International Journal of Physical Sciences Vol. 2 (7), pp. 185-187, July, 2007
Available online at http://www.academicjournals.org/IJPS
ISSN 1992 - 1950 © 2007 Academic Journals

Short Communication

On asymptotic behaviour of a second order delay differential equation

M. S. Zaki

Mathematics Department, Faculty of Science for Girls, Al Azhar University, Nasr City, Cairo – 11754, Egypt. E-mail: mervat_zaki2007@yahoo.com

Accepted 22 June, 2007

In this paper, the relation between the oscillatory behavior of the solutions of second order differential equation and the oscillatory behaviour of the solutions of the corresponding delay differential equation is established. Also a necessary and sufficient condition for every bounded solution of the delay differential equations to have nonooscillatory solutions is given._A.M.S.C.: 34C10, 34C11, 34K15.

Key words: Zeros, oscillatory and nonoscillatory solutions.

INTRODUCTION

The main interest in obtaining qualitative information for second order functional differential equation is due to the fact that they often provide mathematical models for physical systems (Lalli and Grace, 1990; Singh 1973; Elabbasy and Saker, 1999; Ladas et al., 1972; Ladas et al., 1984; Erbe and Kong, 1992; Das, 1994; Staikos and Petsoulas, 1970; Stavroulakis, 2005).

Of particular importance however has been the study of oscillations of delay differential equation, which are generated by the retarded argument and which do not appear in the corresponding differential equations without delay.

A step in the direction, of establishing oscillation results generated by delays, was taken by (Ladas et al., 1984) where they proved that every bounded solutions of retarded equation

$$y''(t) - p(t)\ y(t-\tau) = 0$$

with $p(t) > 0$, $p'(t) \le 0$, $\tau^2 p(t) \ge 2$ is oscillatory.

Noting that (1.1) is nonoscillatory for τ = 0.

In this paper, we investigate the oscillatory behaviour for the solution of the delay differential equation

$$\left(r(t)y'(t)\right)' + p_1(t)\,y(t) = f(t)$$

in relation to the solution of the delay equation

$$\left(r(t)x'(t)\right)' + p_2(t)x(t) = -p_3(t)x\left(t-\tau(t)\right)$$

where

$p(t)$, $p_j(t)$, $j = 1, 2, 3, f(t), r(t)$ *and* $\tau(t)$ are functions

$$[\tau, \infty) \to \mathbf{R},\ \ \tau > 0$$

Also, we denote $x\left(t-\tau(t)\right)$ *by* $x_\tau(t)$.

It will be assumed in this paper that *p(t), r(t)* and *r'(t)* are bounded, real and continuous defined on (—∞, ∞), with $r(t) > 0$, $r'(t) \ge 0$, $P_i(t) > 0$, $i = 1, 2, 3$. *f(t)* is eventually positive on some half line $[\tau, \infty) \to \mathbf{R}$, $\tau > 0$.

Definition (1): A solution is oscillatory of there is an increasing sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathbf{R}$ such that

$$\lim_{n\to\infty} t_n = \infty \text{ and } x(t_n) = 0 \quad \text{for all } n \in \mathbf{N}.$$

Definition (2): The solution *x(t)* of (1.2) or (1.3) is called oscillatory if it has no last zero.

Equations (1.2) or (1.3) are called oscillatory if all continuous nontrivial solutions oscillatory.

In this paper, we shall give the conditions which are sufficient for the second order differential equation with delay to be oscillatory or nonoscillatory respectively.

RESULTS

Theorem 2.1. Let $\lim_{t\to\infty}(t-\tau(t))=+\infty$. Assume that there is a constant $\beta \in [0,1]$ such that

$$\left(\frac{p_1(t)}{p_3(t)}\right) \le \beta < 1$$

for $t \geq t_0$ and t_0 large enough. Assume also that $p_1, p_3 > 0$. If all solutions of (1.2) are oscillatory, then all solutions of (1.3) are oscillatory.

Proof: Suppose that the solutions of equation (1.2) are oscillatory, let *y* one of them and suppose that there is a solution *a x* of equation (1.3) which is nonoscillatory and nonnegative. Let $T_0 \ge t_0$ a real number large enough. So that $x(t) > 0$ for all $t > T_0$.

Let $T = T_0 + M$, where *M* is a positive constant such that $0 < x(t) < M$, then $x_\tau(t)$ is strictly positive for $t \geq T$.

Let $t_1 > T$, $t_2 > t_1$ be two zeros of *y* such that $y(t) > 0$ for $t \in (t_1, t_2)$, $y'(t_1) > 0$ and $y'(t_2) < 0$.

