Full Length Research Paper

Hydromagnetic flows of a mixture of two Newtonian fluids between two parallel plates

S. Barış and M. Ş. Demir*

Department of Mechanical Engineering, Faculty of Engineering, Istanbul University, 34320 Avcilar-Istanbul, Turkey.

The paper aims to study the flow of a binary mixture of electrically conducting, incompressible and viscous fluids between two parallel plates in the presence of a transverse uniform magnetic field. The solution of such a flow model has many applications in magnetohydrodynamic (MHD) power generators, MHD pumps, MHD accelerators, and MHD flowmeters. Exact solutions have been obtained for the following four problems: (1) steady hydromagnetic Couette flow, (2) unsteady hydromagnetic Couette flow, (3) steady hydromagnetic Poiseuille flow, (4) unsteady hydromagnetic Poiseuille flow. The mean velocity of the mixture is drawn for different values of magnetic parameters and results are interpreted with the aid of graphs. The previous solutions involving single Newtonian fluid appear as the special cases of the present analysis.

Key words: Binary mixture, Newtonian fluid, magnetohydrodynamics (MHD), steady/unsteady flow, Couette/Poiseuille flow.

INTRODUCTION

The mixture theory finds important applications in various branches of engineering and technology. Familiar examples are suspensions, emulsions, multigrade oils, polycrystalline aggregates, granular media, bubbly liquids, liquid crystals, fluid filled porous elastic solids, composite elastic solids and alloys (Srivastava et al., 1982). The inadequacy of the basic theory for a single continuous media in predicting the behavior of such substances leads to developments in the continuum theory of mixtures. Historical discussion on the development of the subject is sufficiently available in the literature. Theoretical research on the modern formulation of the thermomechanics of interacting continua was initiated by Truesdell (1957). He presented the balance of mass, momentum, energy and the second law of thermodynamics in the context of the continuum theory. Review articles on the mixture theory by Bowen (1976), Atkin and Craine (1976) and Bedford and Drumheller (1983) are of particular interest. We also refer the reader

to the books by Truesdell (1984), Samohyl (1987) and Rajagopal and Tao (1995) regarding the historical development of the theory and detailed analysis of various results on this subject.

Adkins (1963) formulated constitutive equations for the stresses in each constituent. He also examined some steady flows of compressible mixtures of non-Newtonian fluids. The continuum theory of compressible mixtures of Newtonian fluids was first considered by Green and Naghdi (1965). Müller (1968) also studied a thermomechanical theory for mixtures of fluids in which there are no chemical reactions. Eringen and Ingram (1965) and Ingram and Eringen (1967) studied mixtures of chemically reacting fluids. The constitutive equations for an incompressible mixture of Newtonian fluids were derived by Mills (1966) using the theory of Green and Naghdi (1965). Craine (1971) examined the flow induced by the steady oscillations of an infinite plate in a mixture of two incompressible Newtonian fluids. In his

*Corresponding author. E-mail: demirms@istanbul.edu.tr. Tel: 902124737070.

subsequent study (Craine, 1973), he considered the same problem for a binary mixture of incompressible Newtonian hemihedral fluids. Wilhelm and Van Der Werff (1977) investigated the flow of two miscible, viscous, incompressible fluids subject to oscillatory pressure gradient in a cylindrical tube. Beevers and Craine (1982) extended the list of known solutions for a mixture of two viscous fluids and discussed in a more detail methods for evaluating the response functions. Some exact solutions for flows of a binary mixture of viscous incompressible fluids in different geometries were obtained by Göğüş (1988, 1991, 1992a, b, 1994 and 1995). Many other authors including Al-Sharif et al. (1993), Chamniprasart et al. (1993), Wang et al. (1993), Barış (2005), and Massoudi (2008) worked on applications of the theory of two miscible fluids to practical problems within the context of the mixture theory. Recently, Barış and Demir (2012) have obtained the exact solutions in series form for the flow of a mixture of two incompressible Newtonian fluids in a semicircular duct.

The present paper aims to study the Hartmann problem for a binary mixture of Newtonian fluids and generate theoretical results. In Hartmann flow, the hydromagnetic analogue of Couette and Poiseuille flows, there is an imposed, uniform magnetic field normal to the surfaces. The flow may be induced by a pressure gradient or by relative motion of the two solid walls. Flows of this type are encountered in a variety of applications such as magnetohydrodynamics (MHD) power generators, MHD pumps, MHD accelerators, and MHD flowmeters, and they can also be expanded into various industrial uses. The study of the flow of immiscible fluids under the influence of a magnetic field was considered by various authors. Shail (1973) studied Hartmann flow of a conducting fluid and a non-conducting fluid layer in a channel. Mitra (1982) analyzed the unsteady flow of two electrically conducting fluids between two parallel plates. Lohrasbi and Sahai (1988) considered MHD two-phase flow and heat transfer in a horizontal channel and obtained analytical solutions for the case where one of two fluids was assumed to be electrically non-conducting. Malashetty and Leela (1992) analytically investigated the problem of two-phase MHD flow and heat transfer in a horizontal channel for which both phases are electrically conducting. Malashetty et al. (2001) examined the two-fluid MHD flow and heat transfer in an inclined channel. Umavathi et al. (2006) presented analytical solutions of an oscillatory Hartmann two-fluid flow and heat transfer in an horizontal channel. Recently, Umavathi et al. (2008), Nikodijevic et al. (2011), and Sivaraj et al. (2012) have studied the two-fluid MHD flow and heat transfer with various geometries. Most of the problems relating to the petroleum industry, geophysics, plasma physics, magneto-fluid dynamics, and so forth involve the two-fluid MHD flow situations.

The present investigations on the two-fluid MHD flow pertain to the mechanics of two immiscible fluids.

Different from all studies mentioned above, the present paper deals with the flow of a binary mixture of viscous fluids between two parallel plates in the presence of a transversely magnetic field. The basic scientific method utilized in the present research is the mixture theory. We obtained the analytical solutions for the MHD Couette and Poiseuille flows of a mixture of two incompressible Newtonian fluids in a parallel plate channel.

BASIC THEORY

A brief review of the notation and basic equations of a mixture containing two incompressible Newtonian fluids is presented in this section. The reader should consult the articles by Atkin and Craine (1976) for more details.

The mixture of two viscous fluids is considered to be a purely mechanical system. That is, thermal effects and chemical reactions are ignored. The fluids in the mixture will be represented $s^{(1)}$ and $s^{(2)}$. If $\mathbf{v}^{(\beta)}$ denotes the velocity of $s^{(\beta)}$, the material time derivative $D^{(\beta)}/Dt$ is defined by

$$\frac{D^{(\beta)}}{Dt} = \frac{\partial}{\partial t} + v_k^{(\beta)} \frac{\partial}{\partial x_k} \tag{1}$$

where x_k's are the spatial coordinates and the superscript β refers the β-th fluid. Here and henceforth β takes the values 1 and 2. The mean velocity $\mathbf{v}$ of the mixture is calculated from

$$\rho\mathbf{v} = \rho_1\mathbf{v}^{(1)} + \rho_2\mathbf{v}^{(2)} \tag{2}$$

where ρ_1 and ρ_2 are the current densities of $s^{(1)}$ and $s^{(2)}$ at time t after mixing. The reference densities ρ_{10} and ρ_{20} before the mixing are related to the current densities through $\rho_1 = \phi_1\rho_{10}$ and $\rho_2 = (1-\phi_1)\rho_{20}$, where ϕ_1 is the volume fraction of $s^{(1)}$. The mixture density ρ is given by the sum $\rho = \rho_1 + \rho_2$. In this work, we shall restrict our attention to a binary mixture of incompressible Newtonian fluids. For such a mixture, we can express the current densities in the form

$$\rho_1 = \frac{\rho_{10}(\rho_{20} - \rho)}{\rho_{20} - \rho_{10}}, \quad \rho_2 = \frac{\rho_{20}(\rho - \rho_{10})}{\rho_{20} - \rho_{10}} \tag{3}$$

Assuming no interconversion of mass between the two fluids, conservation of mass for the two fluids are

$$\frac{\partial \rho_1}{\partial t} + (\rho_1 v_k^{(1)})_{,k} = 0, \qquad \frac{\partial \rho_2}{\partial t} + (\rho_2 v_k^{(2)})_{,k} = 0 \tag{4}$$

where a comma denotes partial differentiation with respect to x_k.

