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Full Length Research Paper

Matrix stability of the difference schemes for nonlocal boundary value problems for parabolic differential equations

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In this work, a first order and second order difference schemes, namely Rothe and Crank-Nicholson, respectively, for solving nonlocal boundary value problems for parabolic differential equations are presented. The stability of the difference schemes are proved by using the matrix stability approach. Numerical results are provided to illustrate the accuracy and efficiency of the schemes.

Key words: Matrix stability, Rothe difference scheme, Crank-Nicholson difference scheme, Kailath Theorem, nonlocal boundary value problems for parabolic differential equations, Schur complement, matrix block inversion.

INTRODUCTION

Many problems in applied science, physics and engineering areas are modeled mathematically by parabolic differential equations. In this work, we consider the nonlocal boundary value problem for one dimensional heat equation:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \ (0<x<1, 0<t<1), \\ u(0,x) = u(1,x) + \rho(x), \ 0 \le x \le 1, \\ u(t,0) = 0, \ \ u(t,1) = 0, \ \ 0 \le t \le 1. \end{cases} \quad (1)$$

Using the backward difference approximation for the time derivative $\frac{\partial u(t,x)}{\partial t}$ and the centered difference approximation for the spatial second derivative $\frac{\partial^2 u(t,x)}{\partial x^2}$ in Equation (1), at the point (t_k, x_n), we obtain the following Rothe difference scheme which is accurate of order $O(\tau + h^2)$:

$$\begin{cases} \dfrac{\left(U_n^k - U_n^{k-1}\right)}{\tau} - \dfrac{U_{n+1}^k - 2U_n^k + U_{n-1}^k}{h^2} = f(t_k, x_n), \quad 1 \le k \le N, 1 \le n \le M-1, \\ U_0^k = 0, U_M^k = 0, 0 \le k \le N, \\ U_n^0 - U_n^N = \rho(x_n), 1 \le n \le M-1. \end{cases} \quad (2)$$

Here U_n^k denotes the numerical approximation to the exact solution $u(t_k, x_n)$ where $t_k = k\tau, \ 0 \le k \le N, \ N\tau = 1$ and $x_n = nh, \ 0 \le n \le M,$ $Mh = 1.$

We can arrange the scheme (2) and obtain the following system:

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$$\begin{cases}\left(-\frac{1}{h^2}\right)U_{n+1}^k+\left(-\frac{1}{\tau}\right)U_n^{k-1}+\left(\frac{1}{\tau}+\frac{2}{h^2}\right)U_n^k+\left(-\frac{1}{h^2}\right)U_{n-1}^k \\ \qquad =f(t_k,x_n), \qquad 1\le k\le N,\ 1\le n\le M-1, \\ U_0^k=U_M^k=0,\ \ 0\le k\le N, \\ U_n^0-U_n^N=\rho(x_n), \quad 1\le n\le M-1. \end{cases} \tag{3}$$

The difference scheme (3) can be written in matrix form:

$$\begin{cases} A\,U_{n+1}+B\,U_n+A\,U_{n-1}=\phi_n, \quad 1\le n\le M-1, \\ U_0=\vec{0},\ U_M=\vec{0}. \end{cases} \tag{4}$$

where $\phi_n=\left[\phi_n^0,\phi_n^1,\phi_n^2,...,\phi_n^N\right]^t$, $\vec{0}=0_{(N+1)\times 1}$, $\phi_n^0=\rho(x_n)$, $1\le n\le M-1$, and $\phi_n^k=f(t_k,x_n)$, $1\le k\le N$, $1\le n\le M-1$. Here A and B are the matrices of the form:

$$A=\begin{bmatrix}0 & & & & \\ & a & & & \\ & & a & & \\ & & & \ddots & \\ & & & & a\end{bmatrix}_{(N+1)\times(N+1)} \quad \text{and}$$

$$B=\begin{bmatrix}1 & & & & -1 \\ b & c & & & \\ & b & c & & \\ & & \ddots & \ddots & \\ & & & b & c\end{bmatrix}_{(N+1)\times(N+1)}$$

where $a=-\frac{1}{h^2}=-M^2$, $b=-\frac{1}{\tau}=-N$, $c=\frac{1}{\tau}+\frac{2}{h^2}=N+2M^2$, $U_n=\left[U_n^0,U_n^1,U_n^2,...,U_n^N\right]^T$.

