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# On topological covering-based rough spaces

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Rough set theory, a mathematical tool to deal with vague concepts, has originally described the indiscernibility of elements by equivalence relations. Covering-based rough sets are a natural extension of classical rough sets by relaxing the partitions arising from equivalence relations to coverings. Recently, some topological concepts such as subbase, neighborhood and separation axioms have been applied to study covering-based rough sets. However, the topological space on covering-based rough sets and the corresponding topological properties on the topological covering-based rough space are not studied. This paper studies some of these problems. We defined open sets, closed sets, rough inclusion, rough equality on covering-based rough sets and some of their properties are studied. Then, we give the definition of topology on covering-based rough sets. Finally, we study the properties of rough homeomorphisms. This research not only can form the theoretical basis for further applications of topology on covering-based rough sets but also lead to the development of the rough set theory and artificial intelligence.

**Key words:** Topology, rough Sets, covering, artificial Intelligence.

## INTRODUCTION

Rough set theory, proposed by Pawlak (1982), is a new mathematical approach to deal with imprecision, vagueness, and uncertainty in data analysis and information systems. The classical rough set theory is based on equivalence relations. However, the requirement of equivalence relations as the indiscernibility relation is too restrictive for many applications. In light of this, a method is the relaxation of the partition arising from equivalence relation to a covering. The covering of a universe is used to construct the lower and upper approximations of any subset of the universe. Since then, a lot of mathematicians, logicians, and researchers of computers have become interested in the rough theory and have done a lot of research work of rough theory in theory and application. Its applications are showed in wide fields such as data mining, process control, pattern recognition and artificial intelligence. Recently, Serkan and Ozelik (2010) provide an application of rough set theory in topology course and Serkan (2010) showed that rough set approximate can be used for the analysis of attitude scales.

We know information play an important role in our life. El-Naschie (1992) pointed to the uncertainty of information in quantum space-time. The works about importance of information analysis in physics can be found in Witten (1996).

The topology is to study the invariance of a given space under topological transformation (homeomorphism Shakeel (2010)). Relations of topology and physics have been appeared in El Naschie (2000). The importance of topology has appeared in many fields of applications such as quantum physics, high energy physics, superstring theory (El-Naschie, 2001; 2003; 2006). Regarding the combination of rough set theory and topology, some research is given in the paper of Karim et al. (2001), Li (2004; 2010), Zhu (2007), Wu et al. (2008) and Ge (2010).

However, in the above literature, the topological space on covering-based rough sets and the corresponding topological properties on the topological covering-based rough space are not studied. This paper studies these problems. The basic notions in rough set theory are the lower and upper approximation operators. By using the lower and upper approximations of decision classes, knowledge hidden in information tables may be unraveled and expressed in the form of decision rules. In this paper,

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we study a kind of covering-based rough sets from the topological point of view. The remainder of this paper is organized as follows: First, we present the fundamental concepts and properties of the Pawlak's rough set theory and covering-based rough sets. In addition, we describe some topological concepts. Second, we give the definition of open sets, closed sets and closure of sets on the covering lower approximation and covering upper approximation of a set. Some of their properties are studied. Then, we construct the topology on the covering-based rough sets and define the topological covering-based rough space. Finally, we study the properties of rough homeomorphisms. This research not only can form the theoretical basis for further applications of topology on covering-based rough sets but also lead to the development of the rough set theory and artificial intelligence.

**PRELIMINARIES**

In this section, we introduce the fundamental ideas behind rough sets and topology. Firstly, let's recall some concepts and properties of the Pawlak's rough sets.

Let  $U$  be a finite set and  $R$  be an equivalence relation on  $U$ .  $R$  generates a partition  $U/R = Y_1, Y_2; \dots, Y_m$

On  $U$  where  $Y_1, Y_2; \dots, Y_m$  are the equivalence classes generated by the equivalence relation  $R$ . In the rough set theory, these are also called elementary sets of  $R$ . For any  $X \subseteq U$ , we can describe  $X$  by the elementary sets of  $R$  and the two sets:  $R_*(X) = \cup Y_i \in U/R | Y_i \subseteq X$   $R^*(X) = \cup Y_i \in U/R | Y_i \cap X \neq \emptyset$  which are called the lower and the upper approximation of  $X$ , respectively.