If $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ denote the partial stress tensors of the two-fluid, then the equations for the balance of linear momentum are given by

$$\rho_1 \frac{D^{(1)}v_k^{(1)}}{Dt} = \sigma_{ik,i}^{(1)} - f_k + \rho_1 F_k^{(1)}, \qquad \rho_2 \frac{D^{(2)}v_k^{(2)}}{Dt} = \sigma_{ik,i}^{(2)} + f_k + \rho_2 F_k^{(2)} \tag{5}$$

where f_k represents the mechanical interaction forces between the fluids and $F_k^{(\beta)}$ represents the body force per unit mass of the β th fluid. With these equations as the basis, in order to solve any problem related to binary mixture of fluids one needs to provide the constitutive relations for the interaction forces and stress tensors. The derivation of the constitutive equations appropriate to a binary mixture of incompressible Newtonian fluids has been outlined in Atkin and Craine (1976). If the mixture is considered to be purely mechanical system, the partial stress tensors in such a mixture are related to the motion in the following manner:

$$\sigma_{ik}^{(1)} = (-p_1 + \lambda_1 d_{jj}^{(1)} + \lambda_3 d_{jj}^{(2)})\delta_{ik} + 2\mu_1 d_{ik}^{(1)} + 2\mu_3 d_{ik}^{(2)} + \lambda_5(w_{ik}^{(1)} - w_{ik}^{(2)})$$
(6)

$$\sigma_{ik}^{(2)} = (-p_2 + \lambda_4 d_{jj}^{(1)} + \lambda_2 d_{jj}^{(2)})\delta_{ik} + 2\mu_4 d_{ik}^{(1)} + 2\mu_2 d_{ik}^{(2)} - \lambda_5(w_{ik}^{(1)} - w_{ik}^{(2)})$$
(7)

with the material coefficients satisfying the inequalities

$$\lambda_5 \geq 0, \ \mu_1 \geq 0, \ \mu_2 \geq 0, \ \lambda_1 + \frac{2}{3}\mu_1 \geq 0, \ \lambda_2 + \frac{2}{3}\mu_2 \geq 0, \ (\mu_3 + \mu_4)^2 \leq 4\mu_1\mu_2,$$

$$\left[\lambda_3 + \lambda_4 + \frac{2}{3}(\mu_3 + \mu_4)\right]^2 \leq 4\left(\lambda_1 + \frac{2}{3}\mu_1\right)\left(\lambda_2 + \frac{2}{3}\mu_2\right)$$
(8)

where p_1, p_2, $d_{ik}^{(\beta)}$, and $w_{ik}^{(\beta)}$ are given by

$$p_1 = (\rho - \rho_{20})\left(\rho_1 \frac{dA_1}{d\rho} + \lambda\right), \qquad p_2 = (\rho - \rho_{10})\left(\rho_2 \frac{dA_2}{d\rho} - \lambda\right) \ (9)$$

$$2d_{ik}^{(\beta)} = v_{i,k}^{(\beta)} + v_{k,i}^{(\beta)}, \qquad 2w_{ik}^{(\beta)} = v_{i,k}^{(\beta)} - v_{k,i}^{(\beta)}$$
(10)

In these equations, λ is a Lagrange multiplier associated with Equations (3) and (4), p_1 and p_2 the mechanical pressures, d_{ik} the deformation rate tensor, w_{ik} the spin tensor and A_1 and A_2 the Helmholtz free energy of the fluids. The mixture Helmholtz free energy A is defined by

$$\rho A = \rho_1 A_1 + \rho_2 A_2$$
(11)

Note that, under isothermal conditions, the material coefficients in Equations (6) and (7) depend only on the mixture density.

Finally, we shall mention the interaction force f_k appearing in Equation (5). Deriving constitutive relations for the interaction forces is one of the most important issues of research in multifluid flows. Massoudi (2003) discussed a variety of possible forms of this term. For instance, for fluid-solid and fluid-fluid mixtures, the interaction force generally depends on densities, temperatures, velocity differences, velocity gradients and possibly other quantities. Such interactions play a very important role in the nature of solutions. To make the theory be of practical utility, we need to simplify the constitutive expression for the interaction force. A good starting point is the inclusion of the effects due to drag and density gradient, that is, (Atkin and Craine, 1976).

$$f_k = \alpha(v_k^{(1)} - v_k^{(2)}) - \lambda\rho_{,k}$$
(12)

where α is the interaction coefficient which is a function of the mixture density. Evaluation of α is indeed a difficult task. One

simple approach for estimating the value of α is to make use of Hadamard-Rybczynski solution as a first approximation (Dai and Khonsari, 1994). Many authors like Craine (1971, 1973), Al-Sharif et al. (1993), Chamniprasart et al. (1993), Wang et al. (1993), Dai and Khonsari (1994), Göğüş (1988, 1991, 1992a, b, 1994, 1995) and Barış and Demir (2012) have benefited from Equation (12) to make calculation for various problems related to binary fluid mixtures.

In the next sections, we derive the dimensionless forms of the governing equations. To gain further insight into the influence of the parameters in these equations, we will analytically solve the simplified equations for some simple hydromagnetic flows of a binary fluid mixture between two long horizontal plates.

Steady hydromagnetic Couette flow

We want to examine the steady flow of binary mixture of electrically conducting incompressible and viscous fluids between two parallel insulated plates separated by a distance H in the presence of a transverse magnetic field. As Figure 1 shows, we select a rectangular Cartesian system with the x-axis in the direction of motion and the y-axis perpendicular to the plates. Two plates of the channel are of infinite extent in the x- and z-directions and the flow is fully developed, so the velocity depends only on y. The flow is caused by the motion of the plate at $y = 0$ with a constant velocity U_0 in the x-direction. An external uniform magnetic field B_0 is applied in the y-direction. The magnetic Reynolds number is assumed to be very small. In this case, the induced magnetic field produced by motion of fluid can be ignored in comparison to the applied one. In addition, the imposed and induced electric fields are assumed to be negligible, thus the electromagnetic body force per unit volume simplifies $\mathbf{F}_{em} = \sigma(\mathbf{v} \times \mathbf{B}) \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic field vector and σ is the electrical conductivity. Due to the assumptions stated above, Maxwell's equations become redundant.

It should be mentioned here that a great deal of interest has been focused on the flow problems in the presence of a uniform applied magnetic field. But it can't be appropriate to make such an assumption in some engineering applications. So far a few researchers have worked on the flow of an electrically conducting fluid under a non-uniform space or time dependent magnetic field, due to the difficulty in obtaining the complete solution for the problem of this type. Recently, such an attempt was made by Asghar and Ahmad (2012). They found the analytical solution for unsteady Couette flow in the presence of an arbitrary non-uniform space dependent applied magnetic field. In this paper, for simplicity of analysis, we confined ourselves only to the uniform magnetic field case. It is felt that the uniform magnetic field findings will be a good starting point for shedding light on more complicated two-fluid MHD flows. We shall seek a solution of the form

$$u_{\beta S} = u_{\beta S}(y), \qquad \rho = \rho(y)$$
(13)

where the function $u_{\beta S}$ denotes the velocity component of the β th fluid in the x-direction for the case of steady flow. Under the above assumptions, substituting Equations (6) and (7) into the equations of motion (5) we obtain

$$M_1 u_{1S}'' + M_2 u_{2S}'' + M_1' u_{1S}' + M_2' u_{2S}' - \alpha(u_{1S} - u_{2S}) - \sigma_1 B_0^2 u_{1S} = 0$$
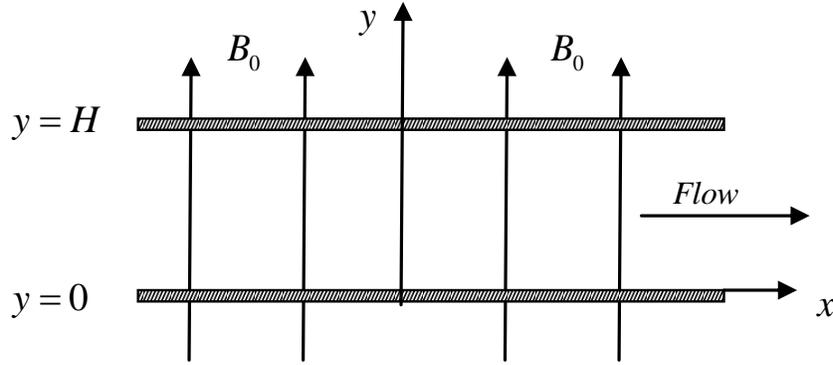(14)

Figure 1. Sketch of flow geometry and coordinate system.

$$\lambda \rho' = p_1' \tag{15}$$

$$M_3 u_{1S}'' + M_4 u_{2S}'' + M_3' u_{1S}' + M_4' u_{2S}' + \alpha(u_{1S} - u_{2S}) - \sigma_2 B_0^2 u_{2S} = 0 \tag{16}$$

$$-\lambda \rho' = p_2' \tag{17}$$

where

$$M_1 = \mu_1 - \frac{\lambda_5}{2}, \quad M_2 = \mu_3 + \frac{\lambda_5}{2}, \quad M_3 = \mu_4 + \frac{\lambda_5}{2}, \quad M_4 = \mu_2 - \frac{\lambda_5}{2} \tag{18}$$

In the above equations, primes denote differentiation with respect to y and σ_β is the electrical conductivity of the β th fluid. The last terms on the left hand sides of Equations (14) and (16) result from electromagnetic body forces. Note that we neglect non-magnetic body forces. With the use of Equations (3), (9) and (11), elimination of λ' between Equations (15) and (17) yields

$$(\rho - \rho_{10})(\rho_{20} - \rho) \, \rho' \, \frac{d^2(\rho A)}{d\rho^2} = 0 \tag{19}$$

