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Full Length Research Paper

A modified variational iteration method for solving generalized Boussinesq equation and Lie'nard equation

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In this paper, we use modified variational iteration method (MVIM) for solving generalized Boussinesq equation and Lie'nard equation. The obtained solutions of these equations using the traditional variational iteration method (VIM) give good approximations only in the neighborhood of the initial position. The main advantage of the MVIM is that it can enlarge the convergence region of iterative approximate solutions. Hence, the solutions obtained using the MVIM give good approximations for a larger interval, rather than a local vicinity of the initial position. Numerical results show that the method is simple and effective.

Key words: Modified variational iteration method, Boussinesq equation, Lie'nard equation.

INTRODUCTION

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.

In this paper, the modified variational iteration method (Geng, 2010) has been applied for solving the generalized Boussinesq equation

$$u_{tt} = (u + u^2 + u_{xx})_{xx} \quad (1)$$

and Lie'nard equation

$$y'' + f(y)y' + g(y) = 0. \quad (2)$$

derived in 1872 to describe shallow water waves (Boussinesq, 1872) has the shortcoming that the Cauchy problem is ill posed. Thus, it cannot be used to analyze wave propagation problems numerically.

A well-known model of nonlinear dispersive waves was proposed by Boussinesq in the generalized form

$$u_{tt} = [f(u)]_{xx} + u_{xxxx} + h(x,t), \qquad -\infty < x < \infty, \qquad t > 0. \quad (3)$$

With u, f and h are sufficiently differentiable functions and f(0)=0 . The initial conditions associated with the Boussinesq Equation (1) are assumed to have the form

$$u(x,0) = a(x), \qquad u_t(x,0) = b(x), \qquad -\infty < x < \infty \quad (4)$$

with a(x) and b(x) given C^∞ functions.

The Boussinesq Equation (3) describes motions of long waves in shallow water under gravity by Hirota (1973, 1973a). It also arises in other physical applications such as nonlinear lattice waves, iron sound waves in plasma, and in vibrations in a nonlinear string.

Ablowitz and Segur (1981) implemented the inverse scattering transform method to handle the nonlinear equations of the physical significance where soliton solutions were developed. Hirota (1973a) constructed the N-soliton solutions of the evolution equation by reducing it to a bilinear form. The approach of Kaptsov (1998) and Andreev et al. (1999) introduced an efficient algorithm to

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handle the Boussinesq equation and developed multi solution solutions. The implementation of Adomian decomposition method (ADM) and its modified form to Boussinesq equation were introduced by Wazwaz (2001). Abbasy et al. (2007) solved the Boussinesq equation by using ADM-Pade technique. Also, variants of the one-dimensional Boussinesq equation with positive and negative exponents are investigated in Wazwaz (2001, 2005, 2006). In these works, Wazwaz used Sine-Cosine method.

In the (2), let f and g be two continuously differentiable functions on R, with f an even function and g and odd function. Then the nonlinear second ordinary differential equation of the form

$$y'' + f(y)y' + g(y) = 0,$$

is called the Lie'nard equation. During the development of radio and vacuum tubes, Lie'nard equations were intensively studied as they can be used to model oscillating circuits. Under certain additional assumptions Lie'nard theorem guarantees the existence of a limit cycle for such a system (Nili Ahmadabadi et al., 2009).

The variational iteration method, which was proposed originally by He (1999, 2006,) and He et al. (2010), has been proved by many authors to be a powerful mathematical tool for addressing various kinds of linear and nonlinear problems (Wazwaz, 2005, 2006; Jafari and Alipoor 2011, Jafari et al., 2010, 2011; Salehpoor, 2011).

The reliability of the method and the reduction in the burden of computational work gave this method wider application.

In this paper, we used MVIM for solving Equations (1) and (2). We obtain an accurate numerical solution. The advantage of the MVIM over the existing methods for solving this problem is that the solution of Equations (1) and (2) obtained using the present method is efficient not only for a smaller value of x but also for a larger value.