In the following discussion, the universe of discourse  $U$  is considered to be infinite.

Now we will list some definitions and results about covering-based rough sets used in this paper, which belongs to Zhu (2002; 2009).

Let  $U$  be a domain of discourse,  $C$  a family of subsets of  $U$ . If none subsets in  $C$  is empty, and  $\cup C = U$ ,  $C$  is called a covering of  $U$ . It is clear that a partition is certainly a covering, so the concept of coverings is an extension of the concept of partitions.

Let  $U$  be a non-empty set,  $C$  a covering of  $U$ . We call the ordered pair  $(U, C)$  a covering approximation space. Assume that  $X \subseteq U$ . If  $X$  is not always accurately described by the elements of the covering  $C$ , that is, it is a covering-based rough set.

Then, we can use two exact sets of  $X$  on  $U$  to approximately describe it. The two exact sets are the covering lower approximation  $X_*$  and the covering upper approximation  $X^*$  of  $X$ , respectively, Which are defined as follows:  $X_* = \cup K \in C | K \subseteq X$   $X^* = \cup K \in C | K \cap X \neq \emptyset$ . The ordered pair  $(X_*, X^*)$  is referred to as the covering-based rough set of  $X$  and labeled  $X = (X_*, X^*)$ .

Let  $K = (U, C)$  be a covering approximation space. Throughout this paper, we use the following notations, where that  $A = A_*, A^* \subseteq U$  and  $B = B_*, B^* \subseteq U$ .

- (1) the intersection operation of covering-based rough sets:  $A \cap B = (A_* \cap B_*, A^* \cap B^*)$ .
- (2) the union operation of covering-based rough sets:  $A \cup B = (A_* \cup B_*, A^* \cup B^*)$ .

Next, we give some concepts of topology. We refer to Engelking (1977), Kelly (1955), for details. A topological space is a pair  $(U, \tau)$  consisting of a set  $U$  and family  $\tau$  of subset of  $U$  satisfying the following conditions:

- (1)  $U \in \tau$  and  $\emptyset \in \tau$ .
- (2)  $\tau$  is closed under arbitrary union.
- (3)  $\tau$  is closed under finite intersection.

The pair  $(U, \tau)$  is called a topological space, the elements of  $U$  are called points of the space, the subsets of  $U$  belonging to  $\tau$  are called open sets in the space, and the complement of the subsets of  $U$  belonging to  $\tau$  are called closed sets in the space; the family  $\tau$  of open subsets of  $U$  is also called a topology for  $U$ .

Let  $(X, \tau)$ ,  $(Y, \delta)$  be two topological spaces; a mapping  $f$  of  $X$  to  $Y$  is called continuous if  $f^{-1}(U) \in \tau$  for any  $U \in \delta$ , that is, if the inverse image of any open subset of  $Y$  is open in  $X$ .

A mapping  $f$  of  $X$  to  $Y$  is called open (closed) if for every open (closed) set of  $A \subseteq X$  the image  $f(A)$  is open (closed) in  $Y$ .

A continuous mapping  $f$  of  $X$  to  $Y$  is called homeomorphism if  $f$  is a single and full mapping and

the inverse mapping  $f^{-1}$  is continuous.

**Lemma 1.**

For a single and full mapping  $f$  of a topological space  $X$  to a topological space  $Y$  the following conditions are equivalent:

- (1) The mapping  $f$  is a homeomorphism.
- (2) The mapping  $f$  is continuous and closed.
- (3) The mapping  $f$  is continuous and open. R. Engelking (1977)

**TOPOLOGICAL COVERING-BASED ROUGH SPACE**

Firstly, we definite open and closed sets in the covering approximation space.