We deduce from the above equation that ρ is a constant. As a result, the coefficients M_1, M_2, M_3 and M_4 are constants. We shall now write the equations of motion in terms of a set of dimensionless variables. If $\bar{f}$ is used to denote the dimensionless form of a quantity f, it follows that

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha H^2}{\mu}, \quad \bar{u}_{\beta S} = \frac{u_{\beta S}}{U_0}, \quad \bar{y} = \frac{y}{H}, \quad Ha_\beta = B_0 H \left(\frac{\sigma_\beta}{\mu} \right)^{1/2} \tag{20}$$

where μ is the viscosity of the mixture. Thus the equations of motion in non-dimensional form become,

$$\bar{M}_1 \bar{u}_{1S}'' + \bar{M}_2 \bar{u}_{2S}'' - \bar{\alpha}(\bar{u}_{1S} - \bar{u}_{2S}) - Ha_1^2 \bar{u}_{1S} = 0 \tag{21}$$

$$\bar{M}_3 \bar{u}_{1S}'' + \bar{M}_4 \bar{u}_{2S}'' + \bar{\alpha}(\bar{u}_{1S} - \bar{u}_{2S}) - Ha_2^2 \bar{u}_{2S} = 0 \tag{22}$$

The magnetic parameter Ha_β in the above equations is often referred to as the Hartmann number. The boundary conditions for the velocity components are

$$\bar{u}_{\beta S}(0) = 1, \qquad \bar{u}_{\beta S}(1) = 0 \tag{23}$$

The solutions of Equations (21) and (22) can be given as

$$\bar{u}_{1S} = C_1 e^{s_1 \bar{y}} + C_2 e^{-s_1 \bar{y}} + C_3 e^{s_2 \bar{y}} + C_4 e^{-s_2 \bar{y}} \tag{24}$$

$$\bar{u}_{2S} = C_5 e^{s_1 \bar{y}} + C_6 e^{-s_1 \bar{y}} + C_7 e^{s_2 \bar{y}} + C_8 e^{-s_2 \bar{y}} \tag{25}$$

where

$$s_1 = \sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}}, \qquad s_2 = \sqrt{\frac{b + \sqrt{b^2 - 4ac}}{2a}} \tag{26}$$

with

$$a = \bar{M}_1 \bar{M}_4 - \bar{M}_2 \bar{M}_3, \quad b = \bar{\alpha}(\bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4) + \bar{M}_1 Ha_1^2 + \bar{M}_4 Ha_2^2,$$
$$c = \bar{\alpha}(Ha_1^2 + Ha_2^2) + Ha_1^2 Ha_2^2 \tag{27}$$

Applying the boundary conditions (23) to Equations (24) and (25), we find the constants $C_1, ..., C_8$ as follows:

$$C_1 = \frac{k_2 - 1}{k_2 - k_1} \frac{e^{-s_1}}{e^{-s_1} - e^{s_1}}, \quad C_2 = \frac{k_2 - 1}{k_2 - k_1} \frac{e^{s_1}}{e^{s_1} - e^{-s_1}}, \quad C_3 = \frac{k_1 - 1}{k_1 - k_2} \frac{e^{-s_2}}{e^{-s_2} - e^{s_2}},$$
$$C_4 = \frac{k_1 - 1}{k_1 - k_2} \frac{e^{s_2}}{e^{s_2} - e^{-s_2}}, \quad C_5 = k_1 C_1, \quad C_6 = k_1 C_2, \quad C_7 = k_2 C_3, \quad C_8 = k_2 C_4 \tag{28}$$

where

$$k_1 = \frac{\bar{\alpha} + Ha_1^2 - \bar{M}_1 s_1^2}{\bar{\alpha} + \bar{M}_2 s_1^2}, \quad k_2 = \frac{\bar{\alpha} + Ha_1^2 - \bar{M}_1 s_2^2}{\bar{\alpha} + \bar{M}_2 s_2^2} \tag{29}$$

Unsteady hydromagnetic Couette flow

Now is the time to investigate unsteady Couette flow of a mixture of two incompressible Newtonian fluids between two parallel plates. A uniform magnetic field is applied in a direction perpendicular to the flow of the binary fluid mixture. The mixture and two plates are initially at rest.

The lower plate is suddenly accelerated from rest and moves in its own plane with constant velocity U_0, while the upper plate is held stationary. It is assumed that the flow is entirely driven by the motion of the lower plate. It seems reasonable to assume that the velocity components and densities of fluids are of the form

$$u_\beta = u_\beta(y,t), \qquad \rho_\beta = \rho_\beta(y,t) \tag{30}$$

Substitution of Equation (30) into Equation (4) yields $\partial \rho_\beta / \partial t = 0$. Thus $\partial \rho / \partial t = 0$ and $\rho = \rho(y)$. As made in the preceding section, elimination of λ' between y - components of the equations of motion give Equation (19), which implies that ρ is a constant. Since ρ has been proved to be a constant, all of the material coefficients are constants. As a result, the dimensionless equations of motions are as follows:

$$\bar{\rho}_1 \frac{\partial \bar{u}_1}{\partial \bar{t}} = \bar{M}_1 \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \bar{M}_2 \frac{\partial^2 \bar{u}_2}{\partial \bar{y}^2} - \bar{\alpha}(\bar{u}_1 - \bar{u}_2) - Ha_1^2 \bar{u}_1 \tag{31}$$

$$\bar{\rho}_2 \frac{\partial \bar{u}_2}{\partial \bar{t}} = \bar{M}_3 \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \bar{M}_4 \frac{\partial^2 \bar{u}_2}{\partial \bar{y}^2} + \bar{\alpha}(\bar{u}_1 - \bar{u}_2) - Ha_2^2 \bar{u}_2 \tag{32}$$

where

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha H^2}{\mu}, \quad \bar{u}_\beta = \frac{u_\beta}{U_0}, \quad \bar{y} = \frac{y}{H}, \quad \bar{\rho}_\beta = \frac{\rho_\beta}{\rho}, \quad \bar{t} = \frac{\mu t}{\rho H^2}, \quad Ha_\beta = B_0 H \left(\frac{\sigma_\beta}{\mu}\right)^{1/2} \tag{33}$$

The boundary and initial conditions are

$$\bar{u}_\beta(0,\bar{t}) = 1; \quad \bar{t} > 0, \qquad \bar{u}_\beta(1,\bar{t}) = 0; \quad \bar{t} \ge 0, \tag{34}$$

$$\bar{u}_\beta(\bar{y},0) = 0; \quad 0 < \bar{y} \le 1. \tag{35}$$

We first have to transform the problem so that the boundary conditions $(34)_1$ are homogeneous. This can be achieved by decomposing $\bar{u}_\beta(\bar{y},\bar{t})$ into the steady Couette velocity field $\bar{u}_{\beta S}(\bar{y})$ and the transient component $f_\beta(\bar{y},\bar{t})$:

$$\bar{u}_\beta(\bar{y},\bar{t}) = \bar{u}_{\beta S}(\bar{y}) - f_\beta(\bar{y},\bar{t}) \tag{36}$$

The transient components satisfy the following differential equations

$$\bar{\rho}_1 \frac{\partial f_1}{\partial \bar{t}} = \bar{M}_1 \frac{\partial^2 f_1}{\partial \bar{y}^2} + \bar{M}_2 \frac{\partial^2 f_2}{\partial \bar{y}^2} - \bar{\alpha}(f_1 - f_2) - Ha_1^2 f_1 \tag{37}$$

$$\bar{\rho}_2 \frac{\partial f_2}{\partial \bar{t}} = \bar{M}_3 \frac{\partial^2 f_1}{\partial \bar{y}^2} + \bar{M}_4 \frac{\partial^2 f_2}{\partial \bar{y}^2} + \bar{\alpha}(f_1 - f_2) - Ha_2^2 f_2 \tag{38}$$

that are consistent with the boundary and initial conditions

$$f_1(0,\bar{t}) = 0, \quad f_2(0,\bar{t}) = 0, \quad f_1(1,\bar{t}) = 0, \quad f_2(1,\bar{t}) = 0 \tag{39}$$

$$f_1(\bar{y},0) = \bar{u}_{1S}(\bar{y}), \quad f_2(\bar{y},0) = \bar{u}_{2S}(\bar{y}) \tag{40}$$

Finite Fourier sine transform will be used to solve the simultaneous partial differential equations (37) and (38) with the conditions (39) and (40). Finite Fourier sine transform of a function $f(y)$ defined for $0 < y < a$ is

$$F_s\{f(y)\} = \tilde{f}(n) = \int_0^a f(y)\sin(ny)dy, \qquad n = 1,2,3,... \tag{41}$$

with inverse transform

$$F_s^{-1}\{\tilde{f}(n)\} = f(y) = \frac{2}{a} \sum_{n=1}^\infty \tilde{f}(n)\sin(ny) \tag{42}$$

