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Analytical and numerical solution of heat generation and conduction equation in relaxation mode: Laplace transforms approach

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In this article analytical solution of one-dimensional heat equation in relaxation mode of heat generation and conduction using Laplace transforms method is presented. The model adopted takes into account finite velocity of heat propagation, and relaxation of heat source capacity. The properties of heat source terms in four different cases are incorporated in the model and investigated. Temperature distributions and variations with conduction mode and relaxation time are analyzed. High relaxation time is observed to lowers the temperature profile, whereas enhanced temperature distribution changes at particular values of α, and τ_g for source capacity proportional to temperature. How the steady state solution is achieved for some selected values of coefficients is also discussed.

Key words: Relaxation time, conduction mode, pulsed heat source, Laplace transforms.

INTRODUCTION

Cattaneo was the first to build an explicit mathematical theory to correct unacceptable properties of Fourier theory of heat diffusion. The arguments used were based on the kinetic theory of gases and second-order correction to propose modification of Fourier law (Cattaneo, 1948), which gives rise to the well-known hyperbolic model of heat conduction. This also leads to suitable heat conduction models that permit the finite speed of heat flow (Ozisik and Tzou, 1994; Joseph and Preziosi, 1990). In most studies of heat propagation in systems the hyperbolic model of heat conduction is used (Jose and Juan, 2011; Al-Nimr et al., 2004; Malinowski, 1993a, Saleh and Al-Nimr, 2008; Cai et al., 2006). For instance in (Malinowski, 1993b) the analytical solutions for the relaxation equation in bodies with low heat resistance, by neglecting temperature gradient were presented. It is shown that differences between parabolic and relaxation solution fluctuate as time elapses. Differences in heat generation and conduction were reported (Lewandowska, 2001) to arise due to the time characteristics of the heat source capacity. For example when the heat is of constant strength this differences slowly decrease for long times. Furthermore, solutions of both hyperbolic and parabolic heat conduction equation for temperature dependent heat source is reported to be use in analyzing normal zones in superconductors

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(Lewandowska and Malinowski, 2002), in which the amount of energy that is dissipated in the zone affect heat production by the heat source capacity which depends on temperature.

Although a lot of works has been done on the hyperbolic and parabolic heat conduction equation under different conditions, yet nobody as far as we know investigate the solutions for one-dimensional relaxation model of heat conduction taking into account the finite velocity of heat propagation, and relaxation of heat source capacity. Matlab program is one of the robust and most widely used program in areas of science and technology (Hübner et al., 2011). Also has many application in script design (Valipour et al., 2012), and in model development (Mohammad and Ali, 2012a; Mohammad et al., 2013). For example in irrigation engineering (Mohammad and Ali, 2012b) used the genetic coding in Matlab environment to determine the effective infiltration parameters in Furrow Irrigation. We use in this paper Matlab environment to write a script to compute the temperature profile in physical domain.

MODEL

By using the modified Fourier law Equation (1), which physically agree for a very short laser pulses and non-infinite speed of heat transport.

$$t_k \frac{\partial q}{\partial t} = -k\nabla\Theta - q, \quad (1)$$

Where t_k, k, q and Θ are the relaxation time of the heat flux, thermal conductivity, heat flux vector, and temperature respectively. Hyperbolic equation of heat conduction is obtained by substitution of Equation (1) into the energy conservation equation.

$$\rho c_p \frac{\partial \Theta}{\partial t} = -\nabla . q + g, \quad (2)$$

In Equation (2) ρ is the density; c_p is specific heat at constant pressure, and g is the capacity of the internal heat source. In this paper we adopt the notion of inert heat source and transient capacity of heat source g_t as seen in Equation (3).

$$t_g \frac{\partial g_t}{\partial t} + g_t = g \quad (3)$$

For the relaxation heat conduction equation that account for both finite speed of heat propagation and the relaxation of heat source capacity Equation (4) is use.