Since  $X_*$  and  $X^*$  are exact sets, we can give the different topology  $\tau_1$  and  $\tau_2$  on  $X_*$  and  $X^*$ , respectively. Then obtain the corresponding topological spaces  $(X_*, \tau_1)$  and  $(X^*, \tau_2)$ . Generally, when  $\tau_1$  and  $\tau_2$  are clearly given,  $(X_*, \tau_1)$  and  $(X^*, \tau_2)$  can be briefly labeled as  $X_*$  and  $X^*$ .

**Definition 1**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $A_1$  and  $A_2$  be the subsets corresponding to the topological space  $X_*$  and  $X^*$ , respectively.  $a_1 \in A_1$  is called an inner point of  $A_1$  inside  $X_*$  if  $a_1$  has a neighborhood  $U_{(a_1)} \subseteq A_1$ ; correspondingly,  $a_2 \in A_2$  is called an inner point of  $A_2$  inside  $X^*$  if  $a_2$  has a neighborhood  $U_{(a_2)} \subseteq A_2$ . The set that consists of all inner points of  $A_1$  inside  $X_*$  is called the interior of  $A_1$  inside  $X_*$ ; likewise, the set that consists of all inner points of  $A_2$  inside  $X^*$  is called the interior of  $A_2$  inside  $X^*$ . These sets are denoted by  $IntA_1$  and  $IntA_2$ , respectively.  $A_1$  is called an open set

of  $X_*$  if  $A_1 = IntA_1$ ;  $A_2$  is called an open set of  $X^*$  if  $A_2 = IntA_2$ . At the same time, if  $A_1$  and  $A_2$  satisfy  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ ,  $A = A_1, A_2$  is called an open set of  $X$ .

**Theorem 1**

Let  $A = A_1, A_2$  be a subset of  $X = (X_*, X^*) \subseteq U$ , and it satisfies  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ .  $A$  is an open set  $\Leftrightarrow$  it is an ordered pair that consists of the union of some neighborhoods.

**Proof;**

By Definition 1 and general topological knowledge (R. Engelking (1977), J.L.Kelly (1955)),  $A_1$  is an open set of  $X_* \Leftrightarrow A_1 = IntA_1$ . Then, for all  $a_1 \in A_1$ , there is a neighborhood  $a_1 \in U_{(a_1)} \subseteq A_1$ . Since  $a_1$  is random, we have  $a_1 \in \bigcup_{a_1 \in A_1} U_{(a_1)}$ . Thus,  $A_1 \subseteq \bigcup_{a_1 \in A_1} U_{(a_1)}$ . However,  $\bigcup_{a_1 \in A_1} U_{(a_1)} \subseteq A_1$ ; then,  $A_1 = \bigcup_{a_1 \in A_1} U_{(a_1)}$ . This means that  $A_1$  is the union of some neighborhoods in  $X_*$ .

Similarly,  $A_2$  is an open set of  $X^* \Leftrightarrow A_2$  is the union of some neighborhoods in  $X^*$ .

By the condition  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ ,  $A = A_1, A_2$  is an open set of  $X = (X_*, X^*) \Leftrightarrow$  it is an ordered pair that consists of the union of some neighborhoods.

**Theorem 2**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $\tau = (\tau_1, \tau_2)$  be a family of open sets that consists of all open sets of  $X$ . Then,  $\tau$  satisfies the followings;

- (1)  $X \in \tau$  and  $\emptyset \in \tau$ , where  $\emptyset = (\emptyset, \emptyset)$  is an empty set.
- (2) If  $O_1, O_2 \in \tau$ , then  $O_1 \cap O_2 \in \tau$ .
- (3) If for any  $\gamma \in \Gamma$ , we have  $O_\gamma \in \tau$ . Then,  $\bigcup_{\gamma \in \Gamma} O_\gamma \in \tau$

**Proof**

(1) Based on the general topological knowledge (Engelking (1977); Kelly (1955)), there are

$X_* \in \tau_1$ , and  $X^* \in \tau_2$ . Again, by  $X_* \subseteq X^*$ , then  $X \in \tau$ . Moreover,