With the aid of Equation (39), application of the Fourier sine transform to Equations (37) and (38) gives

$$\frac{d\tilde{f}_1}{dt} = -l_1 \tilde{f}_1 - l_2 \tilde{f}_2 \tag{43}$$

$$\frac{d\tilde{f}_2}{dt} = -l_3 \tilde{f}_1 - l_4 \tilde{f}_2 \tag{44}$$

where

$$l_1 = \frac{\bar{M}_1 n^2 \pi^2 + \bar{\alpha} + Ha_1^2}{\bar{\rho}_1}, \quad l_2 = \frac{\bar{M}_2 n^2 \pi^2 - \bar{\alpha}}{\bar{\rho}_1}, \quad l_3 = \frac{\bar{M}_3 n^2 \pi^2 - \bar{\alpha}}{\bar{\rho}_2}, \quad l_4 = \frac{\bar{M}_4 n^2 \pi^2 + \bar{\alpha} + Ha_2^2}{\bar{\rho}_2} \tag{45}$$

Taking the Fourier sine transform of Equation (40) results in:

$$\tilde{f}_1(n,0) = \frac{n\pi}{s_1^2 + n^2\pi^2}\left\{(1-(-1)^n e^{s_1})C_1 + (1-(-1)^n e^{-s_1})C_2\right\}$$
$$+ \frac{n\pi}{s_2^2 + n^2\pi^2}\left\{(1-(-1)^n e^{s_2})C_3 + (1-(-1)^n e^{-s_2})C_4\right\} \tag{46}$$

$$\tilde{f}_2(n,0) = \frac{n\pi}{s_1^2 + n^2\pi^2}\left\{(1-(-1)^n e^{s_1})C_5 + (1-(-1)^n e^{-s_1})C_6\right\}$$
$$+ \frac{n\pi}{s_2^2 + n^2\pi^2}\left\{(1-(-1)^n e^{s_2})C_7 + (1-(-1)^n e^{-s_2})C_8\right\} \tag{47}$$

Solving Equations (43) and (44) simultaneously and using the conditions (46) and (47), we obtain

$$\tilde{f}_1(n,\bar{t}) = \frac{(l_1 + r_2)\tilde{f}_1(n,0) + l_2 \tilde{f}_2(n,0)}{r_2 - r_1} e^{r_1 \bar{t}} - \frac{(l_1 + r_1)\tilde{f}_1(n,0) + l_2 \tilde{f}_2(n,0)}{r_2 - r_1} e^{r_2 \bar{t}} \tag{48}$$

$$\tilde{f}_2(n,\bar{t}) = \frac{l_1 + r_1}{l_2} \frac{(l_1 + r_2)\tilde{f}_1(n,0) + l_2 \tilde{f}_2(n,0)}{r_1 - r_2} e^{r_1 \bar{t}} - \frac{l_1 + r_2}{l_2} \frac{(l_1 + r_1)\tilde{f}_1(n,0) + l_2 \tilde{f}_2(n,0)}{r_1 - r_2} e^{r_2 \bar{t}} \tag{49}$$

where

$$r_{1,2} = -\frac{l_1 + l_4}{2} \mp \frac{\sqrt{(l_1 - l_4)^2 + 4l_2 l_3}}{2} \qquad (50)$$

With the help of Equation (42), inverting Equations (48) and (49) and then substituting of the results into Equation(36), we find the following solution for $\bar{u}_\beta(\bar{y}, \bar{t})$

$$\bar{u}_\beta(\bar{y}, \bar{t}) = \bar{u}_{\beta S}(\bar{y}) - 2\sum_{n=1}^{\infty} \tilde{f}_\beta(n, \bar{t}) \sin(n\pi \bar{y}) \qquad (51)$$

Steady hydromagnetic Poiseuille flow

In this section we consider the steady flow of the binary mixture under consideration between two parallel plates in the presence of a transverse magnetic field. The flow is driven by an externally imposed constant pressure gradient in the x-direction, namely $-\partial p / \partial x = p_x > 0$. We seek solutions in which the velocity of the β th fluid and the mixture density are assumed to have the form:

$$w_{\beta S} = w_{\beta S}(y), \quad \rho = \rho(y) \qquad (52)$$

As previously stated, it is proved that the total density and material coefficients become constants. Consequently, the equations of motion in the x-direction reduce to

$$\bar{M}_1 \bar{w}_{1S}'' + \bar{M}_2 \bar{w}_{2S}'' - \bar{\alpha}(\bar{w}_{1S} - \bar{w}_{2S}) - Ha_1^2 \bar{w}_{1S} = -\phi_1 \qquad (53)$$

$$\bar{M}_3 \bar{w}_{1S}'' + \bar{M}_4 \bar{w}_{2S}'' + \bar{\alpha}(\bar{w}_{1S} - \bar{w}_{2S}) - Ha_2^2 \bar{w}_{2S} = \phi_1 - 1 \qquad (54)$$

where

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha H^2}{\mu}, \quad \bar{w}_{\beta S} = \frac{w_{\beta S}\mu}{p_x H^2}, \quad \bar{y} = \frac{y}{H}, \quad Ha_\beta = B_0 H \left(\frac{\sigma_\beta}{\mu}\right)^{1/2} \qquad (55)$$

The adherence boundary conditions of the problem are

$$\bar{w}_{\beta S}(0) = 0, \qquad \bar{w}_{\beta S}(1) = 0 \qquad (56)$$

The velocity fields can be obtained by solving Equations (53) and (54) under the relevant boundary conditions as follows:

$$\bar{w}_{1S} = D_1 e^{s_1 \bar{y}} + D_2 e^{-s_1 \bar{y}} + D_3 e^{s_2 \bar{y}} + D_4 e^{-s_2 \bar{y}} + D_5 \qquad (57)$$

$$\bar{w}_{2S} = k_1 D_1 e^{s_1 \bar{y}} + k_1 D_2 e^{-s_1 \bar{y}} + k_2 D_3 e^{s_2 \bar{y}} + k_2 D_4 e^{-s_2 \bar{y}} + D_6 \qquad (58)$$

where

$$D_1 = \frac{(D_6 - D_5 k_2)(1 - e^{-s_1})}{(k_2 - k_1)(e^{s_1} - e^{-s_1})}, \qquad D_2 = \frac{(D_6 - D_5 k_2)(e^{s_1} - 1)}{(k_2 - k_1)(e^{s_1} - e^{-s_1})},$$

$$D_3 = \frac{(D_6 - D_5 k_1)(1 - e^{-s_2})}{(k_1 - k_2)(e^{s_2} - e^{-s_2})}, \qquad D_4 = \frac{(D_6 - D_5 k_1)(e^{s_2} - 1)}{(k_1 - k_2)(e^{s_2} - e^{-s_2})} \qquad (59)$$

with

$$D_5 = \frac{\bar{\alpha} + \phi_1 Ha_2^2}{\bar{\alpha}(Ha_1^2 + Ha_2^2) + Ha_1^2 Ha_2^2}, \qquad D_6 = \frac{\bar{\alpha} + (1 - \phi_1)Ha_1^2}{\bar{\alpha}(Ha_1^2 + Ha_2^2) + Ha_1^2 Ha_2^2} \qquad (60)$$

Unsteady hydromagnetic Poiseuille flow

Finally, we study the problem of unsteady flow of a mixture of two incompressible Newtonian fluids between two parallel plates. There is an external magnetic field of constant strength in the y-direction. The mixture is initially at rest. The mixture begins to flow due to the sudden imposition of a constant pressure gradient in the x-direction. We look for a solution of the form

$$w_\beta = w_\beta(y, t), \quad \rho_\beta = \rho_\beta(y, t) \qquad (61)$$

As made in the case of unsteady hydromagnetic Couette flow, it is proved ρ is a constant. This is why all the coefficients in the constitutive equations are constants. Thus, the dimensionless governing equations are as follows:

$$\bar{\rho}_1 \frac{\partial \bar{w}_1}{\partial \bar{t}} = \bar{M}_1 \frac{\partial^2 \bar{w}_1}{\partial \bar{y}^2} + \bar{M}_2 \frac{\partial^2 \bar{w}_2}{\partial \bar{y}^2} - \bar{\alpha}(\bar{w}_1 - \bar{w}_2) - Ha_1^2 \bar{w}_1 + \phi_1 \qquad (62)$$

$$\bar{\rho}_2 \frac{\partial \bar{w}_2}{\partial \bar{t}} = \bar{M}_3 \frac{\partial^2 \bar{w}_1}{\partial \bar{y}^2} + \bar{M}_4 \frac{\partial^2 \bar{w}_2}{\partial \bar{y}^2} + \bar{\alpha}(\bar{w}_1 - \bar{w}_2) - Ha_1^2 \bar{w}_2 + 1 - \phi_1 \qquad (63)$$

where

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha H^2}{\mu}, \quad \bar{w}_\beta = \frac{w_\beta \mu}{p_x H^2}, \quad \bar{y} = \frac{y}{H}, \quad \bar{\rho}_\beta = \frac{\rho_\beta}{\rho}, \quad \bar{t} = \frac{\mu t}{\rho H^2}, \quad Ha_\beta = B_0 H \left(\frac{\sigma_\beta}{\mu}\right)^{1/2} \qquad (64)$$