Using the idea of the modified Gauss-elimination method, we can convert the Equation (4) into the following form:

$$U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}, \quad n=M-1,...,2,1,0. \tag{5}$$

This way, the two-step form of the difference scheme in (4) is transformed to one-step method as in (5). Now, we need to determine the matrices α_{n+1} and β_{n+1} satisfying the last equality. Since $U_0=\alpha_1U_1+\beta_1=0$, we can select $\alpha_1=0_{(N+1)\times(N+1)}$ and $\beta_1=0_{(N+1)\times 1}$.

Combining the equalities $U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}$, and $U_{n-1}=\alpha_nU_n+\beta_n$ and the matrix Equation (4), we have

$$\left(A+B\alpha_{n+1}+A\alpha_n\alpha_{n+1}\right)U_{n+1}+(B\beta_{n+1}+A\alpha_n\beta_{n+1}+A\beta_n)=\phi_n. \tag{6}$$

Then, we write

$$\begin{cases} A+B\alpha_{n+1}+A\alpha_n\alpha_{n+1}=0 \\ B\beta_{n+1}+A\alpha_n\beta_{n+1}+A\beta_n=\phi_n, \end{cases}$$

where $1\le n\le M-1$.

So, we obtain the following pair of formulas:

$$\begin{cases} \alpha_{n+1}=-(B+A\alpha_n)^{-1}A, \\ \beta_{n+1}=(B+A\alpha_n)^{-1}(\phi_n-A\beta_n) \end{cases}$$

where $1\le n\le M-1$.

We note that the stepwise stability is proved in (Ivanauskas et al., 2009) for Bitsadze-Samarskii type nonlocal problems. Asymptotic stability of numerical methods are studied in (Tian, 2008) for linear delay parabolic differential equations. The stability of the both schemes is proved by analyzing the behavior of the each iteration matrix that is called the matrix stability. To show the matrix stability of this method, we give some remarks and lemmas.

Remark 1

If *X*, *Y*, *Z*, *T* are square block matrices and the matrices X and S are invertible, then

$$\begin{bmatrix} X & Y \\ Z & T\end{bmatrix}^{-1}=\begin{bmatrix} X^{-1}+X^{-1}YS^{-1}ZX^{-1} & -X^{-1}YS^{-1} \\ -S^{-1}ZX^{-1} & S^{-1}\end{bmatrix} \tag{7}$$

where S is the Schur (Duncan, 1944) complement of the block inversion and $S=\left(T-ZX^{-1}Y\right)$.

Remark 2

The symbol $\|.\|$ denotes the infinity norm which is

$$\|A_{n\times n}\|=\|A_{n\times n}\|_\infty=\max_{1\le i\le n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\} \tag{8}$$

where $A = [a_{ij}]_{n \times n}$.

Remark 3

$Z_{n \times n}$ is called a strictly diagonally dominant (Morača, 2008), if $\left| Z_{i,i} \right| > r_i(Z)$, $1 \le i \le n$, where $r_i(Z)$ is the sum of the absolute value of nondiagonal elements on the *i*-th row of Z.

Remark 4

If $Z_{n \times n}$ is strictly diagonally dominant matrix then it is not singular (Savioli et al., 1997) and

$$\left\| Z^{-1} \right\| \le \frac{1}{\min\limits_{1 \le i \le n} (\left| Z_{i,i} \right| - r_i(Z))}. \tag{9}$$

STABILITY OF THE FIRST ORDER METHOD

Lemma 1

If A and B are matrices given in (4) then $\left\| B^{-1} A \right\| \le \frac{1}{2}$.

Proof

Let matrix B be partition be into subblocks to find its inverse that is

$$B^{-1} = \left[\begin{array}{c|cccc} 1 & & & & -1 \\ \hline b & c & & & \\ & b & c & & \\ & & \ddots & \ddots & \\ & & & b & c \end{array}\right]^{-1} = \left[\begin{array}{c|c} X & Y \\ \hline Z & T \end{array}\right]^{-1}$$

where $X = I_{1x1} = [1]$, $Y = [0, \ldots, 0, -1]$, $Z = [b, 0, \ldots, 0]^t$,

$$T = \begin{bmatrix} c & & & \\ b & c & & \\ & \ddots & \ddots & \\ & & b & c \end{bmatrix}.$$