Multiplying (1.2) by *x(t)* and (1.3) by *y(t)* and subtracting we get

$$\left(r(t)y'(t)\right)' x(t) - \left(r(t)x'(t)\right)' y(t) + \left(p_1(t)-p_2(t)\right)x(t)y(t) = x(t)f(t) + p_3(t)x_\tau(t)y(t) \quad (2.1)$$

$$\frac{d}{dt}\left[r(t)\left(y'(t)x(t)-x'(t)y(t)\right)\right] = x(t)f(t) + \left(p_2(t)-p_1(t)\right)x(t)y(t) + p_3(t)x_\tau(t)y(t)$$

Since $x(t) > 0$ for $t \geq T$, we have

$$\frac{p_3(t)}{p_1(t)}\frac{x_\tau(t)}{x(t)} \ge \frac{1}{\beta}\frac{x_\tau(t)}{x(t)}$$

According to [9], there is $T_1 > T$ large enough, such that

$$\frac{x_2(t)}{x(t)} > \beta$$

for $t > T_1$

From (2.3) and (2.4) for $t > T_1$, it follows

$$\frac{p_3(t)}{p_1(t)}\frac{x_2(t)}{x(t)} > \frac{1}{\beta}\frac{x_2(t)}{x(t)} > \beta\frac{1}{\beta} = 1$$

Now using (2.5) in (2.2), we have

$$\frac{d}{dt}\left[r(t)\left(y'(t)x(t)-x'(t)y(t)\right)\right] > x(t)f(t) + p_2(t)x(t)y(t) \quad (2.6)$$

choosing t_1 and t_2 greater than T_1 and integrating (2.6) between t_1 and t_2, we get

$$r(t)y'(t_2)x(t_2) - r(t_1)x(t_1)y'(t_1) > \int_{t_1}^{t_2} x(t)f(t)dt + \int_{t_1}^{t_2} p_2(t)x(t)y(t)dt \quad (2.7)$$

Since the right hand side of (2.7) is integer and nonnegative, and the left hand side of (2.7) is nonpositive because $y'(t_2) < 0$, $y'(t_1) > 0$, therefore a contradiction. This completes the proof.

Theorem 2.2. Let
Assume that all of the solutions of (1.2) are nonoscillatory. Let *y* one of them

(i) Let *x* be a solution of (1.3) such that $|x(t_0)| > 0$. Then there is $\delta > 0$ such that *x(t)* has constant sign on $(t_0-\delta, t_0+\delta)$ and there is $t_1 \ge T$ such that $t-\tau(t) \in (t_0-\delta, t_0+\delta)$ for all $t > t_1$.
$\lim_{t\to\infty}(t-\tau(t)) = t_0$, t_0 finite.

(ii) $\inf_{[T,\infty)} |p_1(t)| = k > 0$

(iii) $p_1(t) = p_2(t)$, $f(t) = 0$

(iv) The zeros of *x(t)* have no limit point.

If all the solutions of (1.2) are nonoscillatory, then all solutions of the equation (1.3) are nonoscillatory.

Proof: Now consider the cases:

(i) Let $|x(t_0)| > 0$. In this case there is some $t_1 \geq T$ such that $(t - \tau(t)) \in (t_0 - \delta, t_0 + \delta)$, for $t > t_1$. Hence $x_\tau(t)$ has a constant sign for $t > t_1$. Assume that for $t > t_1$, the sign of *y(t)* is constant. Then also $\frac{d}{dt}\left[r(t)\left(y'(t)x(t)-x'(t)y(t)\right)\right]$ has a constant sign and hence there is some $t_3 > t_1$ such that

$$\left[r(t)\left(y'(t)x(t)-x'(t)y(t)\right)\right] \ne 0 \text{ for } t > t_3 > t_1.$$

If $x(t)$ oscillatory, then there exist points $a > t_3$, $b > t_3$, such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ when $t \in (a, b)$. Since

$$\left[r(t)\left(y'(t)x(t)-x'(t)y(t)\right)\right]\neq 0 \quad \text{in} \quad [a, b]$$

The solution $y(t)$ has a zero on (a, b).

(ii) Let x a solution such that $x(t_0) \neq 0$ in each interval which contains t_0. From the assumption of the theorem it follows that t_0 cannot be the limit point of zero of $x(t)$. Hence we have either $(t-\tau(t)) \in (t_0 - \delta, t_0)$ for all $t > t_1 \geq T$ or $(t-\tau(t)) \in (t_o, t_o+\delta)$ for all $t > t_1$, and hence $x_\tau(t)$ has a constant sign for $t > t_1$. We proceed with the proof similarly as in the case (i). This completes the proof.