$$t_k t_g \frac{\partial^3 \Theta}{\partial t^3} + (t_k + t_g)\frac{\partial^2 \Theta}{\partial t^2} + \frac{\partial \Theta}{\partial t} = t_k a \frac{\partial \nabla^2 \Theta}{\partial t} + \mu\nabla^2\Theta + \frac{1}{\rho c_p}\left(t_k \frac{\partial g}{\partial t} + g\right) \quad (4)$$

Where μ is thermal diffusivity, t_g is relaxation time of source capacity, the length of which depends on nature of the source. The dimensionless forms of Equations (1) to (3), are given below respectively as adopted in Lewandowska (2001), which is necessary to ensure temperature variation as a function of dimensionless displacement.

$$\frac{\partial \Phi}{\partial \tau} = -\nabla . \Phi + 2\Psi_t, \quad (5)$$

$$\frac{\partial \theta}{\partial \tau} + 2\theta = -\nabla\Phi, \quad (6)$$

$$\tau_g \frac{\partial \Psi_t}{\partial \tau} + \Psi_t = \Psi. \quad (7)$$

Transformation of Equations (5) to (7) yields the equation of heat conduction below, which permits a finite speed of heat propagation and relaxation of heat source capacity.

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\nabla^2\Phi + \nabla^2 + 2\frac{\partial \Psi}{\partial \tau} + 4\Psi \quad (8)$$

Where $\tau_g = \frac{t_g}{2t_k}$, and t_k is relaxation time due to delay of heat flux as a result of temperature gradient. Equation (8) is treated, considering the temperature gradient as a function of a dimensionless Cartesian co-ordinate, for $\frac{\partial^2 \Phi}{\partial Y^2} = \frac{\partial^2 \Phi}{\partial Z^2} = 0$. Thus, we obtained Equation (9).

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\left(\frac{\partial^2 \Phi}{\partial X^2}\right) + \frac{\partial^2 \Phi}{\partial X^2} + 2\frac{\partial \Psi}{\partial \tau} + 4\Psi. \quad (9)$$

Equation (9) can be reduced to the classical hyperbolic equation of heat conduction for one dimensional case, when the relaxation time of heat capacity is set to zero.

$$\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial X^2} + 2\frac{\partial \Psi}{\partial \tau} + 4\Psi. \quad (10)$$

We take into account the finite speed of heat propagation, relaxation of heat source capacity and heat conduction equations. We also consider temperature gradient to be a function of dimensionless displacement, and assumed high heat resistance.

MATHEMATICAL ANALYSIS

The boundary value problem of Equation (9) was solved after including four different source terms namely: (i) source with constant capacity, (ii) source capacity proportional to temperature, (iii) Dirac delta energy pulse, and (iv) source capacity proportional to time. This gives respectively,

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\left(\frac{\partial^2 \Phi}{\partial X^2}\right) + \frac{\partial^2 \Phi}{\partial X^2} + 2\frac{\partial \Psi}{\partial \tau} + 4\Psi_c \quad (11)$$

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\left(\frac{\partial^2 \Phi}{\partial X^2}\right) + \frac{\partial^2 \Phi}{\partial X^2} + 2\alpha \frac{\partial \Phi}{\partial \tau} + 4\alpha\Phi \quad (12)$$

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\left(\frac{\partial^2 \Phi}{\partial X^2}\right) + \frac{\partial^2 \Phi}{\partial X^2} + 2\beta \frac{\partial(\tau)}{\partial \tau} + 4\beta\,\partial(\tau) \quad (13)$$

$$\tau_g \frac{\partial^3 \Phi}{\partial \tau^3} + (2\tau_g + 1)\frac{\partial^2 \Phi}{\partial \tau^2} + 2\frac{\partial \Phi}{\partial \tau} = \tau_g \frac{\partial}{\partial \tau}\left(\frac{\partial^2 \Phi}{\partial X^2}\right) + \frac{\partial^2 \Phi}{\partial X^2} + 2\beta \frac{\partial(\sin\tau)}{\partial \tau} + 4\beta\sin(\tau) \quad (14)$$