(2) By (1), if  $O_1 \cap O_2 = \emptyset$ , then it is an open set. Let  $O_1 \cap O_2 = (O_{1*} \cap O_{2*}, O_{1*}^* \cap O_{2*}^*)$ . Since  $O_{1*}$  is an open set,  $O_{1*}$  and  $O_{1*}^*$  are also open sets and satisfy  $O_{1*} \subseteq X_* \subseteq O_{1*}^* \subseteq X^*$ ; Similarly,  $O_{2*}$  is an open set,  $O_{2*}$  and  $O_{2*}^*$  are also open sets and satisfy  $O_{2*} \subseteq X_* \subseteq O_{2*}^* \subseteq X^*$ . Then,  $O_{1*} \cap O_{2*}$  and  $O_{1*}^* \cap O_{2*}^*$  are all open sets. By the above assumptions, the two sets satisfy  $O_1 \cap O_2 \subseteq X_* \subseteq O_{1*} \cap O_{2*} \subseteq X^*$ . Therefore  $O_1 \cap O_2 \in \tau$ .

(3) Use the same method above, we can prove the union of any open sets that are members of  $\tau$  still belongs to  $\tau$ .

**Definition 2**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $A_1$  and  $A_2$  be the subsets corresponding to the topological space  $X_*$  and  $X^*$ , respectively.  $A_1$  and  $A_2$  are called the closed sets of  $X_*$  and  $X^*$ , respectively, if the complementary sets  $X_* \setminus A_1$  and  $X^* \setminus A_2$  of  $A_1$  and  $A_2$  are the open sets of  $X_*$  and  $X^*$ , respectively. At the same time, if  $A_1$  and  $A_2$  satisfy  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ ,  $A = A_1, A_2$  is called the closed set of  $X$ .

**Theorem 3**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $\nu = (\nu_1, \nu_2)$  be a family of closed sets that consists of all closed sets of  $X$ . Then,  $\nu$  satisfies the followings.

- (1)  $X \in \nu$  and  $\emptyset \in \nu$ , where  $\emptyset = (\emptyset, \emptyset)$  is an empty set.
- (2) If  $F_1, F_2 \in \nu$ , then  $F_1 \cup F_2 \in \nu$ .
- (3) If for any  $\gamma \in \Gamma$ , we have  $F_\gamma \in \nu$ . Then,  $\bigcap_{\gamma \in \Gamma} F_\gamma \in \nu$ .

**Proof**

$X = (X_*, X^*) \in \nu$ . Again,  $(X_* \setminus \emptyset, X^* \setminus \emptyset) =$

$(X_*, X^*) = X \in \tau$ . Thus,  $\emptyset \in \nu$ .

(2) Assume that  $F_1 = F_{1*}, F_{1*}^* \in \nu$ ,  $F_2 = F_{2*}, F_{2*}^* \in \nu$  and satisfy  $F_{1*} \subseteq X_* \subseteq F_{1*}^* \subseteq X^*$  and  $F_{2*} \subseteq X_* \subseteq F_{2*}^* \subseteq X^*$ . Because

$$X_* \setminus F_{1*} \cup F_{2*} = X_* \setminus F_{1*} \cap X_* \setminus F_{2*},$$

$$X^* \setminus F_{1*}^* \cup F_{2*}^* = X^* \setminus F_{1*}^* \cap X^* \setminus F_{2*}^*, \quad X_* \setminus F_{1*}^*$$

,  $X_* \setminus F_{2*}^*$ ,  $X^* \setminus F_{1*}^*$  and  $X^* \setminus F_{2*}^*$  are all open

sets. Then  $X_* \setminus F_{1*} \cup F_{2*}$  and  $X^* \setminus F_{1*}^* \cup F_{2*}^*$  are open sets by Theorem 2(2). Thus, by definition 2,  $F_{1*} \cup F_{2*}$  and  $F_{1*}^* \cup F_{2*}^*$  are the closed sets of  $X_*$  and  $X^*$ , respectively. By the known assumption, we have  $F_{1*} \cup F_{2*} \subseteq X_* \subseteq F_{1*}^* \cup F_{2*}^* \subseteq X^*$ . From the union operation of covering-based rough sets again, we have  $F_1 \cup F_2 = (F_{1*} \cup F_{2*}, F_{1*}^* \cup F_{2*}^*) \in \nu$ .