The boundary and initial conditions are

$$\bar{w}_\beta(0, \bar{t}) = 0; \quad \bar{t} \geq 0, \qquad \bar{w}_\beta(1, \bar{t}) = 0; \quad \bar{t} \geq 0 \qquad (65)$$

$$\bar{w}_\beta(\bar{y}, 0) = 0; \quad 0 \leq \bar{y} \leq 1 \qquad (66)$$

Note that all of the above conditions are homogeneous, yet there exists a non-trivial solution, since the governing equations are non-homogeneous. We attempt to find a solution of the form

$$\bar{w}_\beta(\bar{y}, \bar{t}) = \bar{w}_{\beta S}(\bar{y}) - g_\beta(\bar{y}, \bar{t}) \qquad (67)$$

The components $g_\beta(\bar{y}, \bar{t})$ must satisfy Equations (37) and (38)

and the boundary conditions (39) by writing $g_\beta(\bar{y}, \bar{t})$ in place of $f_\beta(\bar{y}, \bar{t})$, but with modified initial conditions which now are:

$$g_\beta(\bar{y}, 0) = \bar{w}_{\beta S}(\bar{y}) \tag{68}$$

The procedure for determining $g_\beta(\bar{y}, \bar{t})$ is the same as that used in the case of unsteady hydromagnetic Couette flow, so it is not repeated here. As expected, the second part of the solution given in Equation (51) is also valid for $g_\beta(\bar{y}, \bar{t})$ provided $\tilde{f}_\beta(n, 0)$ is replaced with $\tilde{g}_\beta(n, 0)$ which are given by the following analytical expressions

$$\tilde{g}_1(n, 0) = \frac{D_1 n\pi}{s_1^2 + n^2\pi^2}(1 - (-1)^n e^{s_1}) + \frac{D_2 n\pi}{s_1^2 + n^2\pi^2}(1 - (-1)^n e^{-s_1})$$
$$+ \frac{D_3 n\pi}{s_2^2 + n^2\pi^2}(1 - (-1)^n e^{s_2}) + \frac{D_4 n\pi}{s_2^2 + n^2\pi^2}(1 - (-1)^n e^{-s_2}) + \frac{D_5(1 - (-1)^n)}{n\pi} \tag{69}$$

$$\tilde{g}_2(n, 0) = \frac{k_1 D_1 n\pi}{s_1^2 + n^2\pi^2}(1 - (-1)^n e^{s_1}) + \frac{k_1 D_2 n\pi}{s_1^2 + n^2\pi^2}(1 - (-1)^n e^{-s_1})$$
$$+ \frac{k_2 D_3 n\pi}{s_2^2 + n^2\pi^2}(1 - (-1)^n e^{s_2}) + \frac{k_2 D_4 n\pi}{s_2^2 + n^2\pi^2}(1 - (-1)^n e^{-s_2}) + \frac{D_6(1 - (-1)^n)}{n\pi} \tag{70}$$

We now obtain the solution for the velocity field of the β th fluid by going back through the various substitutions:

$$\bar{w}_\beta(\bar{y}, \bar{t}) = \bar{w}_{\beta S}(\bar{y}) - 2\sum_{n=1}^{\infty} \tilde{g}_\beta(n, \bar{t})\sin(n\pi\bar{y}) \tag{71}$$

NUMERICAL RESULTS AND DISCUSSION

Some simple unidirectional hydromagnetic flows of a binary mixture of Newtonian fluids between two parallel plates are studied theoretically. The two miscible fluids are assumed to be incompressible and electrically conducting, having different viscosities and electrical conductivities. The resulting differential equations are solved analytically. The analytical solutions are made possible under very special conditions when all material properties are assumed to be constants, and the only interaction force is drag resulting from relative velocity in a linear fashion. Removal of these assumptions will make the governing equations highly nonlinear and necessitate a complex numerical solution.

To make predictions based on the foregoing analysis, it is necessary to know material coefficients in the constitutive equations. For a mixture composed of water and oil with water volume fraction ϕ_1, we benefit from the formulae suggested by Sampaio and Williams (1977) to assign the reasonable values to the material coefficients. In all the computations presented here, the following values of the dimensionless parameters are used (Barış and Demir, 2012):

$$\bar{M}_1 = 0.4868, \quad \bar{M}_2 = \bar{M}_3 = 0.2497, \quad \bar{M}_4 = 0.5132,$$
$$\bar{\alpha} = 10^6, \quad \bar{\rho}_1 = 0.8108, \quad \bar{\rho}_2 = 0.1892, \quad \phi_1 = 0.75 \tag{72}$$

Now we want to discuss the reliability of the series solutions given in Equations (51) and (71). As expected, these series are rapidly convergent for large values of time but slowly convergent for small values of time. However, it is important to note that these series solutions can also be used for small values of time provided number of terms in the series expansions is enough to yield satisfactory accuracy. For example, in the case of unsteady Couette flow with $Ha_1/Ha_2 = 1000$, $Ha_1 = 2$ and $\bar{t} = 0.1$, the fourth term is the first one in the series expansion, absolute value of which is less than 10^{-12}. Therefore, the sum of the first four term will give the velocity values of the fluids with an error of less than 10^{-12}. On the other hand, it is necessary to take the first nine term into account for the same order of accuracy in the case of $\bar{t} = 0.02$.

The analytical solutions in the present work include those corresponding to pure Newtonian fluid as a special case. If one sets $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$, $\bar{\rho}_1 = \bar{\rho}_2 = \phi_1 = 1/2$, and $Ha_1 = Ha_2 = Ha/\sqrt{2}$ in Equations (51) and (71), these are obtained as follows:

Unsteady hydromagnetic Couette flow between two parallel plates:

$$\bar{u}_N(\bar{y}, \bar{t}) = \frac{\sinh[Ha(1 - \bar{y})]}{\sinh[Ha]} - 2\sum_{n=1}^{\infty} \frac{n\pi}{n^2\pi^2 + Ha^2}\sin[n\pi\bar{y}]e^{-(n^2\pi^2 + Ha^2)\bar{t}} \tag{73}$$

Unsteady hydromagnetic Poiseuille flow between two parallel plates:

$$\bar{w}_N(\bar{y}, \bar{t}) = \frac{1 - \cosh[Ha\bar{y}]}{Ha^2} + \sinh[Ha\bar{y}]\frac{\cosh[Ha] - 1}{Ha^2\sinh[Ha]} - 2\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi(n^2\pi^2 + Ha^2)}\sin[n\pi\bar{y}]e^{-(n^2\pi^2 + Ha^2)\bar{t}} \tag{74}$$