By using the Laplace transforms technique, with boundary conditions $\Phi(X,0) = \frac{\partial \Phi}{\partial \tau}(X,0) = \frac{\partial^2 \Phi}{\partial \tau^2}(X,0) = 0$, for the four different source terms respectively, solutions of Equations (11) to (14) yields:

$$\frac{d^2\bar{\Phi}}{dX^2} - \frac{A(S)}{B(S)}\bar{\Phi} + \frac{4\Psi_c}{(\tau_g S^2 + S)} = 0 \quad (15)$$

$$\frac{d^2\bar{\Phi}}{dX^2} - \frac{A(S)}{B(S)}\bar{\Phi} + \frac{4\alpha}{B(S)S^2} = 0 \quad (16)$$

$$\frac{d^2\bar{\Phi}}{dX^2} - \frac{A(S)}{B(S)}\bar{\Phi} + \frac{2\beta}{B(S)}(S + 2) = 0 \quad (17)$$

$$\frac{d^2\bar{\Phi}}{dX^2} - \frac{A(S)}{B(S)}\bar{\Phi} + 2\beta(1 + 2\tau) = 0 \quad (18)$$

Where $A(S) = \tau_g S^3 + 2\tau_g S^2 + S^2 + 2S$, and $B(S) = \tau_g S + 1$ for Equations (11) to (14). However, in Equation (12), $A(S) = \tau_g S^3 + 2\tau_g S^2 + S^2 + 2S - 2\alpha S$. The α and β in Equations (16) to (18) are dimensionless coefficients that corresponding to the corresponding sources term. The solution of Equations (15) to (18) satisfying the conditions $\bar{\Phi} = 0$ for $x = 1$ and $x = 0$ is:

$$\bar{\Phi}(X,S) = K_i\left[\frac{\cosh\sqrt{\frac{A(S)}{B(S)}}}{\sinh\sqrt{\frac{A(S)}{B(S)}}} - \frac{1}{\sinh\sqrt{\frac{A(S)}{B(S)}}}\right]\sinh X\sqrt{\frac{A(S)}{B(S)}} - K_i\cosh X\sqrt{\frac{A(S)}{B(S)}} + K_i. \quad (19)$$

Where i = 1, 2, 3, and 4, $A(S)$ and $B(S)$ are as defined above with the same condition and

$$K_1 = \frac{4\Psi_c}{A(S)S} \quad (20)$$

$$K_2 = \frac{4\alpha}{A(S)S^2} \quad (21)$$

$$K_3 = \frac{2\beta(S+2)}{A(S)} \quad (22)$$

$$K_4 = \frac{2\beta}{A(S)S}\tau \quad (23)$$

The Equation (19) in Laplace transformed field is inverted for values of i = 1, 2, 3, and 4 in order to determine the temperature in physical time domain. Riemann-sum approximation (Basant and Clement, 2013) is used for the inversion of the sets of Equation (19). It involves a single summation for the numerical process. In this case the function in $\bar{\Phi}(X,S)$ is inverted to the time field.

$$\Phi(X,\tau) \cong \frac{e^{\epsilon\tau}}{\tau}\left[\frac{1}{2}\bar{\Phi}(X,\epsilon) + Re\sum_{n=i}^{N}\bar{\Phi}\left(X,\epsilon + \frac{in\pi}{\tau}\right)(-1)^n\right] \quad (24)$$