(3) Use the same method above, we can prove the intersection of any closed sets that are members of  $\nu$  still belongs to  $\nu$ .

The rough inclusion and rough equality are defined as follows.

**Definition 3**

The rough inclusion of  $X = (X_*, X^*)$  and  $Y = (Y_*, Y^*)$  is defined as  $X \prec Y \Leftrightarrow X_* \subseteq Y_*$  and  $X^* \subseteq Y^*$ . The rough equality is defined as  $X \approx Y \Leftrightarrow X_* = Y_*$  and  $X^* = Y^*$ .

Based on Definition 3, the following property exists.

**Proposition 1**

- (1)  $X \approx Y \Leftrightarrow X \prec Y$  and  $Y \prec X$ .
- (2)  $X \prec Y$  and  $X \approx Z \Rightarrow Z \prec Y$ .

**Definition 4**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $A_1$  and  $A_2$  be the subsets corresponding to the topological space  $X_*$  and  $X^*$ , respectively. If  $A_1$  and  $A_2$  satisfy:

$A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ ,  $\bar{A} = \bar{A}_1, \bar{A}_2$  is called the closure of  $A = A_1, A_2$  in  $X$ , where  $\bar{A}_1, \bar{A}_2$  are the smallest closed set containing  $A_1, A_2$  in  $X_*$  and  $X^*$ , respectively.

**Theorem 4**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $A = A_1, A_2$  be a subset of  $X$  and satisfy  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ .  $A$  is the closed set of  $X \Leftrightarrow A \approx \bar{A}$ .

**Proof**

If  $A$  is a closed set, then  $A_1$  and  $A_2$  are the closed sets of  $X_*$  and  $X^*$ , respectively. Therefore,  $\bar{A}_1 = A_1$  and  $\bar{A}_2 = A_2$ . So we have  $A \approx \bar{A}$ .

On the contrary,  $A \approx \bar{A}$ , then  $\bar{A}_1 = A_1$  and  $\bar{A}_2 = A_2$  in  $X_*$  and  $X^*$ , respectively. And by  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$  thus,  $A = A_1, A_2$  is the closed set of  $X$ .

**Theorem 5**

Let  $K = (U, C)$  be a covering approximation space. Assume that  $X = (X_*, X^*) \subseteq U$ . Let  $A$  and  $B$  be the subsets of  $X$ , the following properties exist.

- (1)  $\overline{\emptyset} = \emptyset$ .
- (2)  $A \prec \bar{A} \prec X$ .
- (3)  $\overline{\bar{A}} \approx \bar{A}$ .
- (4)  $\overline{A \cup B} \approx \bar{A} \cup \bar{B}$ .

**Proof**

Properties (1) and (2) follow directly from the definition of closure, and property (3) from the fact that  $\bar{A}$  is a closed set.

Let  $A = (A_1, A_2) \subset X$  and  $B = (B_1, B_2) \subseteq X$ , where  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$  and  $B_1 \subseteq X_* \subseteq B_2 \subseteq X^*$ . By  $A_1 \subseteq A_1 \cup B_1$ ,  $A_2 \subseteq A_2 \cup B_2$ , and the definition of

closure, we have  $\bar{A}_1 \subseteq \overline{A_1 \cup B_1}$  and  $\bar{A}_2 \subseteq \overline{A_2 \cup B_2}$ . Similarly, we have  $\bar{B}_1 \subseteq \overline{A_1 \cup B_1}$  and  $\bar{B}_2 \subseteq \overline{A_2 \cup B_2}$ . Therefore, we get  $\overline{A_1 \cup B_1} \subseteq \overline{A_1 \cup B_1}$ ,  $\overline{A_2 \cup B_2} \subseteq \overline{A_2 \cup B_2}$ . Again,  $\overline{A \cup B} = \overline{A_1 \cup B_1, A_2 \cup B_2}$ ,  $\overline{A \cup B} = \overline{A_1 \cup B_1, A_2 \cup B_2}$ , so  $\overline{A \cup B} \prec \overline{A \cup B}$ . By (2),

$A \prec \bar{A}$  and  $B \prec \bar{B}$ , so that  $A_1 \cup B_1 \subseteq \overline{A_1 \cup B_1}$ . Since the last set is closed, being the union of two closed sets, from the definition of closure it follows that  $\overline{A_1 \cup B_1} \subseteq \overline{A_1 \cup B_1}$ .