Theorem 2.3. Suppose there exists a twice continuously differentiable eventually negative and bounded function $h(t)$ that satisfies

$$\left(r(t)h'(t)\right)' + p(t)h(t) > 0 \quad , \quad p(t) = p_1(t) + p_2(t)$$

Also, let

$$\lim_{t\to\infty}\left(t-\tau(t)\right) = +\infty$$

Then (2.8) is necessary and sufficient for every bounded solution of (1.3) to the nonoscillatory.

Proof: Assume that

$$\left(r(t)x'(t)\right) + p_2(t)x(t) + p_3(t)x_\tau(t) = 0$$

is nonoscillatory and $x(t)$ is any bounded nonoscillatory solution of (1.3). Without loss of generality we can assume that $y(t)$ is eventually negative. Let $h(t) = x(t)$. Choose T_1 large enough so that

$$h_\tau(t) = x_\tau(t) < 0 \qquad \text{for} \qquad t > T_1$$

$$\begin{aligned}\left(r(t)h'(t)\right)' + p(t)x(t) &= \left(r(t)h'(t)\right)' + p_1(t)x(t) + p_2(t)x(t)\\ &= \left(r(t)x'(t)\right)' + \frac{p_1(t)}{p_3(t)}p_3(t)x_\tau(t)\frac{x(t)}{x_\tau(t)} + p_2(t)x(t)\end{aligned} \tag{2.9}$$

$$> \left(r(t)x'(t)\right)' + \beta p_3(t)x_\tau(t)\frac{x(t)}{x_\tau(t)} + p_2(t)x(t)$$

where β is given in Theorem 2.1

Since $\lim_{t\to\infty}\left(t-\tau(t)\right) = +\infty$, $\lim_{t\to\infty}\frac{x(t)}{x_\tau(t)} = 1$. since

$$\left(r(t)x'(t)\right)' = -p_2(t)x(t) - p_3(t)x_\tau(t) > 0$$

and $r, p_2, p_3, -x > 0$, x is increasing. so T_1 can be taken so large such that

$$\frac{x(t)}{x_\tau(t)} > 1 \text{ for any } t > T_1.$$

Then from (2.10) and (2.11) we get

$$\left(r(t)x'(t)\right)' + p_1(t)x(t) + p_2(t)x(t) >$$
$$\left(r(t)x'(t)\right)' + p_2(t)x(t) + p_3(t)x_\tau(t) = 0$$

Now suppose

$$\left(r(t)h'(t)\right)' + p(t)h(t) > 0$$

for some eventually negative twice differentiable and bounded function $h(t)$. From (2.12) where $h'(t) < 0$ for $t > T_2$ and continuous on $r(t)$ it follows that

$$\int_{T_2}^{\infty} t\, p(t)dt < \infty$$

This, in turn, due to conditions on $p_1(t)$ and $p_2(t)$ implies

$$\int_{T_2}^{\infty} t\, p_1(t)dt < \infty$$

$$\int_{T_2}^{\infty} t\, p_3(t)dt < \infty$$

By Singh (1973), the proof is complete.

REFERENCES

Das P (1994). Oscillations of mixed neutral equations caused by several deviating arguments. J. Call. Math. Soc. 86: 85-146.

E-Elabbasy E, Saker S (1999). Oscillation of nonlinear delay differential equations with several positive and negative coefficients. Kyungpook. Math. J. 39: 367-377.

Erbe L, Kong Q (1992). Oscillation results for second order neutral differential equations. Funkcialaj Ekvacioj. 35: 545-555.

Ladas G, Ladde G, Papadakes J (1972). Oscillations of functional differential equations generated by delays. J. Diff. Eqn. 12: 385-395.

Ladas G, Sficas Y, Stavrolakis I (1984). Necessary and sufficient conditions for oscillations of higher order delay differential equations. Trans. Am. Math. Soc. 2(85): 81-90.

Lalli B, Grace S (1990). Asymptotic and oscillatory behavior of a class of second order functional differential equations. Rocky Mt. J. Math. 20 1063-1077.

Singh B (1973). A necessary and sufficient conditions for the oscillation of an even order non-linear delay differential equation. S. Can. J. Math. 25: 1078-1089.

Staikos V, Petsoulas A (1970). Some oscillation criteria for second order nonlinear delay differential equations. J. Math. Anal. Appl. 30: 695-701.

Stavroulakis IP (2005). Oscillation Criteria for functional differential equations 2004 Conference on Differential Equations and Applications in Mathematics and Biology, Nanaimo, BC, Canada. Electron. J. Differential Equations, Conference 12: 171180. ISSN: 1072-6691.