Where Re is the real part, i = $\sqrt{-1}$ is the imaginary number, N is the number of terms used in the Riemann-sum approximation. The accuracy of this method depends on the value of ϵ and the truncation error dictated by N. The ϵ is real part of Bromwich contour that is used in inverting Laplace transforms, its value must be selected so that the Bromwich contour encloses all the branch points (Tzou, 1997; Karniadakis and Beskok, 2002). For faster convergence, and reasonable results the quantity $\epsilon\tau$ should be approximately 4.7 (Vernotte, 1961). This shortens the computational time as compared to other tested values. The numerical solution is validated by considering steady state solution of Equation (25) for the first case, compared with the solution of Equation (15) for i=1 and the two results satisfy the boundary conditions x(0)= $\Phi(0)$, and x(1)= $\Phi(0)$. The pulsed energy source shows quasi-steady state behavior as the Dirac delta tends to unity. This indicates the flow is partly driven by buoyancy. This also agrees with source capacity that is proportional to time.

$$\frac{d^2\Phi}{dX^2} + 4\Psi_c = 0, \quad (25)$$

$$\frac{d^2\Phi}{dX^2} + 4\alpha\Phi = 0, \quad (26)$$

$$\frac{d^2\Phi}{dX^2} + 4\beta\delta = 0, \quad (27)$$

$$\frac{d^2\Phi}{dX^2} + 4\beta\tau = 0. \quad (28)$$

RESULT AND DISCUSSION

The results of the calculations are shown in (Figures 1 to 6). Figures 1 to 4 show the temperature profiles for the source terms that follow $\Psi = \Psi_c$, $\Psi = \alpha\Phi$, $\Psi = \beta\delta(\tau)$, and $\Psi = \beta\tau$. Using the solutions of Equations (11) to (14) for the four source terms, we write scripts that solve Equation (19) for i=1, 2, 3, and 4 by using MATLAB program in order to compute and generate the graphs. This is necessary in order to get a clear insight into the physics of the model. Different values of Ψ_c from 0.1 to 1 are used, while higher values in some cases enhance temperature profile distribution similar to the trend observed in the semi-infinite system with a time-dependent pulse energy source (Lewandowska, 2001). The resulting values of the temperature profiles $\Phi(X,\tau)$ are observed to increase for dimensionless temperature versus dimensionless time in conduction mode for the heat source capacity of constant strength. The values of τ_g are set to 1, 3, and 6

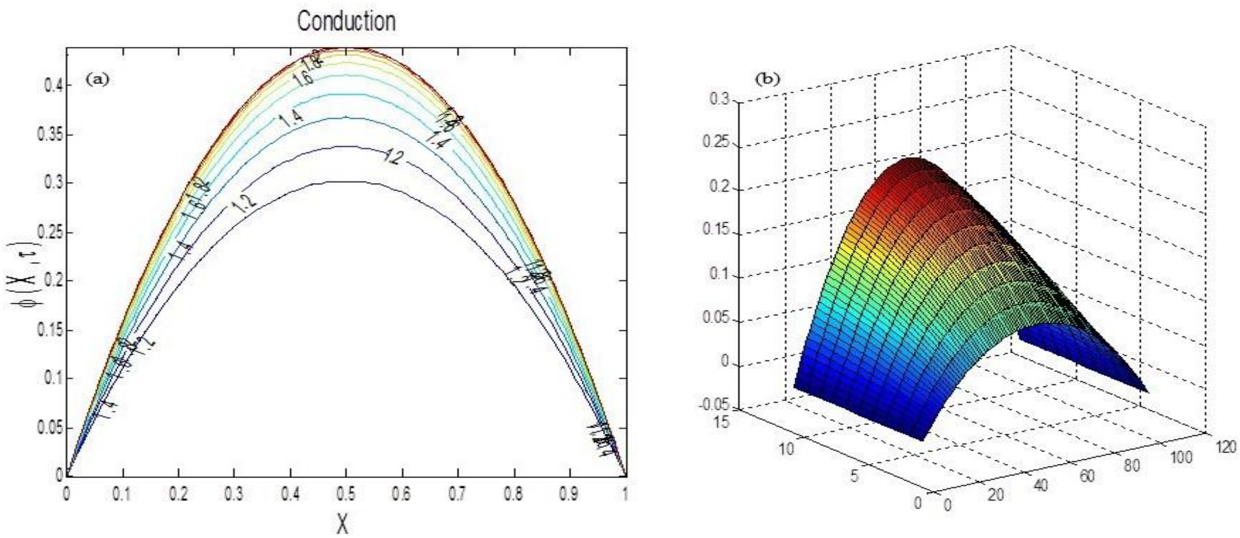

Figure 1. (a) Temperature distributions in conduction mode for constant heat source capacity, for $\tau_g = 1.$ (b) Temperature profile for $\tau_g = 3$ of the first source term.

