International Journal of the Physical Sciences Vol. 6(3), pp. 386-390, 4 February, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.071
ISSN 1992 - 1950 ©2011 Academic Journals

Full Length Research Paper

A re-weighted algorithm for designing data dependent sensing dictionary

Anmin Huang[1]*, Gui Guan[1,2], Qun Wan[1] and Abolfazl Mehbodniya[2]

[1]Department of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China.
[2]Department of Electrical and Communication Engineering, Graduate School of Engineering, Tohoku University, Sendai, 980-8579, Japan.

Accepted 3 February, 2011

The modified orthogonal matching pursuit (OMP) algorithm based on sensing dictionary, shows significant improvement for the performance of sparse recovery, especially in the case of highly coherent dictionary. Assuming a signal to be decomposed, a good sensing dictionary should depend not only on the ordinary dictionary but also the observed data. In this paper, a re-weighted algorithm for designing data dependent sensing dictionary is proposed by introducing the effective posteriori knowledge obtained from the observed data. Simulation results are presented to demonstrate the superior performance of data dependent sensing dictionary designed by the proposed algorithm.

Key words: Coherent dictionary, data dependent sensing dictionary, modified orthogonal matching pursuit, sparse recovery.

INTRODUCTION

Consider a redundant dictionary Φ composed of N vectors $\phi_i \in C^{M\times 1}$ ($i \in \Omega = \{1,...,N\}$) with $M < N$. In general, these vectors are normalized and called atoms. Assume a signal $y \in C^{M\times 1}$ can be exactly represented as a linear combination of a small number of atoms over this dictionary, that is, $y = \Phi x = \Phi_{\Lambda_{opt}} x_{opt}$, where $\Phi_{\Lambda_{opt}}$ is a sub-dictionary whose columns are the optimal atoms with the index set Λ_{opt} . If $\left|\Lambda_{opt}\right| = K$, y is called K exact-sparse signal, where $K<<N$ and $|\cdot|$ returns the cardinality of a set. The goal of sparse recovery is to find the sparsest solution to the inverse problem $y = \Phi x$. Unfortunately, the problem is non-deterministic polynomial-time hard (Natarajan, 1995) and some suboptimal methods have been developed to obtain the sparsest solution.

One such method is orthogonal matching pursuit (OMP) (Pati et al., 1993). The simplicity of implementation of this algorithm makes it attractive for sparse signal recovery (Gribonval and Krstulovic, 2006; Gribonval and Vandergheynst, 2006). A sufficient condition for OMP to identify the correct atom is

$$\left\| \Phi^{\dagger}_{\Lambda_{opt}} \Phi_{\overline{\Lambda}_{opt}} \right\|_{1,1} < 1, \tag{1}$$

where $\dagger$ denotes the pseudo-inverse operation, $\overline{\Lambda}_{opt}$ is the complementary of Λ_{opt} in Ω and $\|\cdot\|_{1,1}$ denotes the maximum l_1 norm of any column of its argument (Tropp, 2004). The cumulative coherence of the redundant dictionary is given by

$$\mu_1(k) = \max_{|J|=k} \max_{i\notin J} \sum_{j\in J} \left|\left\langle \phi_i, \phi_j \right\rangle\right|, \tag{2}$$

Where $\left|\langle \cdot, \cdot \rangle\right|$ denotes the absolute value of Hilbert inner

*Corresponding author. E-mail: hgam2003@163.com.

product, a more useful condition can be obtained. Tropp has proven that (1) is satisfied, if $\mu_1(K)+\mu_1(K-1)<1$ holds and OMP can be guaranteed to recover the optimal representation of K exact-sparse signal in this case (Tropp, 2004). But in most situations, redundant dictionaries for the decomposition of practical signals may be highly coherent and OMP may fail to identify the correct atoms in these cases. Recently, Schnass and Vandergheynst modified OMP algorithm and proposed a method based on alternating projection (AP) to design sensing dictionary (Schnass and Vandergheynst, 2008).

However, the sensing dictionary obtained by this method only depends on the ordinary dictionary and does not take into account any additional information about the observed data. In this paper, we concentrate on the problem of designing sensing dictionary for the modified OMP algorithm. A re-weighted method for designing a data dependent sensing dictionary is proposed by introducing effective posteriori knowledge obtained from the observed data.

