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Full Length Research Paper

Using homotopy analysis method to obtain approximate analytical solutions of wave equations

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The homotopy analysis method (HAM) is proposed for solving wave equations. By using the HAM, an approximate solution is as series which contains the auxiliary parameter $\hbar$. In this way, $\hbar$ provides us with a simple way to adjust and control the convergence region of series solution. This method has a reasonable residual error and in some cases has an exact solution. Some examples are employed to illustrate validity and flexibility of this method.

Key words: Homotopy analysis method, wave equation, analytic methods.

INTRODUCTION

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. It is still difficult to obtain accurate solutions of nonlinear problems and often more difficult to get an analytic approximation than a numerical one of a given nonlinear problem. Recently, a powerful analytic method for nonlinear problems, namely the homotopy analysis method (HAM), has been developed by Liao (2003). This method provides us with a simple way to ensure the convergence of the series solution, so that we can get approximations with enough accuracy. Further more, this technique doesn't have restriction of non-perturbation methods, such as Lyapunov's artificial small parameter method, the δ-expansion method and Adomian's decomposition method (Jafari and Daftardar_Gejji, 2006; Bildik and Konuralp, 2007). In summary, this technique provides us with a convenient way to adjust with convergence region and the rate of approximation series. HAM has been applied successfully to many nonlinear problems in science and engineering (Abbasbandy et al., 2009: Babolian et al., 2009; Abbasbandy, 2006, 2007; Paripour et al., 2010; Liao, 1992, 2003, 2004, 2005; Liao et al., 2003; Fu et al., 2001; Ezzati et al., 2011: Ezzati and Tajdini, 2010; Biazar et al., 2011, Mohyud-Din et al., 2011; Jafari et al., 2011, Jafari and Momani 2007). In this paper HAM is being used to find solutions of wave equations as follows:

$$u_{tt} = \left(F(u)u_x\right)_x, \tag{1}$$

where $F(u)$ is a function of $u(x,t)$; and is considered in the following types:

- $F(u) = u^n \qquad n \geq 0,$
- $F(u) = u^n \qquad n < 0,$
- $F(u) = \dfrac{1}{u^2+1}$
- $F(u) = \dfrac{1}{u^2-1}$

THE BASIC RULES OF THE HAM

In most cases, a nonlinear problem can be described by a set of governing equations with initial and /or boundary conditions. Let us consider the non-linear equation in a general form as follows:

$$N\left[u(r,t)\right] = 0,$$

where, N is a non-linear operator, $u(r,t)$ is an unknown function and r and t denote spatial and temporal

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Abbrevaition: HAM, Homotopy analysis method.

independent variable, respectively. Using homotopy concept , (Liao et al., (2003)) has constructed the so-called zeroth-order-deformation equation:

$$(1-q)L[\phi(r,t,q)-u_0(r,t)] = q\hbar H(r,t)N[\Phi(r,t,q)], \quad (2)$$

where, $q \in [0,1]$ is an embedding parameter, $u_0(r,t)$ denotes an initial guess of the exact solution $u(r,t)$, $\hbar \neq 0$ is an auxiliary parameter, $H(r,t) \neq 0$ is an auxiliary function, $\Phi(r,t,q)$ is an unknown function and L is an auxiliary linear operator with the property:

$$\Phi(r,t,0) = u_0(r,t),$$

$$\Phi(r,t,1) = u(r,t).$$

Clearly, as q increases from 0 to 1, the solution $\Phi(r,t,q)$ varies from $u_0(r,t)$ to the exact solution $u(r,t)$. Liao (2009) named the auxiliary parameter $\hbar \neq 0$ as the convergence-control parameter. HAM, because of having $H(r,t)$ and $\hbar$, is more general than the traditional ones. $H(r,t)$ and $\hbar$ play important roles with in the frame of the HAM (Liao, 1995). By expanding $\Phi(r,t,q)$ in a Taylor series with respect to q we have:

$$\Phi(r,t,q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) q^m, \quad (3)$$

where,

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \Phi(r,t,q)}{\partial q^m}\bigg|_{q=0}.$$

With property chosen of $L, u_0(r,t), \hbar, H(r,t)$, the series (3) will converge at $q = 1$ and the power series (3) becomes:

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t), \quad (4)$$

which must be one of the solutions of $N[u(r,t)] = 0$. In short, defining the vector as:

$$\overrightarrow{u_n} = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\}.$$

According to the definition of $u_m(r,t)$, it can be derived from the zero-order deformation equation (2). Differentiating the equation (2) m-times respective to the embedding parameter q and then dividing it by $m!$ and finally setting $q = 0$, we have the so-called mth-order deformation equation:

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\overrightarrow{u_{m-1}}, r, t), \quad (5)$$

where, χ_m is defined by:

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m \leq 1, \end{cases} \quad (6)$$

and

$$R_m(\overrightarrow{u_{m-1}}, r, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\Phi(r,t,q)]}{\partial q^{m-1}}\bigg|_{q=0}. \quad (7)$$

Clearly, Equation 5 is a linear one. Therefore a nonlinear equation could be transformed to a system of linear ones which can be easily solved using an iterative procedure and this is the main consequence of the HAM. After solving Equation 5, we can substitute $u_m(r,t)$ in Equation 5 and obtain an approximation of arbitrary order. In (Liao, 2003), the authors consider that convergence-control parameter $\hbar$ provides a simple way to adjust and control the convergence region and rate of the approximation series. So we can always find a proper value for $\hbar$ to ensure the convergence of series solution (Liao, 1995). By plotting the so-called $\hbar$-curves, we can adjust and control the convergence region and rate of approximation series and also choose $\hbar$. For more examples and details one can refer to (Paripour et al., 2010; Liao, 2003; Liao et al., 2003).

APPLICATION OF THE HAM

In this section, we apply the HAM to solve different kinds of wave equations. Although in this method we have great freedom to choose the initial approximation $u_0(x,t)$, the auxiliary linear operator L , the auxiliary function $H(x,t)$ and the auxiliary parameter $\hbar$; for simplicity we use auxiliary function $H(x,t) = 1$ and the auxiliary linear operator $L = \frac{\partial^2}{\partial t^2}$ with the property

$$L[A(x)t + B(x)] = 0,$$

where A and B are integral constants. For choosing $u_0(x,t)$, suppose that the following wave equation is given:

$$u_{tt} = (F(u)u_x)_x,$$

with initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

We suggest $u_0(x,t) = f(x) + tg(x)$, where f, g are function of (x,t). For solving the examples of this section, we apply the so-called mth-order deformation equation, as follows:

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\overrightarrow{u_{m-1}}, r, t), \quad (8)$$

with initial conditions:

$$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$$

where χ_m is defined in Equation 6.

Example 1. Consider the non-linear equation:

$$u_{tt} = (u^{-1}u_x)_x, \quad (9)$$

subject to the initial conditions

$$u(x,0) = \frac{1}{(1+x)^2}, \quad u_t(x,0) = 0.$$

According to above explanations, we have $u_0(x,t) = \frac{1}{(1+x)^2}$. Successively, using the so-called mth-order deformation Equation 8 with initial conditions:

$$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$$

one has:

$$u_1(x,t) = \frac{-\hbar t^2}{(1+x)^2},$$

$$u_2(x,t) = \frac{-\hbar(1+\hbar)t^2}{(1+x)^2},$$

$$u_n(x,t) = \frac{-\hbar(1+\hbar)^{n-1}t^2}{(1+x)^2}.$$

So, using Equation 4, the series solution which is in terms of $\hbar$, will be obtained:

$$u(x,t) = \frac{1}{(1+x)^2} + \frac{-ht^2}{(1+x)^2}\sum_{n=1}^{\infty}(1+h)^{n-1}$$

The series will be converge when $|1+\hbar| < 1$, so

$$u(x,t) = \frac{1+t^2}{(1+x)^2}.$$

is the exact solution.

Example 2. Consider the wave equation:

$$u_{tt} = (u^{-2}u_x)_x, \quad (10)$$

subject to the initial conditions:

$$u(x,0) = \frac{1}{2+x}, u_t(x,0) = \frac{-1}{2+x}.$$

Frequently solving Equation 8 with initial conditions:

$$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$$

and choosing $u_0(x,t) = \frac{1-t}{2+x}$, we can obtain $u_i(x,t), i = 1,2,\cdots$, as follows :

$$u_0(x,t) = \frac{1-t}{2+x},$$

$$u_1(x,t) = 0,$$

$$u_2(x,t) = 0,$$

$$u_3(x,t) = 0,$$

: Now according to Equation 4 the solution is

$$u(x,t) = \frac{1-t}{2+x},$$

which is the exact solution.

Example 3. Consider the wave equation

$$u_{tt} = (uu_x)_x, \quad (11)$$

subject to the initial conditions

$$u(x,0) = x, u_t(x,0) = 0.$$

Clearly, for solving Equation 11 with its initial conditions by using the HAM, we choose $u(x,t) = x$. Now, solving Equation 8 with initial conditions

$$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$$

we can obtain $u_i(x,t), i = 1,2,\cdots$, as follows:

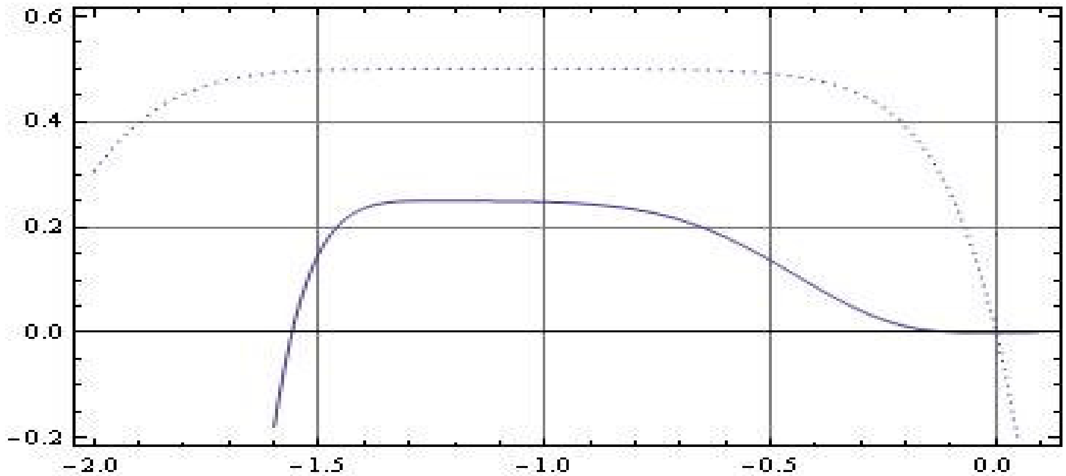

Figure 1. $\hbar$-curves according to $Approx10$ of example 4. Dotted curve: $u_x(1,1)$; solid curve: $u_{tt}(1,1)$.

$$u_1(x,t) = \frac{-1}{2}\hbar t^2,$$

$$u_2(x,t) = \frac{-\hbar t^2}{2}(\hbar + 1),$$

$$u_3(x,t) = \frac{-\hbar t^2}{2}(\hbar + 1)^2,$$

$$u_n(x,t) = \frac{-\hbar t^2}{2}(\hbar + 1)^{n-1}.$$

So, according to Equation 4, we can approximate $u(x,t)$ as follows:

$$u(x,t) = x + \left(\frac{-t^2}{2}\right)\hbar\sum_{i=1}^{\infty} (1+\hbar)^{n-1}.$$

Imposing $|1+\hbar| < 1$ the series converges to

$x + \left(\frac{-t^2}{2}\right)\hbar\left(\frac{1}{1-(1+\hbar)}\right) = x + \frac{t^2}{2}$, which is the exact solution .

Example 4 (Ezzati and Tajdini, 2010), Consider the wave equation

$$u_{tt} = \left(u^2 u_x\right)_x, \qquad (12)$$

subject to the initial condition

$$u(x,0) = x, u_t(x,0) = -x.$$

Frequently, solving Equation 8 with initial conditions

$$u_m(x,0) = 0, \quad u_{m_t}(x,0) = 0,$$

and using the initial guess $u_0(x,t) = x - tx$, we obtain:

$$u_1(x,t) = \frac{1}{10}\hbar t^2(-10 + 10t - 5t^2 + t^3)x,$$
$$u_2(x,t) = \frac{1}{10}\hbar t^2(-10 + 10t - 5t^2 + t^3)x -:$$

$$\frac{1}{120}\hbar^2 t^2(120 - 120t + 96t^3 - 84t^4 + 36t^5 - 9t^6 + t^7)x,$$

We use the 10-term approximation and set

$$Approx10 = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_9(x,t).$$

By plotting the $\hbar$-curves, we obtain the interval $R_\hbar = [-1.3, -0.9]$ as the valid region for $\hbar$ (Figure 1). Testing different values of $\hbar$ in the $R_\hbar$, it can be concluded that the value $\hbar = -1.111$ is the best one with the minimum error. Figure 2 shows error curves in $x = 1,2,3$ for $t \in [0,2]$. Tables 1 and 2 show the absolute errors of $Approx10$ for $\hbar = -1.111$ and $\hbar = -1$, respectively. Clearly, residual error of $Approx10$ for $\hbar = -1.111$ is better than for $\hbar = -1$, (the HAM with $\hbar = -1$ is the homotopy perturbation method).

Example 5. Consider the wave equation;