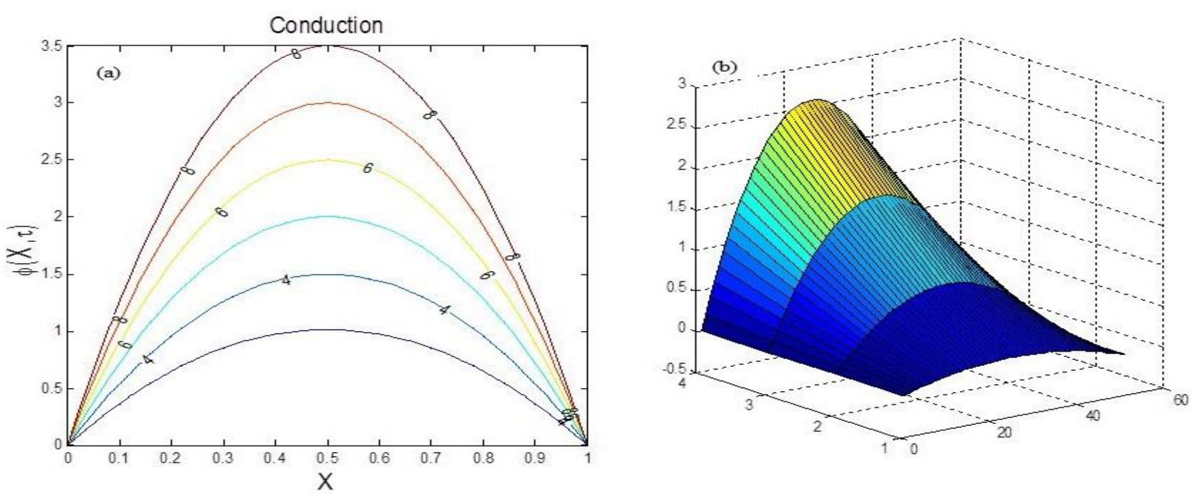

Figure 2. (a) Temperature distributions in conduction mode for source capacity proportional to temperature for for $\tau_g = 1, and\ \alpha = 1.0$, (b) Temperature profile for $\tau_g = 3$ of the second source term.

for all the four cases. In Figure 1(b), temperature profile rise as the dimensionless time slowly drop toward the direction of heat flow (Vedavarz et al., 1994), when $\tau_g = 3$ at constant source capacity. This increase of temperature in the system is caused generally by the heat generation process. Hence, dimensionless temperature distribution is indirectly proportional with the flow of heat flux as indicated in Figure 1(a). Energy is concentrated at the intermediate X for $\tau_g = 1,3,6$, in the case of conduction mode for constant heat source capacity and source capacity proportional to temperature. However, for the pulsed heat source and source capacity proportional to time shown in Figures 3(a and b) and 4(a and b), the energy is less concentrated at $\tau_g = 3.$ Figure 2(a and b) display temperature distributions when the source capacity is proportional to temperature, and for $\tau_g = 3$ respectively. Our results for this case show enhanced temperature distribution at increase value of α, and τ_g . The gradual reduction in temperature along the direction of heat flow is expected to explain the well