The limiting solutions mentioned above give us confidence regarding our analytical calculations. To demonstrate the influence of the applied magnetic field on the velocity profiles, numerical evaluations of the analytical solutions are performed and results are plotted in Figures 2 to 5. In these figures the material parameters $\bar{\rho}_1, \bar{\rho}_2, \phi_1, \bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4, \bar{\alpha}$ and the ratio Ha_1/Ha_2 are kept the constant values. It is clear from these figures that the main effect of the magnetic field on the flow is to decrease the velocity. This is expected since the application of a transverse magnetic field normal to the flow direction has a tendency to create a drag-like
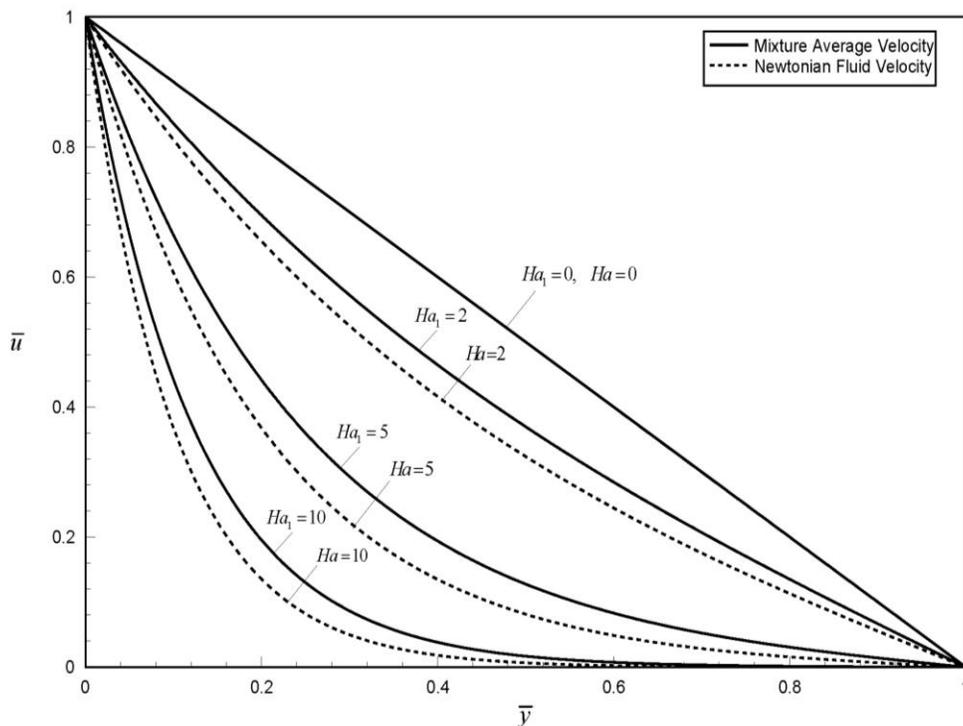
Figure 2. Velocity profiles of steady hydromagnetic Couette flow between parallel plates for $Ha_1/Ha_2 = 1000$.
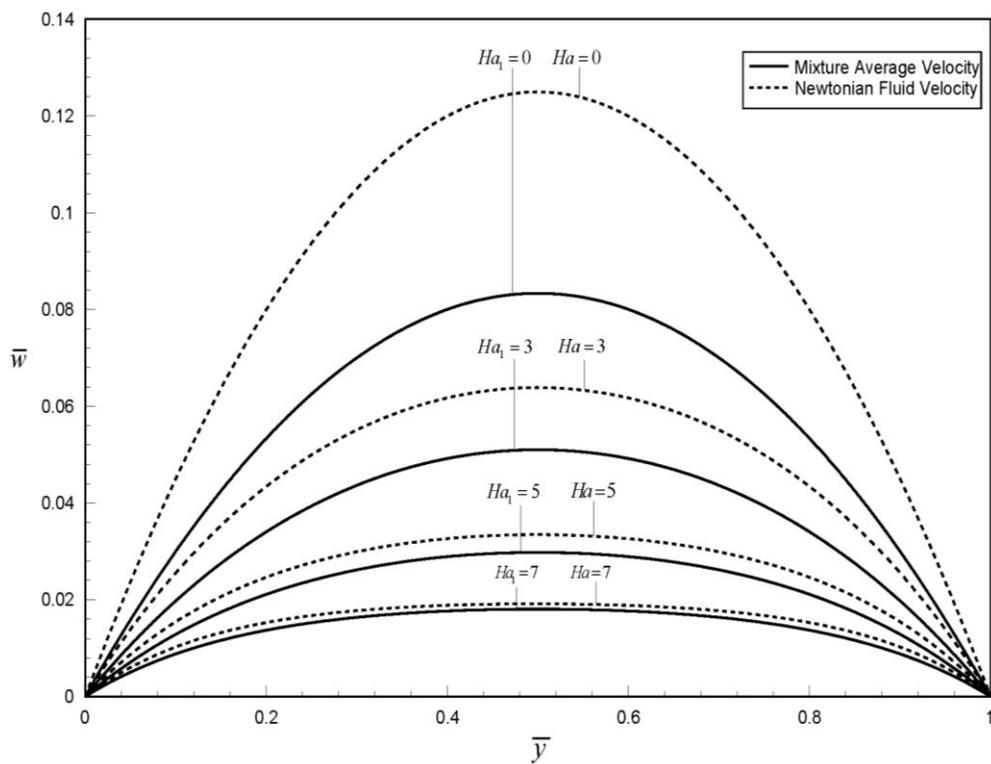

Figure 3. Velocity profiles of steady hydromagnetic Poiseuille flow between parallel plates for $Ha_1/Ha_2 = 1000$.
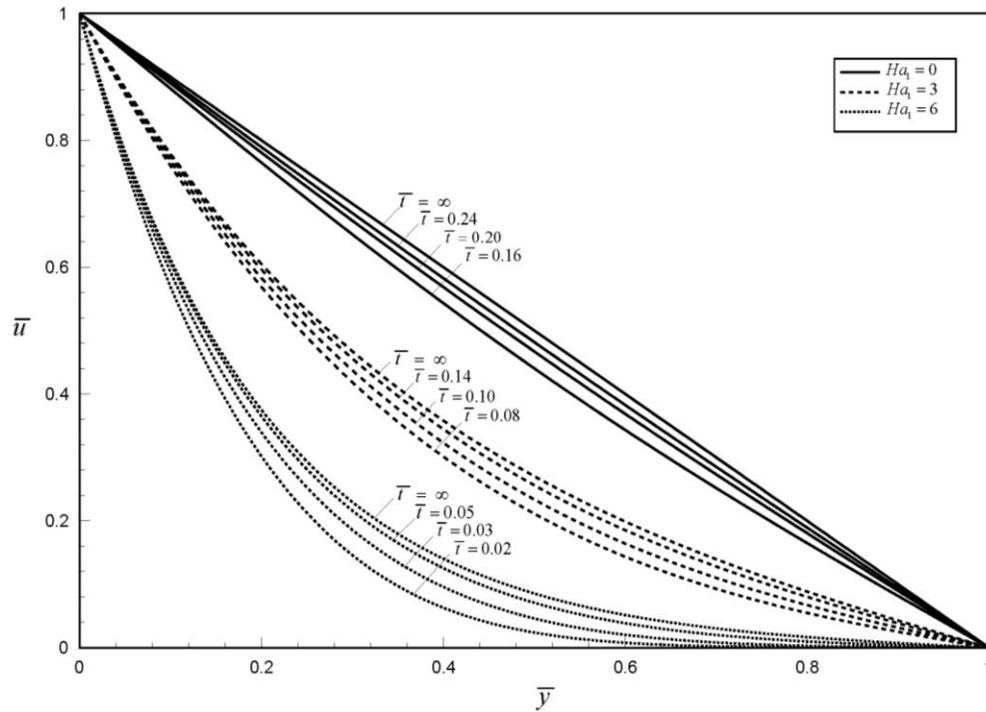
Figure 4. Velocity profiles of unsteady hydromagnetic Couette flow between parallel plates for $Ha_1/Ha_2 = 1000$.
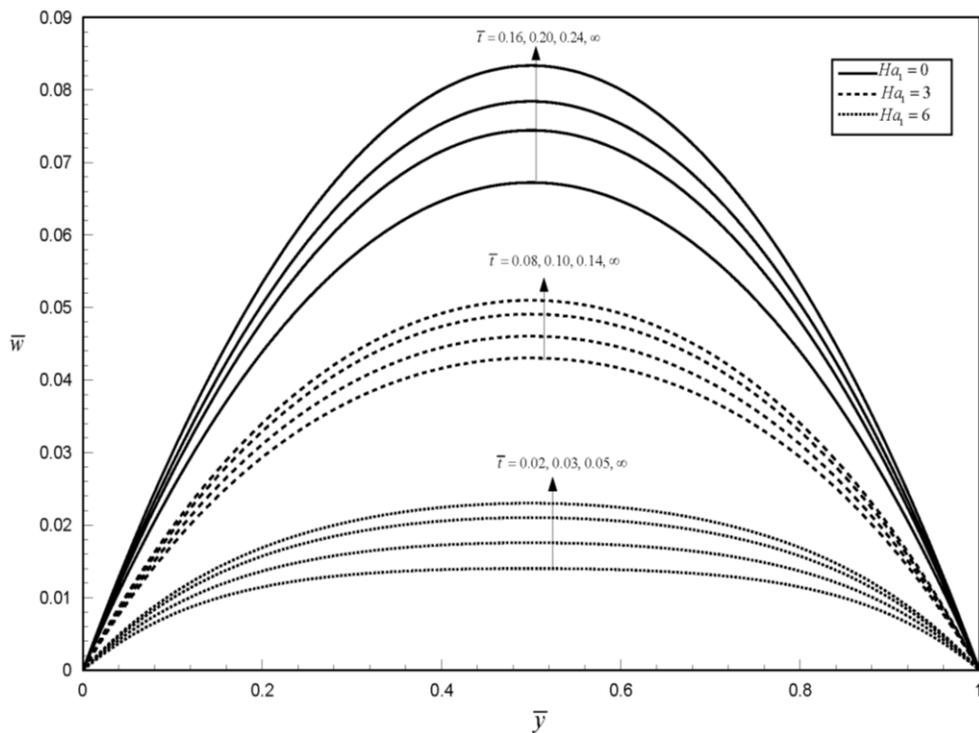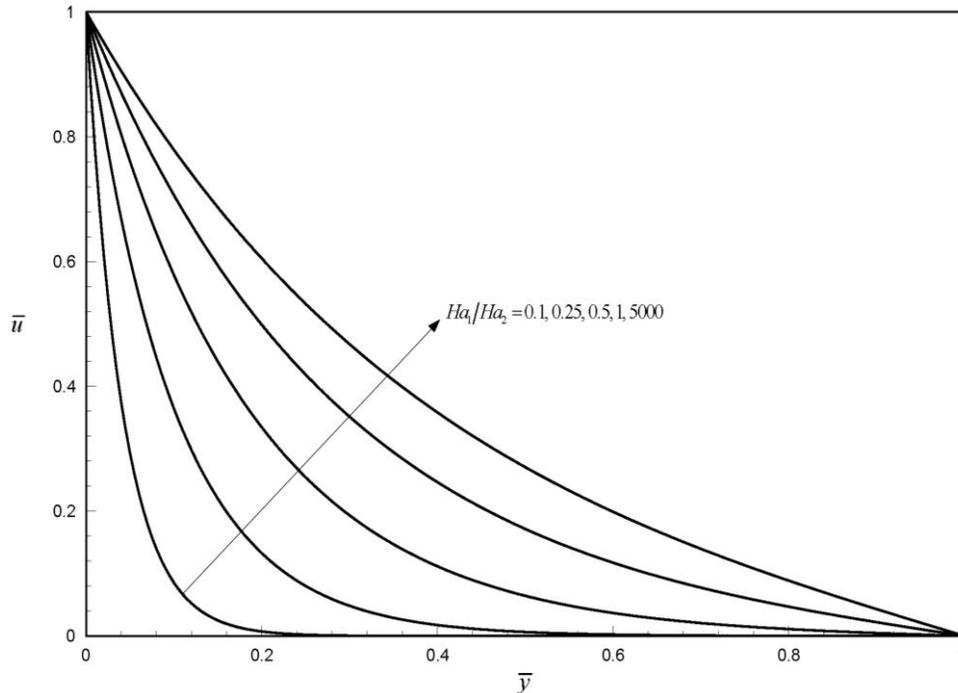

Figure 5. Velocity profiles of unsteady hydromagnetic Poiseuille flow between parallel plates for $Ha_1/Ha_2 = 1000$.