$$u_{tt} = \left(\frac{1}{u^2-1}u_x\right)_x, \qquad (13)$$

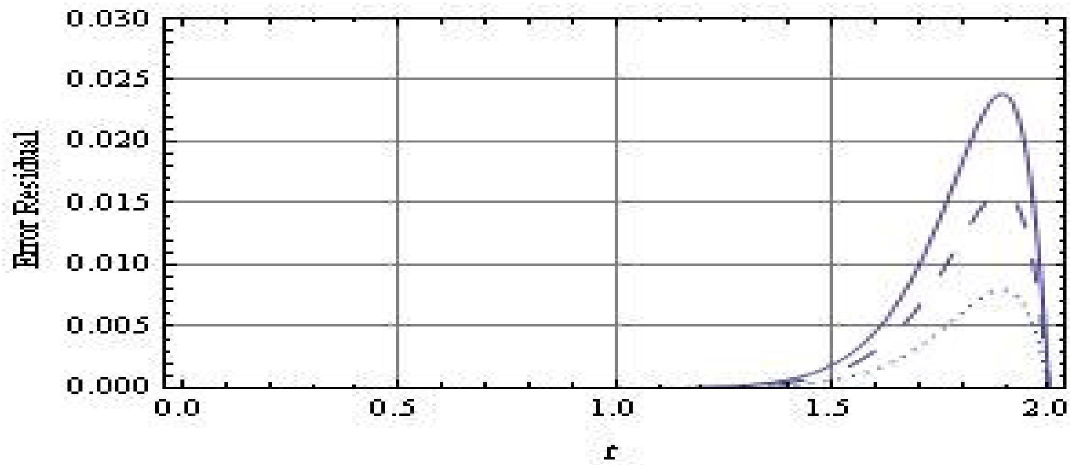

Figure 2. The residual error of Example 4 for the $Approx10$. Solid curve: x = 3; dashed curve: x = 2; dotted curve: x = 1.

Table 1. Absolute errors of $Approx10$ of Example 4, obtained using the HAM with $\hbar = -1.111$.

x	t = 0.5	t = 1	t = 1.5	t = 2
-4	6.60195 E-10	3.08115 E-6	2.49995 E-3	7.70863 E-5
-3	4.95136 E-10	2.31086 E-6	1.87496 E-3	5.78147 E-5
-2	3.30098E-10	1.54057 E-6	1.24997 E-3	3.85432 E-5
-1	1.65049E-10	7.70287 E-7	6.24986 E-3	1.92716 E-5
1	1.65049E-10	7.70287 E-7	6.24986 E-4	1.92716 E-5
2	3.30098E-10	1.54057 E-6	1.24997 E-3	3.85432 E-5
3	4.95136E-10	2.31086 E-6	1.87496 E-3	5.78147 E-5
4	6.60195E-10	3.08115 E-6	2.49995 E-3	7.70863 E-5
10	1.65044E-9	7.70287 E-6	6.24986 E-3	1.92716 E-5

Table 2. Absolute errors of $Approx10$ of Example 4, obtained using the HAM with $\hbar = -1$.

x	t = 0.5	t = 1	t = 1.5	t = 2
-4	1.05843 E-9	4.41821 E-5	5.81078 E-3	5.20989E-2
-3	7.93815 E-10	3.31366 E-5	4.35809 E-3	3.90742E-2
-2	5.29216 E-10	2.2091 E-5	2.90539 E-3	2.60494E-2
-1	2.64608 E-10	1.10455 E-5	1.4527 E-4	1.30247E-5
1	2.64608 E-10	1.10455 E-5	1.4527 E-4	1.30247E-5
2	29216 E-10	2.2091 E-5	2.90539 E-3	2.60494E-2
3	7.93815 E-10	3.31366 E-5	4.35809 E-3	3.90742E-2
4	1.05843 E-10	4.41821 E-5	5.81078 E-3	5.20989E-2
10	2.64606 E-9	1.10455 E-4	1.4527 E-2	1.30247E-1

subject to

$$u(x,0) = \tanh(x), \quad u_t(x,0) = 0.$$

Solving the Equation 8 with initial conditions:

$$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$$

and considering initial guess $u_0(x,t) = tanh(x)$, we can calculate $u_i(x,t), i = 1,2,\cdots$ as follows:

$u_1(x,t) = 0,$
$u_2(x,t) = 0,$
$u_3(x,t) = 0,$

: According to Equation 4, the series solution expression can be written in the form of

$$u(x,t) = u_0(x,t) + u_1(x,t) + \cdots = \tanh(x)$$

which is the exact solution.

Example 6. Consider the wave equation:

$$u_{tt} = \left(\frac{1}{1+u^2} u_x\right)_x, \qquad (14)$$

subject to

$$u(x,0) = cot(x), u_t(x,0) = 0.$$

Solving the Equation 8 with initial conditions

$u_m(x,0) = 0, u_{m_t}(x,0) = 0,$ and considering initial guess $u_0(x,t) = cot(x),$ we can successively calculate

$u_1(x,t) = 0,$

$u_2(x,t) = 0,$

$u_3(x,t) = 0,$

Hence, according to Equation 4, the approximate solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + \cdots = cot(x),$$

which is the exact solution.

CONCLUSION

In this paper, the HAM is applied to obtain the approximated analytical solutions of wave equations in different forms. A fundamental qualitative difference between the HAM and another analytical method is that the HAM provides us with a convenient way to control and adjust convergent regions of approximated solutions. Examples show the flexibility and potential of the HAM for complicated nonlinear problems.

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