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Full Length Research Paper

Automorphisms of real four dimensional Lie algebras

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We are calculating the automorphisms of real Lie algebras in four dimensional matrix method. At first, we introduced the adjoint matrices and we reduced Jacobi and mix Jacobs identities to matrix form, then we presented the automorphisms of real lie algebras. At the end we compared our results to another article.

Key words: Lie algebras, Lie brackets, automorphism, adjoint representations.

INTRODUCTION

In this paper, we introduce automorphism of real four dimensional Lie algebras. Before exhibiting the results on the automorphisms, we briefly recall some basic elements of the theory of Lie algebras. Lie (1893) enumerated all complex four-dimensional algebras in his classic work. Kruchkovich (1954) adapted these results to the real case, and Petrov (1969) and Ellis and Sciama (1966) completed this classification independently. Also, Patera et al. (1976) enumerated all solvable Lie algebras and modified the forms of Mubarakzyanov (1963) in several papers (Patera et al., 1976; Patera and Winternitz, 1977; Patera and Zassenhaus, 1990). At last, the general commutation relations have been achieved by Rastgo and Mehdipour (2006). MacCallum (1999) compared all works and presented a new classification which used the algebraic method like classification of Bianchi algebras in three dimensions.

Automorphisms

Automorphisms of 3-dimensional real Lie algebras (Harvey, 1979) have been proven to be a powerful tool for analyzing the dynamics of 3+1 Bianchi cosmological models (Heckman and Sch¨ucking 1962). At the classical level, time-dependent automorphisms inducing diffeomorphisms can be used to simplify the line element, and thus the Einstein's field equations without loss of generality (Jantzen, 1979). They also provide an algorithm for counting the number of essential constants; the results obtained agree for all Bianchi types with the preexisting results (Christodoulakis et al., 2001), but unlike these, the algorithm can be extended to four or more dimensions. At the quantum level, outer automorphisms provide integrals of motion of the classical Hamiltonian dynamics; their quantum analogues can be used to reconcile quantum Hamiltonian dynamics with the kinematics of homogeneous 3-spaces (Christodoulakis and Papadopoulos, 2002).

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METHODOLOGY

This method differs from the other works. At first adjoint representations of algebras were introduced and then the relations of automorphisms were reduced to the form of adjoint representations. At last, automorphisms was calculated by computer program (maple software) from matrices relations.

At first, we call some notations and relations that would be used. The following notation was used. *L* denotes a Lie algebra, and L_n a *Lie* algebra of dimension *n*. If $\{X_i;\ i = 1, 2,\dots, n\}$ is a basis of L_n;

$$[X_i, X_j] = f_{ij}^{k}\ X_k \qquad (1)$$

where [;] is the Lie product (commutator) and summation over repeated indices is implied, defines the structure constants f_{ij}^{k} .

If $\{X_i';\ i = 1; 2; \dots ; n\}$ is a basis automorphisms algebras; then

$$[X'_\ell, X'_m] = f_{\ell m}^{n}\ X'_n \qquad (2)$$

Preserve the Lie brackets for automorphisms of algebras:

$$X'_\ell = o^i_\ell \ X_i \, . \qquad (3)$$

Now adjoint representations are presented in this manner:

$$\begin{aligned} (\chi_i)_j^{\ k} &= -f_{ij}^{\ \ k} \\ (Y^k)_{ij} &= -f_{ij}^{\ \ k} \end{aligned} \qquad (4)$$

where $(\chi_i)_j^{\ k}$ are the adjoint representations of algebra's basis and $(Y^k)_{ij}$ are the antisymmetric matrices. Then, the X_i or Y_i matrices were found for all Lie algebras.

By applying Equation 3 in 2, the general matrix form of the elements of the automorphism groups of the algebras can be calculated:

$$[X'_\ell, X'_m] = f^n_{\ell n} \ X'_m \Rightarrow [O^i_\ell \ X_i, O^j_m \ X_j] = f^n_{\ell m} \ O^k_n \ X_k$$

$$\Rightarrow O^i_\ell \ O^j_m \ f^k_{ij} \ X_k = f^n_{\ell m} \ O^k_n \ X_k \Rightarrow$$

$$\Rightarrow \begin{cases} O^i_\ell \ O^j_m (\chi_i)^k_j = (\chi_\ell)^n_m \ O^k_n \\ \\ O^i_\ell \ O^j_m (Y^k)_{ij} = (Y^n)_{\ell m} \ O^k_n \end{cases}$$

$$\Rightarrow \begin{cases} O^i_\ell \ O \chi_i = \chi_\ell \ O \\ \\ (O^t Y^k O)_{\ell m} = (Y^n O^k_n)_{\ell m} \Rightarrow O^t Y^k O = Y^n O^k_n \end{cases} \qquad (5)$$

Any of them can be used. The first one is in terms of adjoint representations and the other one is in terms of antisymmetric matrices. At last, the six equations have been released from any matrices relation that have been solved earlier. The results are matrices of automorphism Lie algebras.