The modified OMP algorithm

In order to guarantee OMP algorithm to identify the correct atoms for highly coherent dictionary, Schnass and Vandergheynst introduced the concept of sensing dictionary and modified the ordinary OMP algorithm (Schnass and Vandergheynst, 2008). This modified OMP algorithm selects an atom at each iteration through solving the optimization $\arg\max_{i\in\Omega}\left|\left\langle \psi_i, r\right\rangle\right|$ rather than $\arg\max_{i\in\Omega}\left|\left\langle \phi_i, r\right\rangle\right|$, where $\psi_i(i=1,...,N)$ are the columns of an appropriate sensing dictionary, Ψ and r denotes the residual produced at the last iteration. It is easy to see that, this algorithm reduces to the ordinary OMP, if the sensing dictionary is selected as $\Psi=\Phi$ for sufficiently incoherent dictionary. Schnass and Vandergheynst generalized the sufficient condition (1) to the case of highly coherent dictionary and proved that, the modified OMP can exactly recover the correct atoms if (Schnass and Vandergheynst, 2008)

$$\left\|(\Phi^*_{\Lambda_{opt}}\Psi_{\Lambda_{opt}})^{-1}\Phi^*_{\Lambda_{opt}}\Psi_{\bar{\Lambda}_{opt}}\right\|_{1,1}<1, \tag{3}$$

where $*$ represents the complex conjugate transpose. Define the cross cumulative coherence

$$\tilde{\mu}_1(k)=\max_{|J|=k}\max_{i\notin J}\sum_{j\in J}\left|\left\langle \psi_i,\phi_j\right\rangle\right|, \tag{4}$$

it has been proved that, the modified OMP can identify a correct atom at each step if

$$\tilde{\mu}_1(K)+\tilde{\mu}_1(K-1)<\beta, \tag{5}$$

holds, where the minimal coherence β is defined as $\beta=\min_i\left|\left\langle \psi_i,\phi_i\right\rangle\right|$ (Schnass and Vandergheynst, 2008). Compared to (2), the inequality (5) shows that, the modified OMP can be guaranteed to recover K exact-sparse signal with a larger K, if cross cumulative coherence is smaller than the cumulative coherence and β is close to one. Motivated by these results, Schnass and Vandergheynst formulated the problem of finding a sensing dictionary, as looking for a matrix Ψ, such that the gram type matrix $G=\Psi^*\Phi$ is closest to a structural constraint Hermitian matrix, and designed sensing dictionary as the solution to the following optimal problem (6)

$$\begin{aligned}&\arg\min_{\Psi\in C^{M\times N}}\left\|H-\Psi^*\Phi\right\|_F^2,\ \text{s.t.}\\ &H\in\chi \triangleq \{C^{N\times N}:H=H^*, H_{ii}=1\ and\ \left|H_{ij}\right|\le\mu\ for\ i\ne j\},\end{aligned} \tag{6}$$

where $\mu=\sqrt{(N-M)/M(N-1)}$ and $\|\cdot\|_F$ denotes the Frobenius norm. The algorithm based on AP, alternately solves the two basic matrices nearness problems.

(1) Given the matrix $\tilde{G}=\tilde{\Psi}^*\Phi$, finding a matrix $\tilde{H}$ such that $\tilde{H}\in\arg\min_{H\in\chi}\left\|H-\tilde{G}\right\|_F$.

(2) Given the Hermitian matrix $\tilde{H}$, finding a matrix $\tilde{\Psi}$ such that $\tilde{\Psi}\in\arg\min_{\Psi\in\mathbb{C}^{M\times N}}\left\|\Psi^*\Phi-\tilde{H}\right\|_F$.

By alternately finding structural constraint Hermitian matrix and sensing dictionary, this algorithm can construct a sensing dictionary conveniently.

