (49), for example, $u_0 = A \cos(\omega t)$, the differential equation for u_1 becomes:

$$u_1'' + \omega^2 u_1 + a_1 u_0 + d_1 A \cos(\omega t) |A \cos(\omega t)| (A \cos(\omega t))^{-1} = 0, \tag{51}$$

$$u_1(0) = 0, u_1'(0) = 0$$

Note that the following Fourier series expansion is valid.

$$\cos(\omega t)^{2n-1} |\cos(\omega t)| = \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) \tag{52}$$

$$= c_1 \cos(\omega t) + c_3 \cos(3\omega t) + \dots$$

Where c_i can be determined by Fourier series, in Equation (52), $n = 0$

$$c_1 = \frac{4}{\pi} \tag{53}$$

We rewriting Equation (50) in the following form:

$$u_1'' + \omega^2 u_1 + a_1 A \cos(\omega t) + d_1 \sum_{k=0}^{\infty} c_{2k+1} \cos((2k+1)\omega t) = 0 \tag{54}$$

Avoiding the presence of a secular term requires that

$$a_1 + \frac{d_1 c_1}{A} = 0 \tag{55}$$

Also, substituting $\varepsilon = 1$, into Equations (47) and (48) gives:

$$0 = \omega^2 + a_1 \quad (56)$$

$$1 = d_1 \quad (57)$$

From Equations (55), (56) and (57), the first-order approximation to the angular frequency is:

$$\omega = \frac{2}{\sqrt{\pi A}} \quad (58)$$

Tao (2008) obtained the same result.

CONCLUSION

In this work, iteration perturbation method has been successfully used to obtain approximate frequencies for nonlinear discontinuity equations. Three examples have been considered in this paper. The comparison of the results obtained by iteration perturbation method with the exact one show that the first iteration of the method led to an excellent solution for the nonlinear oscillators with discontinuities. In general, we conclude that this method is efficient for calculating periodic solutions for nonlinear oscillatory systems, and the author suggest the iteration perturbation method as a powerful method and has a great potential and could be applied to other strong nonlinear oscillators.

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