Figure 6. Velocity profiles of steady hydromagnetic Couette flow between parallel plates for different values of Ha_1/Ha_2 ($Ha_1 = 3$).

Lorentz force. This force has a decreasing effect on the velocity.

We observe from Figure 2 that when the Hartmann number increases, the velocity gradient at the moving plate increases and hence the force necessary to move this plate is greater. Figure 3 shows that as the strength of the applied magnetic field increases, the velocity profiles are flattened over the greater part of the cross-section. In other words, the magnetic field causes the shear stresses in the fluid in the vicinity of the plates to become larger. Figures 4 and 5 illustrate the time histories of hydromagnetic Couette and Poiseuille flows for various values of the Hartmann number, respectively. These figures exhibit the same transient behavior, namely the velocity gradually increases with in time and it reaches the steady-state values. Again from these figures, we arrive at the conclusion that the application of the magnetic field speeds up the transition from the unsteady-state to the steady-state. For example, in the case of Poiseuille flow with $Ha_1 = 6$, the transient behavior lasts about one-sixth as long as it does in the non-mhd case. The effect of the ratio of Hartmann numbers (Ha_1/Ha_2) on the velocity field is shown in Figures 6 to 9. It is found that the effect of decreasing Ha_1/Ha_2 is to decrease the velocity field.

Validations of the results presented in this paper can be judged by comparing them with the experimental results.

Unfortunately, no comparisons with experimental data were performed due to a lack of existence of such data. For this reason, it is not possible to comment with any certainty on the relative merits of the constitutive equations used here. It is hoped that the exact results presented in this paper can be useful as a benchmark for validating the numerical solutions to more complicated two-fluid MHD flows.

Conclusions

Couette and Poiseuille flows of a binary fluid mixture between two infinitely long parallel plates in the presence of a transverse magnetic field were investigated. Under very special conditions stated previously, steady-state and transition solutions were obtained analytically by the usual methods of solving these kinds of equations. Numerical evaluations of the analytical solutions were performed and graphical results for the velocity distributions were presented to illustrate the influence of the magnetic field on the solutions. It was found that, owing the presence of a transverse magnetic field, increases in the values of the Hartmann number have the tendency to slow the motion of the fluid mixture. In addition, as the Hartmann number increased, the transition to the steady-state became faster. It should be noted that the velocity distributions for the fluid mixture
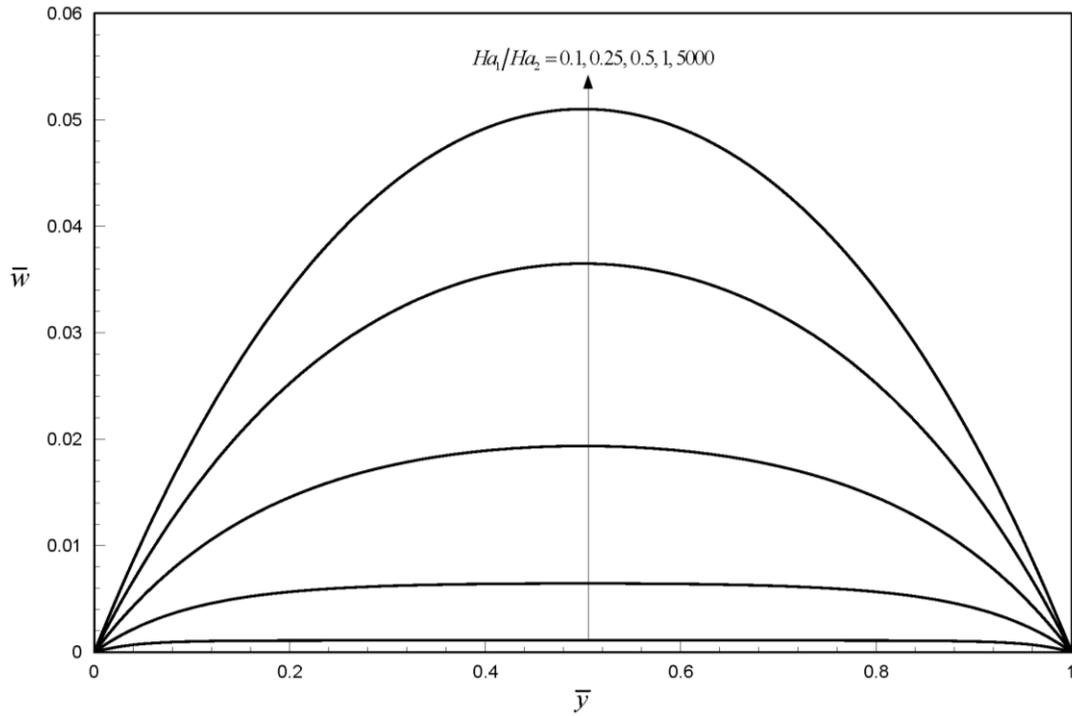
Figure 7. Velocity profiles of steady hydromagnetic Poiseuille flow between parallel plates for different values of Ha_1/Ha_2 ($Ha_1 = 3$).
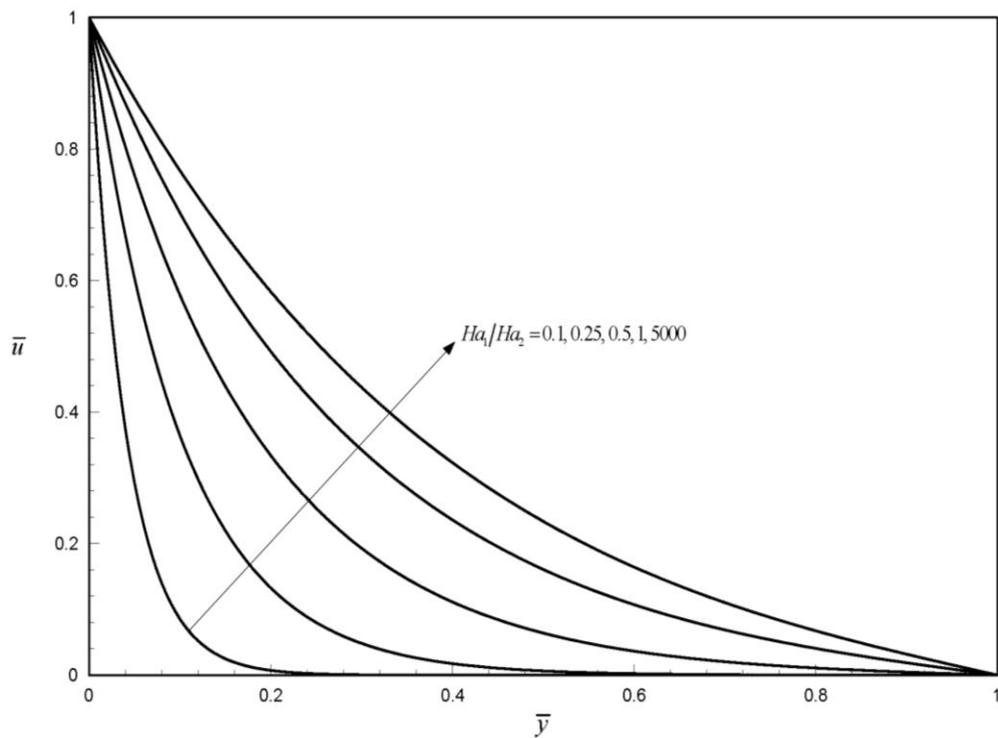