Similarly, we have  $\overline{A_2 \cup B_2} \subseteq \overline{A_2 \cup B_2}$ . Therefore,  $\overline{A \cup B} \prec \overline{A \cup B}$ .

Based on the above discussion, there is  $\overline{A \cup B} \approx \overline{A \cup B}$ . At last, we gave the definition of the topological space on the covering-based rough sets.

**Definition 5**

Assume that  $X$  is a covering-based rough set. Let  $\tau$  be a family of subset of  $X$  and its members satisfy the results in Theorem 1, i.e., (1)  $X \in \tau$  and  $\emptyset \in \tau$ , where  $\emptyset = (\emptyset, \emptyset)$  is an empty set.

- (2) If  $O_1, O_2 \in \tau$ , then  $O_1 \cap O_2 \in \tau$ .
  - (3) If for any  $\gamma \in \Gamma$ , we have  $O_\gamma \in \tau$ . Then,  $\bigcup_{\gamma \in \Gamma} O_\gamma \in \tau$ .
- Then,  $\tau$  is called a topology on  $X$ . The members of  $\tau$  are called the open sets of  $X$ . The covering-based rough set  $X$  together with its topology  $\tau$  is called the topological covering-based rough space, labeled  $(X, \tau)$ . Generally, when the topology  $\tau$  is clearly given,  $(X, \tau)$  can be briefly labeled as  $X$ .

**ROUGH HOMEOMORPHISMS**

**Definition 6**

Let  $X = (X_*, X^*)$  and  $Y = (Y_*, Y^*)$  be the two topological covering-based rough spaces. If  $f_1 : X_* \rightarrow Y_*$  is a continuous mapping and  $f_2 : X^* \rightarrow Y^*$  is also a continuous mapping, then  $f = f_1, f_2$  is called a rough continuous mapping from  $X$  to  $Y$  and is labeled as  $f : X \rightarrow Y$ .

**Definition 7**

Let  $X = (X_*, X^*)$  and  $Y = (Y_*, Y^*)$  be two topological covering-based rough spaces. Assume that  $f_1 : X_* \rightarrow Y_*$  is a single and full mapping, and  $f_2 : X^* \rightarrow Y^*$  is also a single and full mapping. At the same time,  $f_1, f_2, f_1^{-1}$  and  $f_2^{-1}$  are continuous mappings. Then,  $f_1$  is called a homeomorphism from  $X_* \rightarrow Y_*$ , and  $f_2$  is called a homeomorphism from  $X^* \rightarrow Y^*$ .  $f_1$  and  $f_2$  are also called the lower rough homeomorphism and the upper rough homeomorphism from  $X$  to  $Y$ , respectively. Here,  $f = f_1, f_2$  is called a rough homeomorphism from  $X$  to  $Y$  and is labeled as  $f : X \rightarrow Y$ . If there is such homeomorphism or rough topological equivalence and labeled as  $f : X \cong Y$  or  $Y \cong X$ . If  $f_1$  is an identity mapping  $1_{X_*}$  from  $X_* \rightarrow Y_*$ , and  $f_2$  is also an identity mapping  $1_{X^*}$  from  $X^* \rightarrow Y^*$ , then  $f$  is called an identity mapping  $(1_{X_*}, 1_{X^*})$  from  $X$  to  $X$ , and we label it as  $1_X = (1_{X_*}, 1_{X^*})$ .