Data dependent sensing dictionary design

To construct a good sensing dictionary, at first we reveal the relationship between the recovery condition and local measurement of cross cumulative coherence. Similar to (4), the local measurement of cross cumulative coherence corresponding to the optimal atoms is defined by

$$\begin{aligned}\tilde{\mu}_1(k,\Lambda_{opt})&=\tilde{\mu}_1(k,\Psi,\Phi_{\Lambda_{opt}})\\ &=\max_{\substack{|J|=k\\ J\subseteq\Lambda_{opt}}}\max_{i\notin J}\sum_{j\in J}\left|\left\langle\psi_i,\phi_j\right\rangle\right|,\ for\quad k\le K.\end{aligned} \tag{7}$$

The left-hand side of (3) can be bound as

$$\begin{aligned}\left\|(\Phi^*_{\Lambda_{opt}}\Psi_{\Lambda_{opt}})^{-1}\Phi^*_{\Lambda_{opt}}\Psi_{\bar{\Lambda}_{opt}}\right\|_{1,1}&\le\left\|(\Phi^*_{\Lambda_{opt}}\Psi_{\Lambda_{opt}})^{-1}\right\|_{1,1}\left\|\Phi^*_{\Lambda_{opt}}\Psi_{\bar{\Lambda}_{opt}}\right\|_{1,1}\\ &=\left\|(\Phi^*_{\Lambda_{opt}}\Psi_{\Lambda_{opt}})^{-1}\right\|_{1,1}\tilde{\mu}_1(K,\Lambda_{opt}).\end{aligned} \tag{8}$$

Based on the definition of β and $\tilde{\mu}_1(k,\Lambda_{opt})$, we can get

$$\begin{aligned}\left\|\Phi^*_{\Lambda_{opt}}\Psi_{\Lambda_{opt}}-I_K\right\|_{1,1}&=\max_{i\in\Lambda_{opt}}\Big(\left|\left\langle\psi_i,\phi_i\right\rangle-1\right|+\sum_{j\ne i,j\in\Lambda_{opt}}\left|\left\langle\psi_i,\phi_j\right\rangle\right|\Big)\\ &=\max_{i\in\Lambda_{opt}}\big(\left|1-\left\langle\psi_i,\phi_i\right\rangle\right|\big)+\max_{i\in\Lambda_{opt}}\sum_{j\ne i,j\in\Lambda_{opt}}\left|\left\langle\psi_i,\phi_j\right\rangle\right|\\ &\le1-\beta+\max_{i\in\Lambda_{opt}}\sum_{j\ne i,j\in\Lambda_{opt}}\left|\left\langle\psi_i,\phi_j\right\rangle\right|\\ &\le1-\beta+\tilde{\mu}_1(K-1,\Lambda_{opt}),\end{aligned} \tag{9}$$

where I_K denotes the identity matrix of size $K \times K$. Using the convergent property of Neumann series, it is easy to get $\|(I_K + A)^{-1}\|_{1,1} \le (1 - \|A\|_{1,1})^{-1}$ whenever $\|A\|_{1,1} < 1$ [5]. Accordingly, if $\tilde{\mu}_1(K-1, \Lambda_{opt}) < \beta$, we can obtain

$$\min_{\Psi \in C^{M \times N}} \|\Psi^* \Phi W\|_F^2, \text{ s.t. } \psi_i^* \phi_i = 1 \text{ for } i \in \Omega, \tag{10}$$

From (3), (8) and (10), we get the sufficient condition for the modified OMP

$$\tilde{\mu}_1(K, \Lambda_{opt}) + \tilde{\mu}_1(K-1, \Lambda_{opt}) < \beta. \tag{11}$$

These results show that, the exact recovery condition for the modified OMP is formulated with less restrictive condition, computed only on the optimal atoms. The aforestated derivation reveals that, a good sensing dictionary should satisfy the condition that the minimal coherence β is large enough, while the local measurement of cross cumulative coherence with respect to the optimal atoms $\tilde{\mu}_1(k, \Lambda_{opt})$ grows slowly.

It is of no use to consider the cross cumulative coherence corresponding to these atoms that do not participate in the representation of sparse signal. Therefore, a good sensing dictionary for the modified OMP should be data dependent, since the set of optimal atoms is dependent on the observed data. We here weight the atoms of the ordinary dictionary by introducing effective posteriori knowledge obtained from the observed data and construct the sensing dictionary Ψ as the solution to the following optimization

$$\min_{\Psi \in C^{M \times N}} \|\Psi^* \Phi W\|_F^2, \text{ s.t. } \psi_i^* \phi_i = 1 \text{ for } i \in \Omega, \tag{12}$$

where $W = diag\{w_1, ..., w_N\}$ $(w_i \in [0,1])$ is the weighting matrix, which provides a way to introduce posteriori knowledge. This method modifies the algorithm in (6) by using a weighting matrix and linear constraints. Based on the definition of matrix norm, we can get

$$\min_{\Psi \in C^{M \times N}} \|\Psi^* \Phi W\|_F^2 = \min_{\Psi \in C^{M \times N}} \sum_{i=1}^{N} \|W \Phi^* \psi_i\|_2^2, \tag{13}$$

Using (12) and (13), we obtain the column vectors of sensing dictionary as

$$\psi_i = \frac{R^{-1}\phi_i}{\phi_i^* R^{-1} \phi_i}, \; i = 1, ..., N, \tag{14}$$

Where $R = \Phi W^2 \Phi^*$.