ANALYSIS OF THE VARIATIONAL ITERATION METHOD

To illustrate basic concepts of VIM, we consider the following general non-linear system:

$$Lu + Nu = g(t), \tag{5}$$

where L is a linear operator, N is a non-linear operator and g(t) is an inhomogeneous term. According to the VIM, we construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda\{Lu_n + N\tilde{u}_n - g\}dt, \tag{6}$$

where λ is a general multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation (He, 1999, 2006; He et al., 2010; Jafari et al., 2011), that is, $\delta\tilde{u}_n$=0.

Following the VIM, the correction variational functional in t-direction for generalized (scalar) Boussinesq equation (4) can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{u_{n\tau\tau} - [f(\tilde{u}_n)]_{xx} - \tilde{u}_{nxx\tau\tau} - h(x,\tau)\}d\tau \tag{7}$$

Making the correction functional, (6), stationary with respect to u_n , we construct

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1+\lambda'(\tau)|_{\tau=t}) + \lambda(\tau)\delta u_{n\tau}|_{\tau=t} + \int_0^t \lambda''(\tau)\delta u_n d\tau, \tag{8}$$

which yields the following stationary conditions:

$$\begin{aligned} \lambda''(\tau) &= 0, \\ \lambda(\tau)|_{\tau=t} &= 0, \\ 1 - \lambda'(\tau)|_{\tau=t} &= 0. \end{aligned} \tag{9}$$

The Lagrange multiplier, therefore, can easily be identified as

$$\lambda(\tau) = \tau - t. \tag{10}$$

Using this expression of λ in (7) we get

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\{u_{n\tau\tau} - [f(\tilde{u}_n)]_{xx} - \tilde{u}_{nxx\tau\tau} - h(x,\tau)\}d\tau. \tag{11}$$

Now we begin with an arbitrary initial approximation: $u_0 = a + bt$. Where a and b are depending on x to be determined on using the initial conditions (4), thus we have

$$u_0(x,t) = a(x) + b(x)t. \tag{12}$$

In the case of Lie'nard equations, we have

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda[y_n''(t) + f(\tilde{y}_n(t))\tilde{y}_n'(t) + g(\tilde{y}_n(t))]dt.$$

Therefore

$$\delta y_{n+1} = (1 - \lambda_t(x,x))\delta y_n + \lambda(x,x)\delta y'_n + \int_0^x \lambda''(x,t)\delta y_n dt.$$

We obtain

$$\begin{cases} \lambda_{tt}(x,t) = 0, \\ \lambda_t(x,x) = 1, \\ \lambda(x,x) = 0, \end{cases}$$

which gives $\lambda(x,t) = t - x$. Therefore our iteration formula becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x (t-x)[y_n''(t) + f(y_n(t))y_n'(t) + g(y_n(t))]dt.$$

The components of the iteration Formulae (11) can be obtained using symbolic packages such as *Maple* or *Mathematica*. As a result, the components $u_\circ, u_1, \ldots$ are identified and the solution is thus entirely determined as the limit of this sequence.

However, in many cases, the exact solution may be obtained in a closed form.

THE MODIFIED VARIATIONAL ITERATION

The main drawback of the standard VIM is that the sequence of successive approximations of the solution obtained can be rapidly convergent only in a small region, which will greatly restrict the application area of such a method.

To enlarge the convergence region of the sequence of successive approximations obtained, Geng (2010) modified the VIM by introducing an auxiliary parameter.

For using MVIM for (5), we rewrite it as

$$Lu - Lu + \gamma[Lu + Nu - g(x)] = 0, \quad (13)$$

where γ is an auxiliary parameter and $\gamma \neq 0$, which is used to adjust the convergence region of the following iterative formula.

A correct functional for (5) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(t) - L\tilde{u}_n(t) + \gamma[L\tilde{u}_n(t) + N\tilde{u}_n(t) - g(t)]\}dt, \quad (14)$$

where λ is a general Lagrangian multiplier, which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

According to the VIM, the following iteration formula can be obtained:

$$u_{n+1}(x) = u_n(x) + \gamma\int_0^x \lambda(t,x)[Lu_n(t) + Nu_n(t) - g(t)]dt, \qquad n = 0,1,2,\ldots \quad (15)$$

From the convergence analysis aforementioned, it is easy to see that smaller the value of $|\gamma|$ is, the wider the convergence region of iterative sequence (15).

In fact, iterative formula (15) gives us vast freedom of choice for some strong nonlinear problems; one can choose a relatively small value of $|\gamma|$ (generally less than 1) to obtain a good approximation in a wider region.