Then,

$$B^{-1} = \begin{bmatrix} X^{-1} + X^{-1} Y S^{-1} Z X^{-1} & -X^{-1} Y S^{-1} \\ -S^{-1} Z X^{-1} & S^{-1} \end{bmatrix} = \begin{bmatrix} 1 + 1 Y S^{-1} Z 1 & -1 Y S^{-1} \\ -S^{-1} Z 1 & S^{-1} \end{bmatrix},$$

where S is the Schur complement of this block inversion and $S = T - Z X^{-1} Y = T - ZY$. Hence, block matrix multiplication gives $B^{-1} A = \begin{bmatrix} 0 & -YaS^{-1} \\ 0 & aS^{-1} \end{bmatrix}$. $\left\| B^{-1} A \right\| = \max\{ \left\| -YaS^{-1} \right\|, \left\| aS^{-1} \right\| \} = \left\| aS^{-1} \right\|$, since YaS^{-1} is exactly the last row of aS^{-1}.

On the other hand,

$$S = \begin{bmatrix} c & & & \\ b & c & & \\ & \ddots & \ddots & \\ & & b & c \end{bmatrix} - \begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix} = \begin{bmatrix} c & & & b \\ b & c & & \\ & \ddots & \ddots & \\ & & b & c \end{bmatrix}.$$

Since $|c| > |b|$, the matrix S is strictly diagonally dominant. From Remark 4, it follows that $\left\| S^{-1} \right\| \le \frac{1}{|c| - |b|}$. Therefore,

$$\begin{aligned} \left\| B^{-1} A \right\| &= \left\| a S^{-1} \right\| \le |a| \left\| S^{-1} \right\| = M^2 \left\| S^{-1} \right\| \\ &\le M^2 \frac{1}{|c| - |b|} = M^2 \frac{1}{(N + 2M^2) - N} \\ &= \frac{1}{2}. \end{aligned}$$

So $\left\| B^{-1} A \right\| \le \frac{1}{2}$.

Lemma 2.

If $\left\| \alpha_n \right\| \le 1$ and $\left\| B^{-1} A \right\| \le \frac{1}{2}$ then $(I + \alpha_n B^{-1} A)$ is strictly diagonally dominant and $\left\| (I + \alpha_n B^{-1} A)^{-1} \right\| \le \frac{1}{1 - \left\| \alpha_n \right\| \left\| B^{-1} A \right\|}$.

Proof

Since $\left\| \alpha_n \right\| \le 1$ and $\left\| B^{-1} A \right\| \le \frac{1}{2}$, we have

$\left\|\alpha_n B^{-1}A\right\| \le \left\|\alpha_n\right\|\left\|B^{-1}A\right\| \le \frac{1}{2}$. Therefore $I+\alpha_n B^{-1}A$ is strictly diagonally dominant. Put $Z = I+\alpha_n B^{-1}A$. Define $r_i(Z)$ as in Remark 4. Then, using the Remark 4, we have

$$\left\|(I+\alpha_n B^{-1}A)^{-1}\right\| = \left\|Z^{-1}\right\| \le \frac{1}{\min_i\left\{\left|Z_{i,i}\right| - r_i(Z)\right\}}$$
$$\le \frac{1}{1-\left\|\alpha_n B^{-1}A\right\|}$$
$$\le \frac{1}{1-\left\|\alpha_n\right\|\left\|B^{-1}A\right\|}.$$

Theorem 1

The matrix Equation (5) which we use to solve the differential Equation (1) is unconditionally stable.

Proof

α_{n+1} is the iteration matrix of the system

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1} \tag{10}$$

and to prove the stability we will show that $\left\|\alpha_{n+1}\right\| \le 1$ for all $0 \le n \le M-1$, as in (Smith, 1993). We prove it by induction.

Since $\alpha_1 = \vec{0}$ then $\left\|\alpha_1\right\| \le 1$. Moreover, $\alpha_2 = -(B+A\alpha_1)^{-1}A = -(B+A\ \ \vec{0})^{-1}A = -B^{-1}A,$ from Lemma 1 we already know $\left\|\alpha_2\right\| = \left\|-B^{-1}A\right\| \le \frac{1}{2} \le 1.$