**Definition 8**

Let  $X = (X_*, X^*)$  and  $Y = (Y_*, Y^*)$  be two topological covering-based rough spaces. If the mapping  $f : X \rightarrow Y$  transforms every open set (closed set) of  $X$  to be the open set (closed set) of  $Y$ , then  $f$  is called the rough open mapping (closed mapping).

**Theorem 6**

$f$  is a rough homeomorphism  $\Leftrightarrow f$  is a single and full rough open continuous mapping (rough closed continuous mapping).

**Proof**

$\Rightarrow$  Assume  $f = f_1, f_2$  is a rough homeomorphism from  $X$  to  $Y$ . Then  $f = f_1, f_2$  is a single and full mapping and  $f_1$  is a homeomorphism from the

topological space  $X_*$  to the topological space  $Y_*$  and  $f_2$  is a homeomorphism from the topological space  $X^*$  to the topological space  $Y^*$ . By lemma 1,  $f_1$  is a continuous and open (or closed) mapping and  $f_2$  is a continuous and open (or closed) mapping. Thus,  $f = f_1, f_2$  is a single and full rough open (or closed) continuous mapping.

$\Leftarrow$  If  $f = f_1, f_2$  is a single and full rough open (or closed) continuous mapping. Then  $f_1$  is a single and full open (or closed) continuous mapping from the topological space  $X_*$  to the topological space  $Y_*$  and  $f_2$  is a single and full open (or closed) continuous mapping from the topological space  $X^*$  to the topological space  $Y^*$ . By lemma 1,  $f_1$  is a homeomorphism from the topological space  $X_*$  to the topological space  $Y_*$  and  $f_2$  is a homeomorphism from the topological space  $X^*$  to the topological space  $Y^*$ . Then  $f = f_1, f_2$  is a rough homeomorphism from  $X$  to  $Y$  by Definition 7.

**Theorem 7**

Let  $X = (X_*, X^*)$  and  $Y = (Y_*, Y^*)$  be two topological covering-based rough spaces. Assume that  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are two rough continuous mappings, and there are  $gf = 1_X$  and  $fg = 1_Y$ , respectively. Then,  $f$  must be the rough homeomorphism.

**Proof**

Assume that  $f = f_1, f_2$  and  $g = g_1, g_2$ . Because  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are rough continuous mappings, then  $f_1 : X_* \rightarrow Y_*, f_2 : X^* \rightarrow Y^*, g_1 : Y_* \rightarrow X_*$ , and  $g_2 : Y^* \rightarrow X^*$  are also the continuous mappings. Since  $gf = 1_X$  and  $fg = 1_Y$ , then  $g_1 f_1 = 1_{X_*}, g_2 f_2 = 1_{X^*}, f_1 g_1 = 1_{Y_*}, f_2 g_2 = 1_{Y^*}$ .

Because  $f_1$  is a mapping,  $\forall x_1, x_2 \in X_*$  and  $x_1 \neq x_2$ ,  $\exists y_1, y_2 \in Y_*$  such that  $y_1 = f_1(x_1)$  and  $y_2 = f_1(x_2)$ .

For  $y_1, y_2 \in Y_*$ , since  $g_1$  is also a mapping and  $g_1 f_1 = 1_{X_*}$ , then  $\exists g_1(y_1), g_1(y_2) \in X_*$  such that  $g_1(f_1(x_1)) = x_1 \in X_*$ ,  $g_1(f_1(x_2)) = x_2 \in X_*$ , i.e.,  $x_1 = g_1(y_1)$  and  $x_2 = g_1(y_2)$ .

Since  $g_1: Y_* \rightarrow X_*$  is a mapping again, if  $y_1, y_2 \in Y_*$ , then  $g_1(y_1) = g_1(y_2)$ . Thus,  $x_1 = x_2$ , which is in contradiction with  $x_1 \neq x_2$ . Therefore, there must be  $y_1 \neq y_2$  when  $x_1 \neq x_2$ . Therefore,  $f_1$  is a single mapping. Similarly,  $f_2$  is also a single mapping.