For example, we mentioned one algebras calculation such as (R $\oplus$ II). If a_{ij} is considered as the elements of automorphism matrix Lie algebras, the six equations can be achieved by solving the Equation 5:

$$\begin{cases} a_{12} = a_{13} = a_{14} = 0 \\ a_{13} \ a_{22} - a_{12} \ a_{23} = 0 \\ a_{13} \ a_{32} - a_{12} \ a_{33} = 0 \end{cases}$$

$$\begin{cases} a_{13} \ a_{42} - a_{12} \ a_{43} = 0 \\ a_{23} \ a_{32} - a_{22} \ a_{33} = -a_{11} \\ a_{23} \ a_{42} - a_{22} \ a_{43} = 0 \\ a_{33} \ a_{42} - a_{32} \ a_{43} = 0 \end{cases}$$

Solving these equations we have these relations in automorphism:

$$\begin{cases} a_{43} = a_{14} = a_{13} = a_{42} = a_{12} = 0 \\ a_{11} = a_{22} \ a_{33} - a_{23} \ a_{32} \end{cases}$$

These mean that the matrix of automorphism is presented in this form:

$$\begin{pmatrix} a_{22} \ a_{33} - a_{23} \ a_{32} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

All matrices have been solved and represented in Table 1.

RESULTS AND DISCUSSION

It is not easy to compare two automorphism matrices, but in apparent, the two matrices have the same structure. For example automorphism matrix $A_{4\ 1}$ algebras in this paper is:

$$\begin{bmatrix} a_{33}a^2{}_{44} & 0 & 0 & 0 \\ a_{32}a_{44} & a_{33}a_{44} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Which is same with automorphism matrix introduced by Christodoulakis and Papadopoulos (2002):

$$\begin{bmatrix} a_{11}a^2{}_{16} & a_7 a_{16} & a_3 & a_4 \\ 0 & a_{11}a_{16} & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$$

These two matrices are triangular. One of them is upper triangular and other is lower triangular. The elements of main diagonal are the same (considering $a_{11} \to a_{33}, a_{16} \to a_{44}$); but the decomposable ones would not have the same results, because the paper is based on Patera algebras but we used Bianchi form in three dimensional. The Bianchi form in three dimensional Lie algebras is more useful than the others. Therefore, this method is right for finding the automorphism algebras. We present all algebras in the two methods in matrix form in Table 1.

Table 1. Calculated automorphisms together with the automorphism for comparing.

Number		Automorphisms (matrix)	Automorphisms for comparing
1	$\mathrm{III} \oplus R:$	$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{33} & a_{32} & -a_{34} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & -a_{42} & a_{44} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ a_9 & 0 & a_{11} & a_{12} \\ a_{13} & 0 & a_{15} & a_{15} \end{bmatrix}$
2	$A_2 \oplus A_2$	$\begin{bmatrix} 0 & 0 & 1 & a_{14} \\ 0 & 0 & 0 & a_{24} \\ 1 & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_7 & a_8 \\ 1 & 0 & 0 & 0 \\ a_{13} & a_{14} & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{15} \end{bmatrix}$
3	$\mathrm{II} \oplus R:$	$\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ a_9 & 0 & a_{11} & a_{12} \\ a_{13} & 0 & a_{15} & a_{15} \end{bmatrix}$
4	$\mathrm{IV} \oplus R:$	$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{15} \end{bmatrix}$
5	$\mathrm{V} \oplus R:$	$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{32} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_5 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$
6	$\mathrm{VI}_0 \oplus R:$	$\begin{bmatrix} 0 & a_{12} & 0 & 0 \\ -a_{12} & 0 & a_{70} & 0 \\ a_{31} & a_{32} & -1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$ $\begin{bmatrix} a_{22} & a_{21} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$ $\begin{bmatrix} 0 & a_2 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$

Table 1. Contd.

7	$VI_a \oplus R:$	$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{33} & a_{32} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$
8	$VII_0 \oplus R:$	$\begin{bmatrix} a_{22} & -a_{21} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & -a_1 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$
		$\begin{bmatrix} a_{22} & -a_{21} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$
9	$VII_a \oplus R:$	$\begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{33} & -a_{32} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{bmatrix}$
10	$A_{4\ 1}$	$\begin{bmatrix} a_{33}a^2_{44} & 0 & 0 & 0 \\ a_{32}a_{44} & a_{33}a_{44} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$	$\begin{bmatrix} a_{11}a^2_{16} & a_7a_{16} & a_3 & a_4 \\ 0 & a_{11}a_{16} & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{bmatrix}$
11	$A^a_{4\ 2}$	$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{33} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
12	$A^1_{4\ 2}$	$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{33} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ a_5 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table 1. Contd.