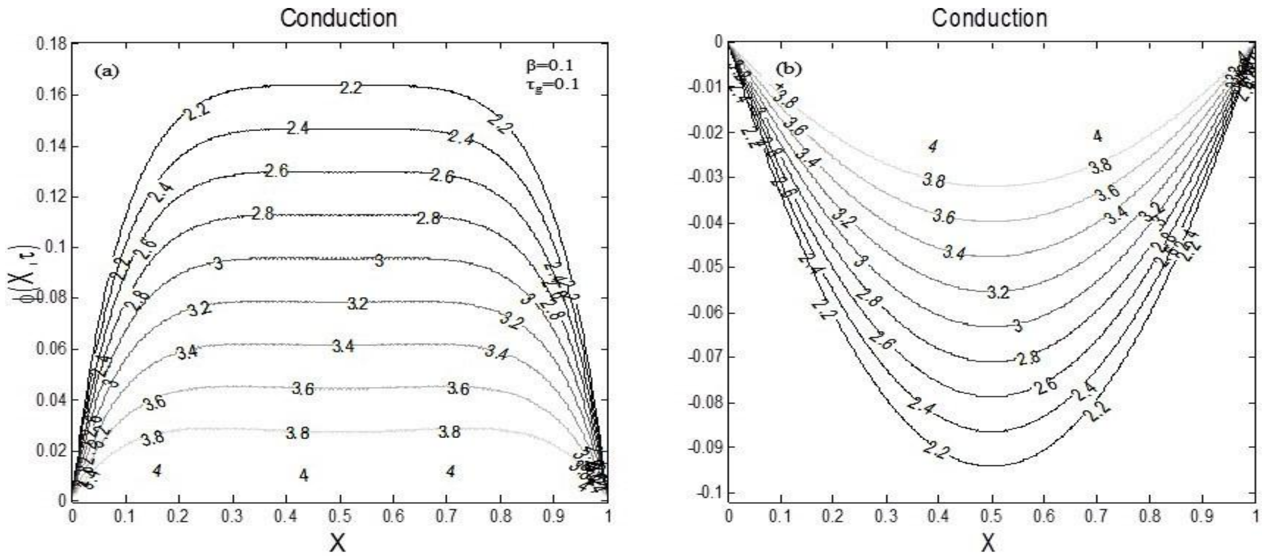

Figure 3. (a) Temperature distributions in conduction mode for pulsed heat source, (b) Temperature distributions for pulsed heat source at $\tau_g = 3$.

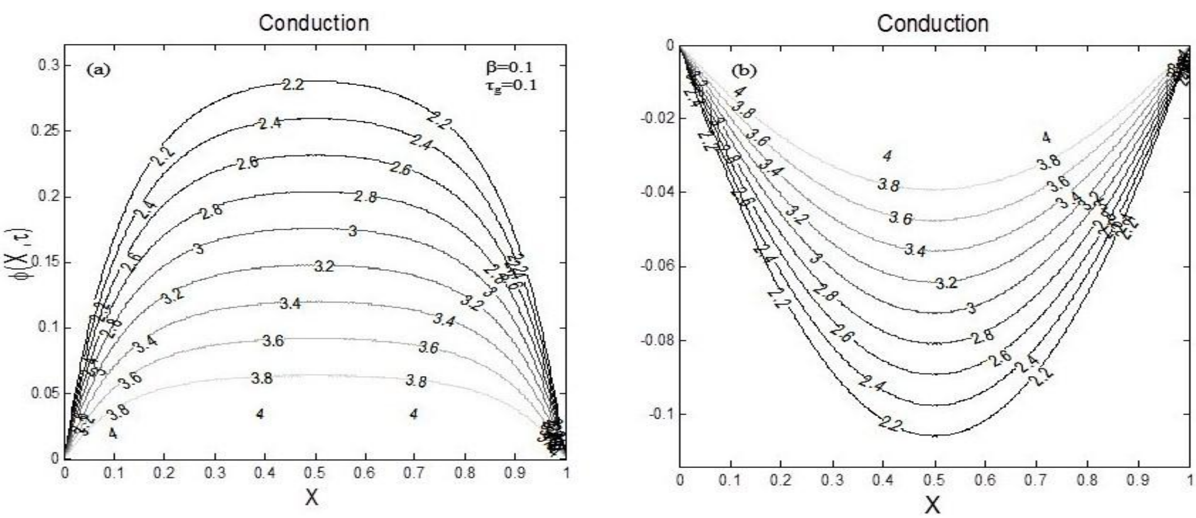

Figure 4. (a) Temperature distributions in conduction mode for source capacity proportional to time. (b) Temperature distributions for source capacity proportional to time at $\tau_g = 3$.

behavior of this model.