Since  $g_1$  is a mapping, for  $\forall y \in Y_*$ ,  $\exists x \in X_*$ , such that  $x = g_1(y)$ . For  $x \in X_*$ , since  $f_1$  is also a mapping and  $f_1 g_1 = 1_{Y_*}$ , then  $\exists f_1(x) \in Y_*$  such that  $f_1(g_1(y)) = y \in Y_*$ , i.e.,  $y = f_1(x)$ . It shows that for any  $\forall y \in Y_*$ , there is an element  $x \in X_*$  so that it corresponds to  $y$ , and let  $y = f_1(x)$ . Therefore,  $f_1$  is a full mapping from  $X_* \rightarrow Y_*$ . Similarly,  $f_2$  is also a full mapping. Thus, we get  $f = f_1, f_2$  is a single and full mapping. Now, we prove that the mapping  $f$  transforms every open set of  $X$  to be the open set of  $Y$ .

Let  $A = (A_1, A_2) \subset X$ , under the mapping  $f$ , there is a corresponding  $B = (B_1, B_2)$  such that  $B = f(A)$ , i.e.,  $B_1 = f_1(A_1)$  and  $B_2 = f_2(A_2)$ . If  $A$  is an open set, then  $A_1$  and  $A_2$  are the open sets and satisfy  $A_1 \subseteq X_* \subseteq A_2 \subseteq X^*$ . For  $\forall x \in A_1$ ,  $\exists y \in B_1$  such that  $y = f_1(x)$ . Again, for  $A_1 \subseteq X_*$ ,  $x \in X_*$ ; also  $y \in Y_*$  and, therefore,  $B_1 \subseteq Y_*$ . Similarly,  $Y_* \subseteq B_2$  and  $B_2 \subseteq Y^*$  are also obtained. Then, there is  $B_1 \subseteq Y_* \subseteq B_2 \subseteq Y^*$  and  $B = (B_1, B_2)$  is a subset of  $Y$ . On the other hand, since  $A_1$  is an open set, for  $\forall x \in A_1$ , there is a neighborhood

$M_x$  of  $x$  such that  $M_x \subseteq A_1$  and  $\exists y \in B_1$  such that  $y = f_1(x)$ . Let  $V_y = f_1(M_x)$ , then,  $y \in V_y \subseteq B_1$ .

Therefore,  $y$  is an inner point of  $B_1$ . Since  $x$  is random, and  $f_1$  is a single and full mapping, then  $y$  is any point of  $B_1$ ; therefore,  $B_1$  is an open set of  $Y_*$ . Similarly,  $B_2$  is an open set of  $Y^*$ . By  $B_1 \subseteq Y_* \subseteq B_2 \subseteq Y^*$ , again,  $B = (B_1, B_2)$  is an open set of  $Y$ . Then  $f_1$  and  $f_2$  are open mappings.

Thus the mapping  $f$  transforms every open set of  $X$  to be the open set of  $Y$ , i.e.,  $f$  is a rough open mapping.

Therefore,  $f$  is a single and full rough open continuous mapping. By Theorem 6, we get  $f = f_1, f_2$  is a rough homeomorphism from  $X$  to  $Y$ .

## CONCLUSION AND FUTURE WORK

In this work we studied the topological structures on covering-based rough sets and rough homeomorphism. It forms the theoretical basis for further applications of topology on covering-based rough sets. However, under topological transformation, some properties of covering-based rough sets and topological theory on it such as separation axioms and covering properties are not studied, this is the limitation of this paper and also one of our future work.

Application of rough set theory to real problems is an important and challenging issue for rough set research. We know that reduction of attributes can be applied in the process of compactification of space time dimensions (Naschie, 1992), we will give an algorithm for knowledge reduction using topological structures in our next work.

Furthermore, it seems a need and possibility to apply the covering-based rough set theory to the computational theory for linguistic dynamic systems, software watermarking and software obfuscation (Zhu, 2009).

Also if rough set theory and topology combine with other soft computing methods such as fuzzy sets (Bayram, (2010)), it will solve many problems of high machine IQ, hybrid intelligent system and incomplete information system. Our work in this paper is a start in these directions and they need further study.

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