In addition, it should be especially pointed out that when $\gamma = 1$, (14) becomes the standard variational iteration formula (14).

For Equation (1), according to the aforementioned MVIM, we construct the correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\{u_{ntt} - \tilde{u}_{ntt}(t) + \gamma[\tilde{u}_{ntt} - \tilde{u}_{xx} - \tilde{u}_{xx}^2 - \tilde{u}_{xxxx}]\}dt, \qquad 0 \le x \le x \quad (16)$$

where $\tilde{u}_n$ is a restricted variation, that is, $\delta\tilde{u}_n = 0$; λ is a general Lagrangian multiplier and can be easily identified as $\lambda = t - x$.

So we can obtain the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \gamma\int_0^x (t-x)[u_{ntt} - u_{nxx} - u_{nxx}^2 - u_{nxxxx}]dt, \quad 0 \le x \le x, \; n = 0,1,2 \quad (17)$$

where $u_\circ(x)$ is an initial approximation satisfying the initial condition of Equation (1).

Theorem 1

Suppose that $u_\circ(x) = \alpha$ and the iterative sequence $\{u_n(x)\}$ obtained from (17) converge to u(x), then u(x) is the solution of Equation (1) (Geng, 2010).

Theorem 2

Define a nonlinear mapping (Geng, 2010)

$$T[u(x)] = u(x) + \gamma\int_\circ^x [u_{tt} - u_{xx} - u_{xx}^2 - u_{xxxx}]dt.$$

A sufficient condition for convergence of the iterative sequence $\{u_n(x)\}$ obtained from (17) is strict contraction of the nonlinear mapping T. Furthermore, the sequence (17) converges to the fixed point of T which is also the solution of Equation (1).

Therefore, according to (17), by choosing a proper γ and initial approximation $u_0(x)$, the successive

Table 1. Numerical results for Example 1.

(x,t)	MVIM		(VIM)		Exact
	$u_2(x,t)$	$u_2(x,t)$	$u_4(x,t)$	$u_2(x,t)$	
(-0.5,0.2)	-6.65264	-5.64433	-4.93865	-7.20312	-7.20312
(-0.5,0.5)	-12.1813	-23.3471	-36.9943	-20	-20
(0,0.3)	-5.87304	-7.83001	-10.26	-7.16327	-7.16327
(0,0.6)	-10.4438	-50.8722	-93.9444	-24	-24
(0.5,0.3)	-4.49565	-8.59412	-12.9882	-5.63265	-5.63265
(0.5,0.6)	-7.8024	-45.1842	-84.2184	-19.3125	-19.3125

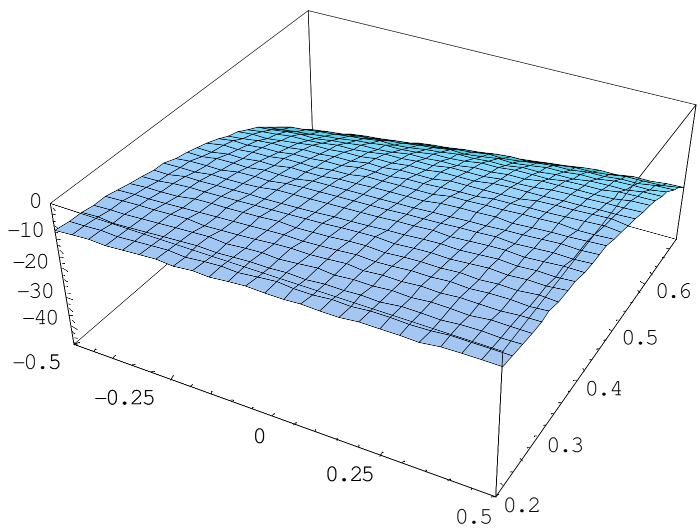

Figure 1. Approximate solution $u(x,t)$.

approximations of the solution to (1) on the entire interval [0, x] can be obtained.

NUMERICAL EXAMPLES

Here, we apply the MVIM to Boussinesq and Lie'nard equation. Numerical results show that the MVIM is very effective.