The problem which remained in the study's method, is how to obtain the effective posteriori knowledge by using the observed data. Note that, the correlation between each atom ψ_i of sensing dictionary and the observed signal y suggests the probability for the corresponding atom ϕ_i to appear in the optimal atoms, one possible choice for the weighting matrix is $W = diag\{|\Psi^* y|\}$. In order to get more effective weights, we alternately update the weights and sensing dictionary at each step. This yields a re-weighted algorithm which can be summarized as:

(1) Initialize. Set $W = diag\{|\Phi^* y|\}$.

(2) Repeat until the stopping criterion is satisfied.

(a) Calculate $R : R = \Phi W^2 \Phi^*$;

(b) Update Ψ: $\psi_i = \frac{R^{-1}\phi_i}{\phi_i^* R^{-1}\phi_i}$ for $i = 1, ..., N$;

(c) Update W: $W = diag\{|\Psi^* y|\}$.

As discussed earlier, this algorithm combines the characteristics of both optimization and knowledge-based methods. Different from the method proposed by Schnass and Vandergheynst (Schnass and Vandergheynst, 2008), the sensing dictionary obtained by this algorithm does not only depend on the ordinary dictionary but also the observed data. In additional, consider that linear constraints $\psi_i^* \phi_i = 1$ $(i \in \Omega)$ are imposed on our algorithm, the sensing dictionary obtained by this algorithm satisfies $\beta = 1$.

SIMULATION RESULTS

To illustrate the performance of data dependent sensing dictionary obtained by this proposed algorithm, simulation results are presented here. We consider a complex random dictionary with the dimensions $M = 60$ and $N = 120$. The entries of this dictionary were drawn independently from standard complex Gaussian distribution and the atoms are normalized. In our experiments, we fix the number of iteration for the modified OMP as the support size K. The number of iteration for the AP algorithm and the proposed method are 40 and 10, respectively. Figure 1 presents local measurement of (cross) cumulative coherence via different support size. Beta denotes the minimal coherence β for the AP algorithm. As shown in this plot, we have $\mu_1(K, \Lambda_{opt}) + \mu_1(K-1, \Lambda_{opt}) > 1$ for $K \ge 3$, which means that, the ordinary OMP can only be guaranteed to exactly recover *K*-sparsity signal with $K \le 2$. With sensing dictionary obtained by the AP algorithm in (Schnass and Vandergheynst, 2008), the modified OMP can be guaranteed to recover sparse signal of up to three atoms since (11) is satisfied when $K = 3$.

Using the sensing dictionary obtained by the proposed algorithm, the modified OMP can be guaranteed to exactly recover sparse signal of up to thirteen atoms due to $\tilde{\mu}_1(13, \Lambda_{opt}) + \tilde{\mu}_1(12, \Lambda_{opt}) < \beta$. This result show that, the modified OMP based on data dependent sensing dictionary, can be guaranteed to identify correct atoms at