Now, assume $\left\|\alpha_n\right\| \le 1$. On the other hand, from the Kailath theorem (Kailath, 1980), we can write $(B+A\alpha_n)^{-1} = B^{-1} - B^{-1}A(I+\alpha_n B^{-1}A)^{-1}\alpha_n B^{-1}.$ Then we obtain,

$$\left\|\alpha_{n+1}\right\| = \left\|-(B+A\alpha_n)^{-1}A\right\|$$
$$= \left\|\left[B^{-1} - B^{-1}A(I+\alpha_n B^{-1}A)^{-1}\alpha_n B^{-1}\right]A\right\|$$
$$= \left\|B^{-1}A - B^{-1}A(I+\alpha_n B^{-1}A)^{-1}\alpha_n B^{-1}A\right\|$$
$$\le \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|\left\|(I+\alpha_n B^{-1}A)^{-1}\alpha_n\right\|\left\|B^{-1}A\right\|$$
$$\le \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|\left\|(I+\alpha_n B^{-1}A)^{-1}\right\|\left\|\alpha_n\right\|\left\|B^{-1}A\right\|$$
$$\le \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|^2\left\|(I+\alpha_n B^{-1}A)^{-1}\right\|\left\|\alpha_n\right\|$$
$$= \left\|B^{-1}A\right\|\left(1+\left\|\alpha_n\right\|\left\|B^{-1}A\right\|\left\|(I+\alpha_n B^{-1}A)^{-1}\right\|\right).$$

If we combine the last inequality and Lemma 2, then we have

$$\left\|\alpha_{n+1}\right\| \le \left\|B^{-1}A\right\|\left(1+\frac{\left\|\alpha_n\right\|\left\|B^{-1}A\right\|}{1-\left\|\alpha_n\right\|\left\|B^{-1}A\right\|}\right) \le \left\|B^{-1}A\right\|\left(\frac{1}{1-\left\|\alpha_n\right\|\left\|B^{-1}A\right\|}\right). \tag{11}$$

Since $\left\|\alpha_n\right\| \le 1$ and $\left\|B^{-1}A\right\| \le \frac{1}{2}$, we have $\left\|\alpha_{n+1}\right\| \le 1$. Therefore $\left\|\alpha_{n+1}\right\| \le 1$ for all $0 \le n \le M$.

THE CRANK-NICHOLSON METHOD

To solve the problem (1), we can also use the Crank-Nicholson difference scheme which is accurate of order $O(\tau^2+h^2)$:

$$\begin{cases} \dfrac{\left(U_n^k - U_n^{k-1}\right)}{\tau} - \dfrac{1}{2}\left[\dfrac{U_{n+1}^k - 2U_n^k + U_{n-1}^k}{h^2} + \dfrac{U_{n+1}^{k-1} - 2U_n^{k-1} + U_{n-1}^{k-1}}{h^2}\right] = f(t_k - \dfrac{\tau}{2}, x_n); \\ \qquad 1 \le k \le N, 1 \le n \le M-1, \\ U_0^k = 0, U_M^k = 0, 0 \le k \le N, \\ U_n^0 - U_n^N = \rho(x_n), 1 \le n \le M-1. \end{cases} \tag{12}$$

We can arrange the scheme (12), and obtain the following system:

$$\begin{cases} \left(-\dfrac{1}{2h^2}\right)U_{n+1}^{k-1} + \left(-\dfrac{1}{2h^2}\right)U_{n+1}^k + \left(-\dfrac{1}{\tau}+\dfrac{1}{h^2}\right)U_n^{k-1} + \left(\dfrac{1}{\tau}+\dfrac{1}{h^2}\right)U_n^k + \left(-\dfrac{1}{2h^2}\right)U_{n-1}^{k-1} \\ \qquad + \left(-\dfrac{1}{2h^2}\right)U_{n-1}^k = f(t_k - \dfrac{\tau}{2}, x_n), 1 \le k \le N, \ 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, \ 0 \le k \le N, \\ U_n^0 - U_n^N = \rho(x_n), 1 \le n \le M-1. \end{cases} \tag{13}$$

The difference scheme (13) can be written in matrix form:

$$\begin{cases} AU_{n+1} + BU_n + AU_{n-1} = \phi_n, \quad 1 \le n \le M-1, \\ \qquad U_0 = \vec{0}, \ U_M = \vec{0}. \end{cases} \tag{14}$$

where $\phi_n = \left[\phi_n^0, \phi_n^1, \phi_n^2, ..., \phi_n^N\right]^t$, $\vec{0} = 0_{(N+1)\times 1}$, $\phi_n^0 = \rho(x_n)$, $1 \le n \le M-1$ and $\phi_n^k = f(t_k - \frac{\tau}{2}, x_n)$, $1 \le k \le N$, $1 \le n \le M-1$.