Figure 3(a and b) show the temperature distribution of the system, in which heat is release from a pulsed energy source. In Figure 3(a) high relaxation time lowers the temperature profile for $\tau = 0.1$, and β =1, but at higher value of τ_g the temperature profile fluctuate within the set boundary, however, the trend remains same. The temperature distribution for source capacity proportional to time in conduction mode compared to the pulsed energy source term as depicted in Figures 3(a and b) and 4(a and b). This is because when $\tau > 0$ the Dirac delta pulse approach unity, which rendered the two terms to be same at that instant that is, when $\delta(\tau) = \tau$. The little difference observed is the variation in temperature distribution, which is enhanced for source capacity proportional to time as compared to the pulsed energy source term.

Figure 5(a and b) and 6(a and b) show calculations results for the four different source terms with respect to dimensionless temperature variation versus relaxation

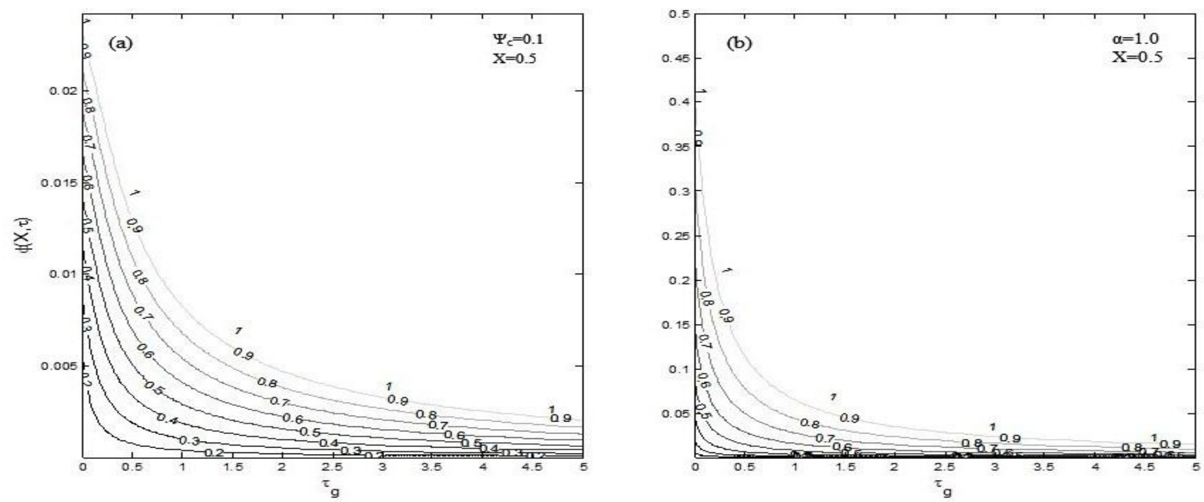

Figure 5. (a) Temperature variation with the relaxation time of source capacity calculated for the constant heat source capacity, (b) Temperature variation with the relaxation time of source capacity calculated for the source capacity proportional to temperature.