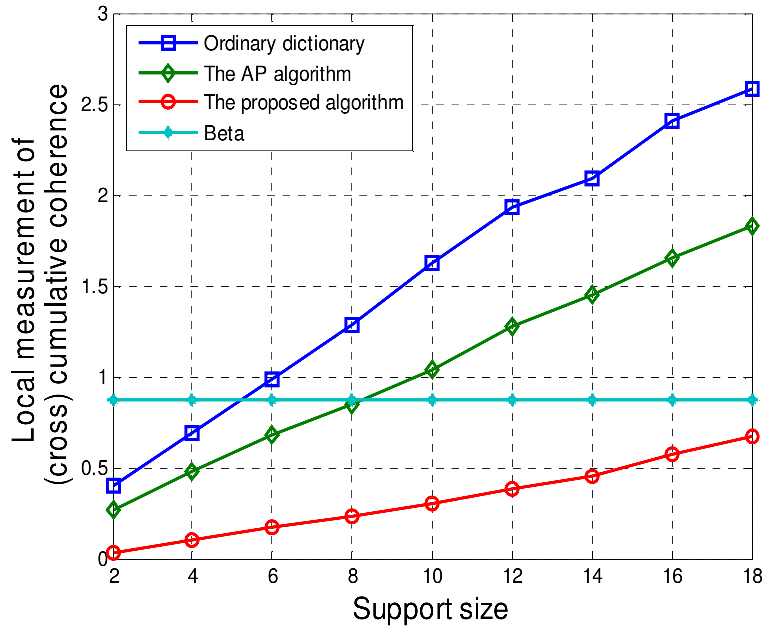

Figure 1. Local measurement of (Cross) cumulative coherence via different support size.

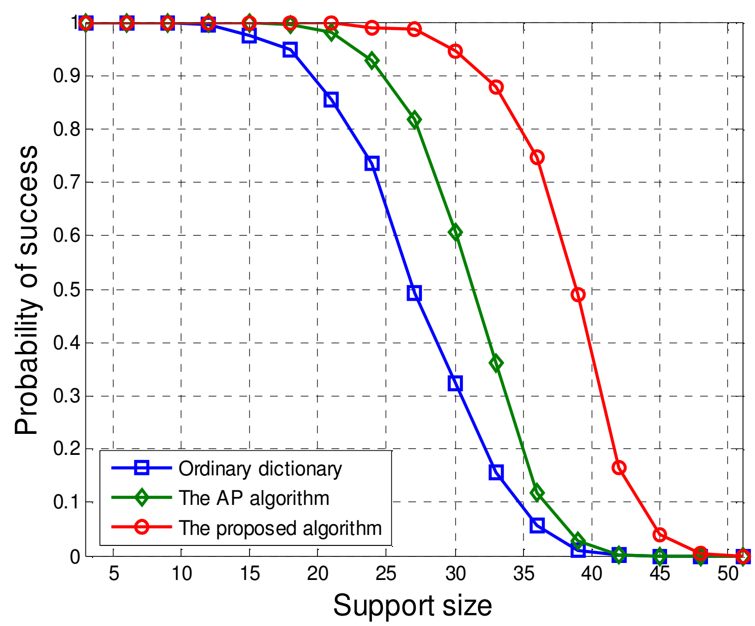

Figure 2. Probability of exact recovery via different support size.

each step for larger support size K, since the local measurement of cross cumulative coherence is significantly lower than the cross cumulative coherence.

In our second experiment, we compare the performance of exact recovery of these methods and simulation results are obtained over 500 independent Monte-Carlo trails. Figure 2 shows the probability of exact recovery via different support size. It is observed that the performance of the modified OMP algorithm based on data dependent sensing dictionary is better than other methods.

Conclusion

In this paper, we provided a re-weighted algorithm for designing data dependent sensing dictionary by introducing posteriori knowledge obtained from the observed data. Through alternately updating sensing dictionary and revising weights, the proposed algorithm can effectively design a data dependent sensing dictionary.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China under grant 60772146, the National High Technology Research and Development Program of China (863 Program) under grant 2008AA12Z306 and in part by the Key Project of Chinese Ministry of Education under grant 109139.

REFERENCES

Gribonval R, Krstulovic S (2006). MPTK: Matching Pursuit made tractable. Proc. Int. Conf. Acoust. Speech Signal Process. (ICASSP'06), 5: 496-499.

Gribonval R, Vandergheynst P (2006). On the exponential convergence of matching pursuits in quasi-incoherent dictionaries. IEEE Trans. Inf. Theory, 52(1): 255-261.

Natarajan BK (1995). Sparse approximate solutions to linear systems. SIAM J. Comput., 24(4): 227-234.

Pati YC, Rezaiifar R, Krishnaprasad PS (1993). Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition. Proc. 27th Annu. Asilomar Conf. Signals Syst. Comput., Pacific Grove, CA, 1: 40-44.

Schnass K, Vandergheynst P (2008). Dictionary Preconditioning for Greedy Algorithms. IEEE Trans. Signal Process., 56(5): 1994-2002.

Tropp J (2004). Greed is good: Algorithmic results for sparse approximation. IEEE Trans. Inf. Theory, 50(10): 2231-2242.