Here A and B are the matrices of the form:

$$A = \begin{bmatrix} 0 & & & & \\ a & a & & & \\ & a & a & & \\ & & \ddots & \ddots & \\ & & & a & a \end{bmatrix}_{(N+1)\times(N+1)}$$

$$B = \begin{bmatrix} 1 & & & & -1 \\ b & c & & & \\ & b & c & & \\ & & \ddots & \ddots & \\ & & & b & c \end{bmatrix}_{(N+1)\times(N+1)}$$

where

$$a = -\tfrac{1}{2h^2} = -\tfrac{M^2}{2}, \; b = -\tfrac{1}{\tau} + \tfrac{1}{h^2} = -N + M^2, \; c = \tfrac{1}{\tau} + \tfrac{1}{h^2} = N + M^2$$

Using the aforementioned approach, we can convert the Equation (14) into the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0. \tag{15}$$

Lemma 3

If A and B are matrices given in (14) and if $N \ge M^2$, then $\left\|B^{-1}A\right\| \le \frac{1}{2}$.

Proof

The matrix B was partition into sub blocks to find its inverse that is

$$B^{-1} = \left[\begin{array}{c|c} X & Y \\ \hline Z & T \end{array}\right]^{-1}$$

where

$$X = I_{1x1} = [1], \quad Y = [0, ..., 0, -1], \quad Z = [b, 0, ..., 0]^t,$$

$$T = \begin{bmatrix} c & & & & \\ b & c & & & \\ & b & c & & \\ & & \ddots & \ddots & \\ & & & b & c \end{bmatrix}.$$

Then,

$$B^{-1} = \begin{bmatrix} X^{-1} + X^{-1}YS^{-1}ZX^{-1} & -X^{-1}YS^{-1} \\ -S^{-1}ZX^{-1} & S^{-1} \end{bmatrix} = \begin{bmatrix} 1 + 1YS^{-1}Z1 & -1YS^{-1} \\ -S^{-1}Z1 & S^{-1} \end{bmatrix},$$

where S is the Schur complement of this block inversion and $S = T - ZX^{-1}Y = T - ZY$.

Hence, block matrix multiplication gives $B^{-1}A = \begin{bmatrix} -YS^{-1}K & -YS^{-1}L \\ S^{-1}K & S^{-1}L \end{bmatrix}$, where K and L are submatrices of A and

$$K = \begin{bmatrix} a \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \; L = \begin{bmatrix} a & & & \\ a & a & & \\ & \ddots & \ddots & \\ & & a & a \end{bmatrix}.$$

$$\left\|B^{-1}A\right\| \le \max\{\left\|-YS^{-1}K\right\| + \left\|-YS^{-1}L\right\|, \left\|S^{-1}K\right\| + \left\|S^{-1}L\right\|\} = \left\|S^{-1}K\right\| + \left\|S^{-1}L\right\|,$$

since $-YS^{-1}K$ is exactly the last row of $S^{-1}K$ and $-YS^{-1}L$ is exactly the last row of $S^{-1}L$. On the other hand,

$$S = \begin{bmatrix} c & & & \\ b & c & & \\ & \ddots & \ddots & \\ & & b & c \end{bmatrix} - \begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} c & & & b \\ b & c & & \\ & \ddots & \ddots & \\ & & b & c \end{bmatrix}.$$

Since $|c| > |b|$, the matrix S is strictly diagonally dominant. From Remark 4 it follows that $\left\|S^{-1}\right\| \le \frac{1}{|c|-|b|}$.

Therefore,

Table 1. The errors for the solutions of Example 1 for some values of *M* and *N*.

M #space node	*N* #time node	The error by Rothe	The error by C-N
6	40	0.000665509259259	0.00003906273582
7	50	0.000516451478552	0.00002448991067
8	70	0.000376674107145	0.00001275524756
9	90	0.000287896492744	0.00000762113037

$$\begin{aligned}\left\|B^{-1}A\right\| &\le \left\|S^{-1}[K,L]\right\| \le \left\|S^{-1}\right\|\left\|[K,L]\right\| \\ &\le \frac{1}{|c|-|b|}(2\,|\,a\,|) \\ &= \frac{1}{(M^2+N)-(M^2-N)}\frac{2M^2}{2}=\frac{M^2}{2N},\end{aligned}$$

where $[K,L]$ is the matrix obtained from A by deleting its first row. So $\left\|B^{-1}A\right\| \le \frac{1}{2}$ if $N \ge M^2$.