Example 1

Consider the Boussinesq equation (Yusufoglu, 2008)

$$u_{tt} = (u - u^2/2)_{xx} + u_{xxtt},$$
$$-\infty < x < \infty, t \geq 0, \tag{18}$$
$$u(x,0) = a(x) = -x^2 + 2x - 3,$$
$$u_t(x,0) = b(x) = -2x^2 + 4x - 8.$$

The exact solution can be easily determined to be

$$u(x,t) = 1 - \frac{x^2 - 2x + 4}{(t-1)^2}. \tag{19}$$

According to (19), taking γ = 0.3, n = 4, the numerical results is shown in Table 1. From Table 1, we find that the solution derived by the VIM (Yusufoglu, 2008) gives a good approximation only in the neighborhood of the initial position, while the present method gives a good approximation in a wider region. In Figures 1 and 2, we plot *u*1*(x, t)* and *u(x, t)* which is the exact solution (Yusufoglu, 2008).

Example 2

Consider the Lie'nard equation

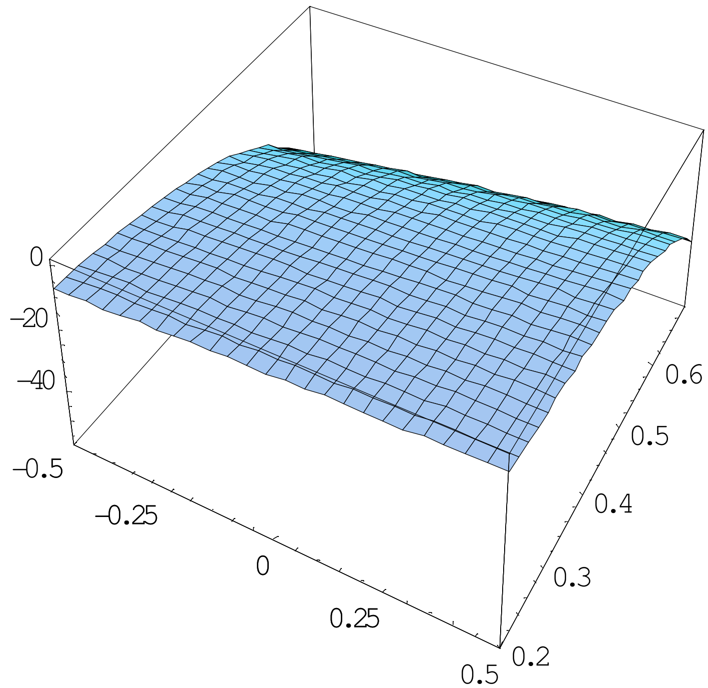

Figure 2. Exact solution $u(x,t)$.

Table 2. Comparison of VIM and MVIM with exact solution for Example 2.

X	(MVIM)		VIM (Nili Ahmadabadi et al., 2009)		Exact
	$u_2(x,t)$	$u_4(x,t)$	$u_2(x,t)$	$u_4(x,t)$	
0.1	0.706576	0.706046	0.705339	0.703571	0.741508
0.3	0.702334	0.697561	0.691197	0.675287	0.803527
0.4	0.698621	0.690136	0.678823	0.650538	0.830647
0.5	0.693849	0.68059	0.662913	0.618718	0.85502
0.7	0.681121	0.655134	0.620486	0.533866	0.895647
0.9	0.66415	0.621193	0.563918	0.420729	0.926363

$$\begin{cases} y'' - y + 4y^3 - 3y^5 = 0, \\ y(0) = \dfrac{\sqrt{2}}{2}, \\ y'(0) = \dfrac{\sqrt{2}}{4}, \end{cases}$$

with exact solution [19]

$$y(x) = \sqrt{\frac{1+\tanh(x)}{2}}.$$

In Table 2, we compare the VIM and MVIM with exact solution.

Conclusion

In this paper, MVIM is used for solving Boussinesq equations and Lie'nard equation. Comparing with the standard VIM results, the results for numerical examples demonstrate that the MVIM can give a more accurate approximation in a larger region. This is also the main advantage of this method. Therefore, the modification of the VIM can overcome the restriction of the application area of the VIM, and then expand its scope of application.

However, generally, when the value of $|\gamma|$ chosen is small, the rate of convergence of the iterative formula is relatively slow, and so more iterative steps are required. This is the drawback of this modification.

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