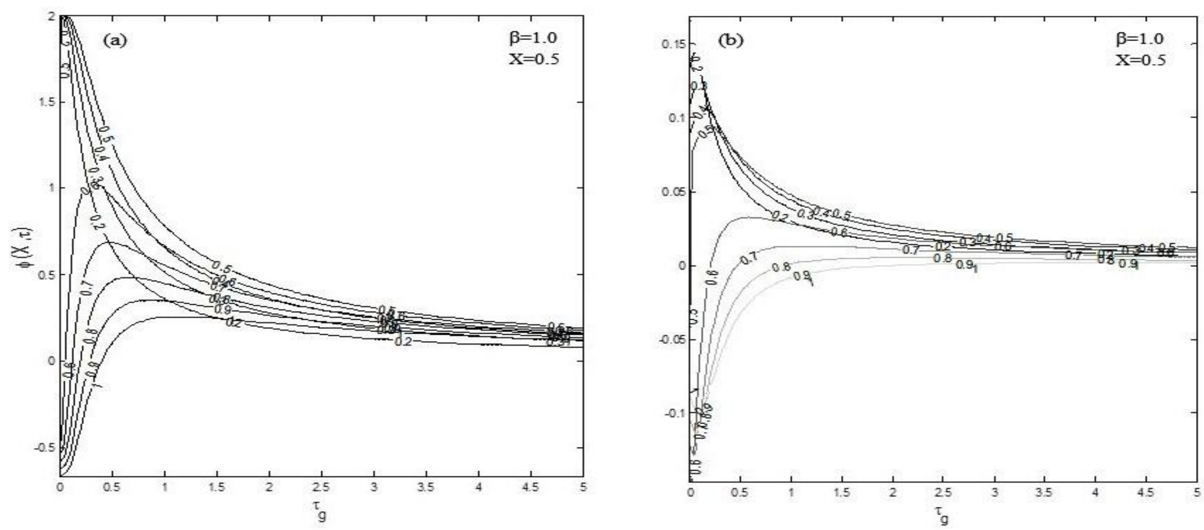

Figure 6. (a) Temperature variation with the relaxation time of source capacity calculated for the pulsed heat source, (b) Temperature variation with the relaxation time of source capacity calculated for the source capacity proportional to time.

time of source capacity. In Figure 5(a and b) uniform temperature variation occurs at shorter duration, it decrease with increase of relaxation time and stabilize at high value of relaxation time, hence the mode of conduction of heat is non-diffusive for extremely short duration. In Figure 5(b) the temperature profile is observed to depend on coefficient α, which causes oscillation at higher value of α. The overall effect is enhancement of temperature profile at high values of the coefficients that is, $\Psi_c = 1, 3, 6.$ In Figure 6(a), temperature variation approaches constant value between the heat pulses of 3.5 to 5, and decrease with increase of the heat pulse from both sides of the temperature profile. The steady state solutions of Equations (26), (27) and (28) agrees with source capacity proportional to temperature, which proves the validity of the Riemann-sum approximation used in this work.

CONCLUSION

The problem of heat conduction equation for the finite velocity of heat propagation, and relaxation of heat source capacity is solved analytically and numerically using Riemann-sum approximation. Four different expressions for dimensionless heat source capacity are considered. The effects of coefficients on temperature distribution, variation, and steady state solution are analyzed. It is observed that the temperature profile decreases when the relaxation time is high, however, at higher value of τ_g the temperature profile fluctuate within the set boundary. Furthermore, the gradual drop in temperature profile along the conduction direction agrees with the natural behavior of heat propagation.

Conflict of Interest

The authors have not declared any conflict of interest.

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NOMENCLATURE

τ: Dimensionless time, t_k: Relaxation time of the heat flux, k: Thermal conductivity, Θ: Temperature,
Ψ: Dimensionless capacity of the internal heat source, ρ: Density, c_p: The specific heat at constant pressure, g: Capacity of the internal heat source, τ_g: Dimensionless relaxation time of source capacity, g_t: Transient heat capacity of the source, Φ: Dimensionless temperature, a: Thermal diffusivity, α: Dimensionless coefficient in expression for the Source capacity proportional to temperature, β: Dimensionless coefficient in expression for the Dirac delta pulse and Source capacity proportional to time, Ψ_c: Dimensionless coefficient in expression for the constant Source capacity, K_i: Coefficients defined by Equations (20-23),
δ: Dirac delta function, μ:Thermal diffusivity.

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