Theorem 2

The matrix Equation (15) which we use to solve the differential Equation (1) is stable, if $N \ge M^2$.

Proof

Assume $N \ge M^2$, then by using the Lemma 3, we obtain $\left\|B^{-1}A\right\| \le \frac{1}{2}$. By following the steps in the proof of the Theorem 1, it is seen that the norms of the iteration matrices are less than unity, that is, $\left\|\alpha_{n+1}\right\| \le 1$ for all $0 \le n \le M$.

NUMERICAL ANALYSIS

Example 1

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2 u(t,x)}{\partial x^2}=2tx-2tx^2-1+2t^2, (0<x<1, 0<t<1), \quad (16)\\ u(0,x)=u(1,x)+x(x-1), 0\le x\le 1,\\ u(t,0)=0,\ u(t,1)=0,\ 0\le t\le 1.\end{cases}$$

Exact solution of this problem is $U(t,x)=(t^2-1/2)x(1-x)$. The errors when solving this problem are listed in Table 1 for various values of time and space nodes. The errors are calculated by the formula $\max\limits_{\substack{0\le n\le M\\ 0\le k\le N}}\left|u(t_k,x_n)-U_n^k\right|$.

The solutions of this problem at $x=1/2$ are compared in Figure 1 and the data given in Table 2. The graph of the solution by the Crank-Nicholson scheme is given in Figure 2

Example 2.

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2 u(t,x)}{\partial x^2}=2t\sin\left(x(1-x)\right)+\left(t^2+1\right)\sin\left(x(1-x)\right)(1-2x)^2\\ \qquad +2\left(t^2+1\right)\cos\left(x(1-x)\right); \qquad (0<x<1, 0<t<1),\\ u(0,x)=u(1,x)-\sin(x(x-1)),\ 0\le x\le 1,\\ u(t,0)=0,\ u(t,1)=0,\ 0\le t\le 1.\end{cases}$$

(17)

Exact solution of this problem is $U(t,x)=(t^2+1)\sin(x-x^2)$. The errors when solving this problem are listed in the Table 3 for various values of time and space nodes. The solutions at $x=1/2$ of this problem are given in Table 4. The graph of the solution by the first order method is given in Figure 3.

CONCLUSION AND FUTURE WORK

Unconditional stability of the first order difference scheme for the nonlocal boundary value problems for parabolic differential equation is proved. A useful sufficient condition is obtained for the stability of the Crank Nicolson difference scheme for the nonlocal boundary value problems. Numerical results are provided to illustrate the accuracy and efficiency of these schemes.

This method gives a very practical way of analyzing the stability of nonlocal boundary value problems for parabolic differential equations.

In the future, it may be studied on the matrix stability of *fractional* nonlocal boundary value problems for parabolic differential equations.