Figure 8. Velocity profiles of unsteady hydromagnetic Couette flow between parallel plates for different values of Ha_1/Ha_2 ($Ha_1 = 3$, $\bar{t} = 0.1$).
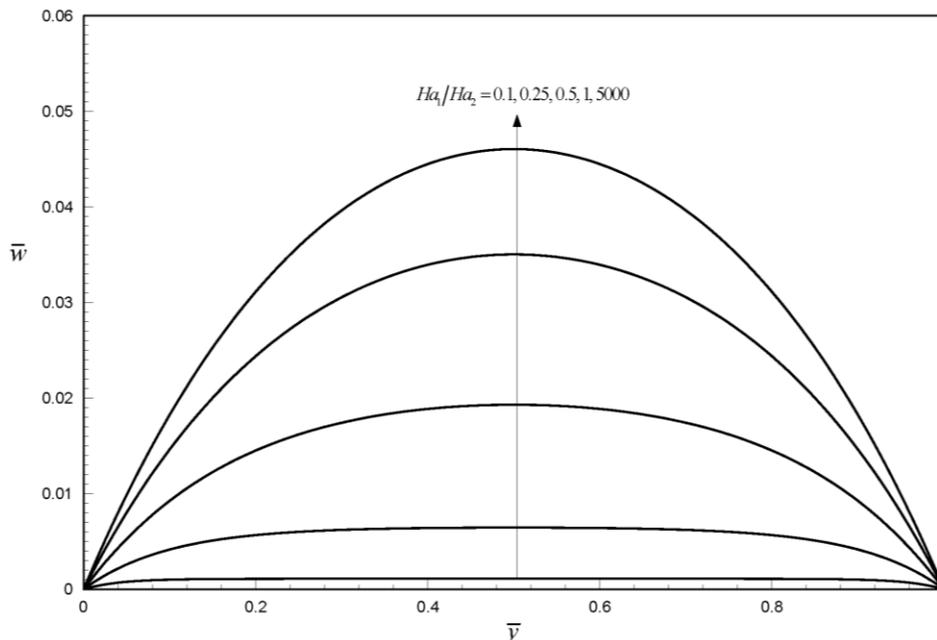
Figure 9. Velocity profiles of unsteady hydromagnetic Poiseuille flow between parallel plates for different values of Ha_1/Ha_2 ($Ha_1 = 3$, $\bar{t} = 0.1$).

have qualitatively the same characteristics as those exhibited by a single Newtonian fluid. Also, all the results corresponding to $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$, $\bar{\rho}_1 = \bar{\rho}_2 = \phi_1 = 1/2$, and $Ha_1 = Ha_2 = Ha/\sqrt{2}$ reduce the classical solutions of a single Newtonian fluid. This provides a useful check. A further check on the validity of the theoretical results presented here can be accomplished by comparisons with experimental data. As far as the authors are aware, such a quantitative comparison has been hampered by a lack of reliable experimental data. For this reason, the researcher of necessity has to rely on the mixture theory to produce the correct results qualitatively at least.

ACKNOWLEDGEMENTS

REFERENCES

Adkins JE (1963). Non-linear diffusion, 1. Diffusion and flow of mixtures of fluids.Phil. Trans. Roy.   Soc. London, A. 255:607-633.
Al-Sharif A, Chamniprasart K, Rajagopal KR, Szeri AZ (1993). Lubrication with binary mixtures: Liquid-liquid emulsion. J. Tribol. 115:46-55.
Asghar S, Ahmad A (2012). Unsteady Couette flow of viscous fluid under a non-uniform magnetic field. Appl. Mat. Lett. 25:1953-1958.
Atkin RJ, Craine RE (1976). Continuum theories of mixtures:
Applications. J. Inst. Maths. Appl. 17:153-207.
Barış S (2005). Unsteady flows of a binary mixture of incompressible Newtonian fluids in an annulus. Int. J. Eng. Sci. 43:1471-1485.
Barış S, Demir MŞ (2012). Flow of a binary mixture of fluids in a semicircular duct. Int. J. Phys. Sci. 7:1333-1345.
Bedford A, Drumheller DS (1983). Theory of immiscible and structured mixtures. Int. J. Eng. Sci. 21:863-960.
Beevers CE, Craine RE (1982). On the determination of response functions for a binary mixture of incompressible Newtonian fluids. Int. J. Eng. Sci. 20:737-745.
Bowen RM (1976). Theory of mixtures.In: A.C. Eringen (Ed.), Continuum Physics, Vol. III, Academic Press, New York.
Chamniprasart K, Al-Sharif A, Rajagopal KR, Szeri AZ (1993). Lubrication with binary mixtures: Bubbly oil. J. Tribol. 115:253-260.
Craine RE (1971). Oscillations of a plate in a binary mixture of incompressible Newtonian fluids. Int. J. Eng. Sci. 9:1177-1192.
Craine RE (1973). An oscillating plate in a binary mixture of hemihedral fluids. Z. Angew. Math. Phys. 24:365-374.
Dai F, Khonsari MM (1994). A theory of hydrodynamic lubrication involving the mixture of two fluids. J. Appl. Mech. 61:634-641.
Eringen AC, Ingram JD (1965). A continuum theory of chemically reacting media-I. Int. J. Eng. Sci. 3:197-212.
Göğüş MŞ (1988). The influence of vibrating plates on Poiseuille flow of a binary mixture.Int. J. Eng. Sci. 26:313-323.
Göğüş MŞ (1991). The influence of a vibrating plate on the flow between parallel plates of a binary mixture. Int. J. Eng. Sci. 29:1651-1659.
Göğüş MŞ (1992a). The influence of a longitudinal vibrating pipe on Poiseuille flow of a binary   mixture. Int. J. Eng. Sci. 30:141-151.
Göğüş MŞ (1992b). The steady flow of a binary mixture between two rotating parallel non-coaxial disks. Int. J. Eng. Sci. 30:665-677.
Göğüş MŞ (1994). The velocity profile in pulsatile flow of a binary mixture. Int. J. Eng. Sci. 32:705-714
Göğüş MŞ (1995). The steady flow of a binary mixture between two rotating parallel coaxial disks. Int. J. Eng. Sci. 33:611-624.
Green AE, Naghdi PM (1965). A dynamical theory of interacting continua. Int. J. Eng. Sci. 3: 231-241.

Ingram JD, Eringen AC (1967). A continuum theory of chemically reacting media-II. Constitutive equations of reacting fluid mixtures. Int. J. Eng. Sci. 4:289-322.

Lohrasbi J, Shai V (1988). Magnetohydrodynamic heat transfer in two-phase flow between parallel plates. App. Sci. Res. 45:53-66.

Malashetty MS, Leela V (1992). Magnetohydrodynamic heat transfer in two-phase flow. Int. J. Eng. Sci. 30:371-377.

Malashetty MS, Umavathi JC, Kumar JP (2001). Convective MHD two-fluid flow and heat transfer in an inclined channel. Heat Mass Trans. 37:259-264.

Massoudi M (2003). Constitutive relations for the interaction force in multicomponent particulate flows. Int. J. Non-Linear Mech. 38:313-336.

Massoudi M (2008). Flow of a binary mixture of linearly incompressible viscous fluids between two horizontal parallel plates. Mech. Res. Comm. 35:603-608.

Mills N (1966). Incompressible mixtures of Newtonian fluids. Int. J. Eng. Sci. 4:97-112.

Mitra P (1982). Unsteady flow of two electrically conducting fluids between two rigid prallel plates. Bull. Calc. Math. Soc. 74:87-95.

Müller I (1968). A thermodynamic theory of mixtures of fluids. Arc. Rat. Mech. Anal. 28:1-39.

Nikodijevic D, Stamenkovic Z, Milenkovic D, Blagojevic B, Nikodijevic J (2011). Flow and heat transfer of two immiscible fluids in the presence of uniform inclined magnetic field. Math. Prob. Eng. Article ID:132302, 18 P.

Rajagopal KR, Tao L (1995). Mechanics of Mixtures, World Scientific Publishing, New Jersey.

Samohyl I (1987). Thermomechanics of Irreversible Processes in Fluid Mixtures, Tuebner, Liepzig.

Sampaio R, Williams WO (1977). On the viscosities of liquid mixtures. Z. Angew. Math. Phys. 28:607-613.

Shail R (1973). On laminar two-phase flows in magnetohydrodynamics. Int. J. Eng. Sci. 11:1103-1108.

Sivaraj R, Kumar BR, Prakash J (2012). MHD mixed convective flow of viscoelastic and viscous fluids in a vertical porous channel. Applic. Appl. Math. (AAM) 7:99-116.

Srivastava BM, Agarval RP, Srivastava VP (1982). Unsteady flow of suspension through a rectangular pipe. Z. Angew Math. Mech. (ZAMM). 62:261-264.

Truesdell C (1957). Sulle basi della thermomeccanica. Rend. Accad. Naz. Lincei. 8:158-166.

Truesdell C (1984). Rational Thermodynamics (second edition), Springer-Verlag, New York.

Umavathi JC, Mateen A, Chamkha AJ, Mudhaf AA (2006). Oscillatory Hartman two-fluid flow and heat transfer in a horizontal channel. Int. J. Appl. Mech. Eng. 11:155-178.

Umavathi JC, Chamkha AJ, Mateen A, Kumar JP (2008). Unsteady magnetohydrodynamic two-fluid flow and heat transfer in a horizontal channel. Int. J. Heat Tech. 26:121-133.

Wang SH, Al-Sharif A, Rajagopal KR, Szeri AZ (1993). Lubrication with binary mixtures: Liquid-liquid emulsion in an EHL conjunction. J. Tribol. 115:515-522.

Wilhelm HE, Van Der Werff TJ (1977). Oscilating flows of miscible fluids. Appl. Sci. Res. 33:339-352.