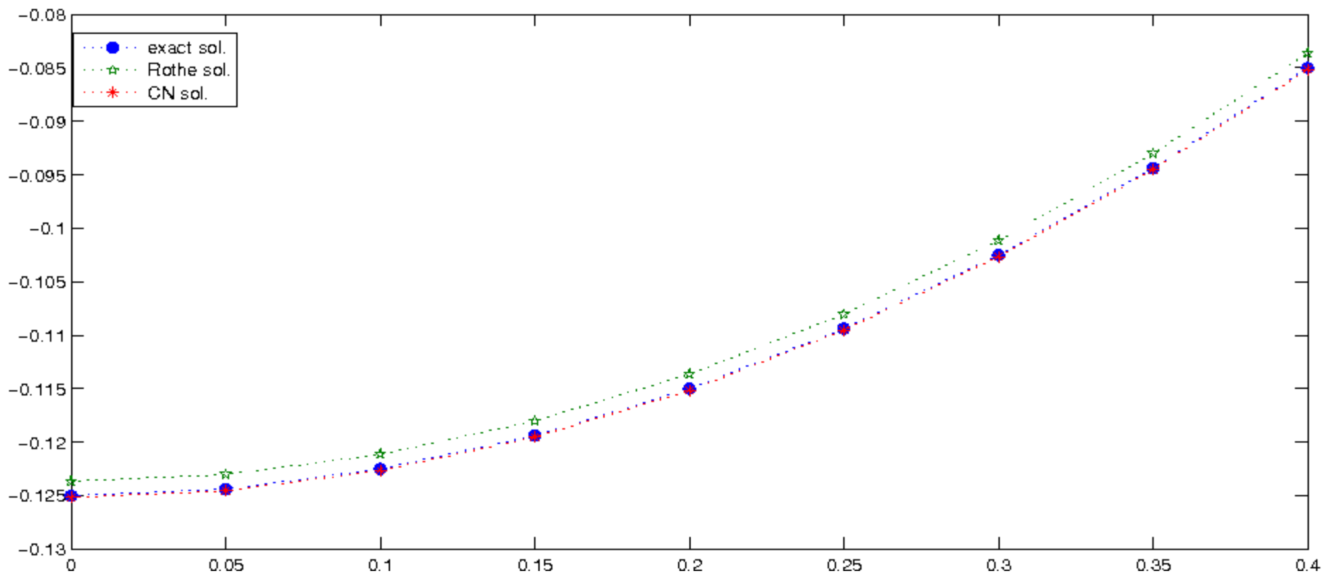

Figure 1. The solutions for the problem (16), when N=20, M=4 at $x=1/2$ and $0 \le t \le 0.4$.

Table 2. The solutions for the problem (16), when N=20, M=4 at $x=1/2$.

t_k	Exact solution	Solution by Rothe	Solution by C-N
0.00	-0.125000	-0.12363281250000	-0.12515625000000
0.05	-0.124375	-0.12300781250000	-0.12453125000000
0.10	-0.122500	-0.12113281250000	-0.12265625000000
0.15	-0.119375	-0.11800781250000	-0.11953125000000
0.20	-0.115000	-0.11363281250000	-0.11515625000000
0.25	-0.109375	-0.10800781250000	-0.10953125000000
0.30	-0.102500	-0.10113281250000	-0.10265625000000
0.35	-0.094375	-0.09300781250000	-0.09453125000000
0.40	-0.085000	-0.08363281250000	-0.08515625000000

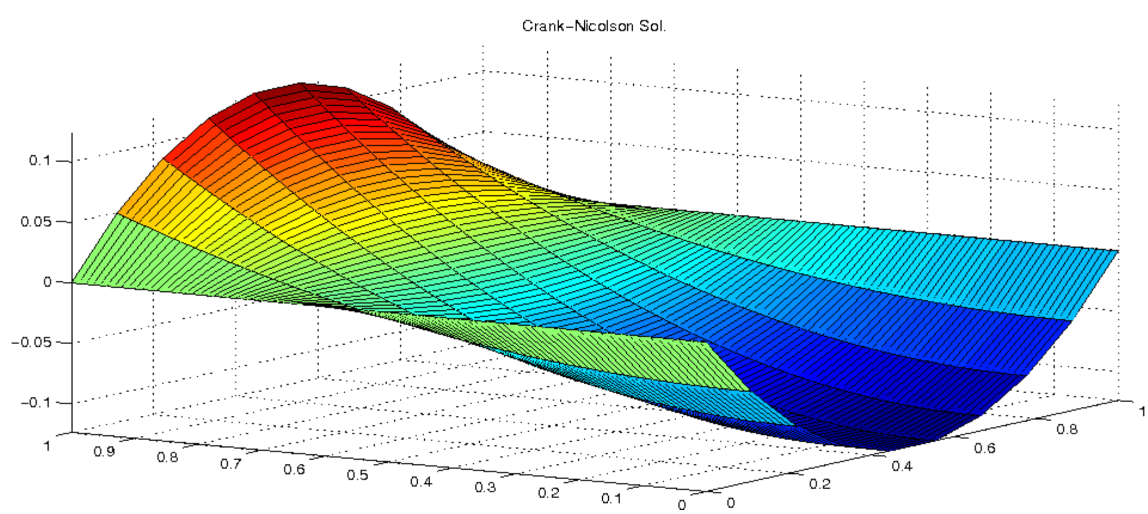

Figure 2. The solutions by the Crank-Nicolson method when N=90, M=9.

Table 3. The errors for the solutions of the problem (17) for some values of M and N.

M #space node	N #time node	The error by Rothe	The error by C-N
9	90	0.0002255180078	0.0001383289906
11	130	0.0001576646963	0.0000894840438
15	250	0.0000841576084	0.0000462926435
25	650	0.0000320973542	0.0000162142703
30	1000	0.0000212795923	0.0000112644031

Table 4. The solutions of the problem (17), when *N*=1000, *M*=30 at $x = 1/2$.

t_k	Exact solution	Solution by Rothe	Solution By C-N
0	0.24740395930000	0.24741861480366	0.24739269489694
0.1	0.24987799890000	0.24989738756506	0.24987147341118
0.2	0.25730011770000	0.25731974802291	0.25729383376890
0.3	0.26967031560000	0.26968984151623	0.26966392653856
0.4	0.28698859280000	0.28700781198754	0.28698189629827
0.5	0.30925494910000	0.30927371304064	0.30924779672858
0.6	0.33646938460000	0.33648756475956	0.33646164784626
0.7	0.36863189940000	0.36864937467387	0.36862345713749
0.8	0.40574249330000	0.40575914560781	0.40573322736750
0.9	0.44780116630000	0.44781687861759	0.44779095959343
1	0.49480791860000	0.49482257410356	0.49479665419710

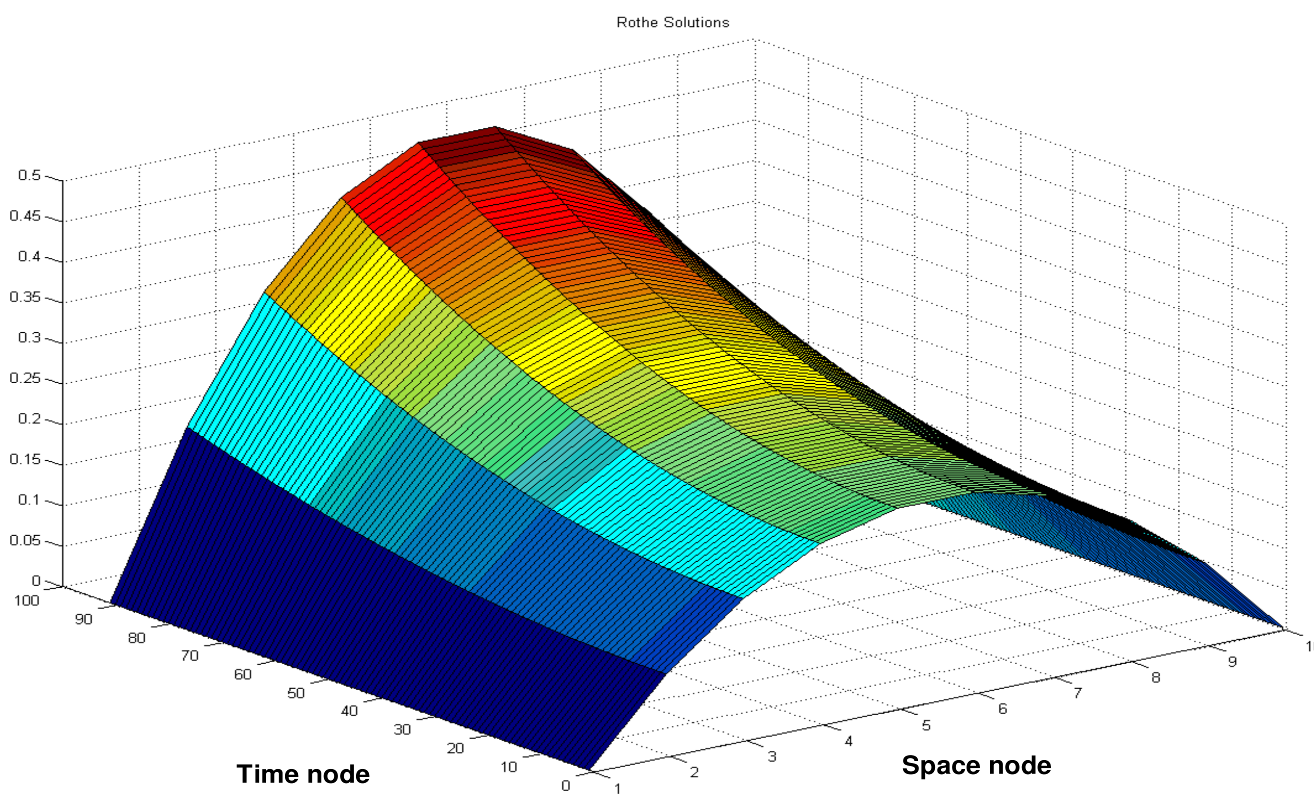

Figure 3. The solutions of the problem (17) by the Rothe method, when *N*=90, *M*=9.

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