13	$A_{4,3}$	$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
14	$A_{4,4}$	$\begin{bmatrix} a_{33} & 0 & 0 & 0 \\ a_{21} & a_{33} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_8 \\ 0 & 0 & a_1 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
15	$A_{4,5}^{a\ b}$	$\begin{bmatrix} a_{33} & 0 & 0 & 0 \\ 0 & a_{33} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
16	$A_{4,5}^{a\ a}$	$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
17	$A_{4,5}^{a\ 1}$	$\begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & a_3 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
8	$A_{4,5}^{1\ 1}$	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_{78} \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
19	$A_{4,7}$	$\begin{bmatrix} {a_{33}}^2 & 0 & 0 & 0 \\ -a_{33}a_{43} & a_{33} & 0 & 0 \\ -a_{33}a_{43}-a_{32}a_{43}+a_{33}a_{42} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a^2{}_6 & -a_{12}a_6 & -a_{12}(a_6+a_7)+a_6a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table 1. Contd.

20	$A_{4,6}^{a,b}$	$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{33} & -a_{32} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_{1} & 0 & 0 & a_{4} \\ 0 & a_{6} & a_{7} & a_{8} \\ 0 & -a_{7} & a_{6} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
21	$A_{4,8}$	$\begin{bmatrix} a_{22}a_{33} & 0 & 0 & 0 \\ a_{22}a_{43} & a_{22} & 0 & 0 \\ a_{33}a_{42} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$ $\begin{bmatrix} -a_{23}a_{32} & 0 & 0 & 0 \\ -a_{23}a_{42} & 0 & a_{23} & 0 \\ -a_{32}a_{43} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & -1 \end{bmatrix}$	$\begin{bmatrix} a_{11}a_{6} & a_{12}a_{6} & a_{11}a_{8} & a_{4} \\ 0 & a_{6} & 0 & a_{8} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -a_{10}a_{7} & -a_{10}a_{8} & -a_{12}a_{7} & a_{4} \\ 0 & 0 & a_{7} & a_{8} \\ 0 & a_{10} & 0 & a_{12} \\ 0 & 0 & 0 & -1 \end{bmatrix}$
22	$A_{4,9}^{b}$	$\begin{bmatrix} a_{22}a_{33} & 0 & 0 & 0 \\ -a_{22}a_{43}/b & a_{22} & 0 & 0 \\ a_{33}a_{42} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} a_{11}a_{6} & -a_{12}a_{6}/\beta & a_{8}a_{11} & a_{4} \\ 0 & a_{6} & 0 & a_{8} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
23	$A_{4,9}^{1}$	$\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & 0 & 0 & 0 \\ a_{23}a_{42}-a_{22}a_{43} & a_{22} & a_{23} & 0 \\ a_{42}a_{33}-a_{43}a_{32} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{5} & a_{6} & 0 & 0 \\ a_{9} & 0 & a_{11} & a_{12} \\ a_{13} & 0 & a_{15} & a_{15} \end{bmatrix}$
24	$A_{4,9}^{0}$	$\begin{bmatrix} a_{22}a_{33} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{33}a_{42} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a_{11}a_{6} & a_{2} & a_{8}a_{11} & a_{4} \\ 0 & a_{6} & 0 & a_{8} \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
25	$A_{4,10}$	$\begin{bmatrix} a_{23}^{2}+.a_{22}^{2} & 0 & 0 & 0 \\ -a_{22}a_{42}-a_{23}a_{43} & a_{22} & a_{23} & 0 \\ -a_{22}a_{43}+a_{23}a_{42} & -a_{23} & a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & -1 \end{bmatrix}$ $\begin{bmatrix} -a_{23}^{2}-.a_{22}^{2} & 0 & 0 & 0 \\ a_{22}a_{42}+.a_{23}a_{43} & a_{22} & a_{23} & 0 \\ -a_{22}a_{43}+a_{23}a_{42} & a_{23} & -a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & -1 \end{bmatrix}$	$\begin{bmatrix} -a_{6}^{2}-a_{7}^{2} & a_{12}a_{7}+a_{6}a_{8} & -a_{12}a_{6}+a_{7}a_{8} & a_{4} \\ 0 & a_{6} & a_{7} & a_{8} \\ 0 & a_{7} & -a_{6} & a_{12} \\ 0 & 0 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} a_{6}^{2}+a_{7}^{2} & a_{12}a_{7}-a_{6}a_{8} & -a_{12}a_{6}-a_{7}a_{8} & a_{4} \\ 0 & a_{6} & a_{7} & a_{8} \\ 0 & -a_{7} & a_{6} & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table 1. Contd.

26	$A_{4,11}^{a}$	$\begin{bmatrix} a_{32}^{2}+a_{33}^{2} & 0 & 0 & 0 \\ -W/(1+\alpha^2) & a_{33} & -a_{32} & 0 \\ -W'/(1+\alpha^2) & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$ $W = a_{32}(\alpha a_{42} - a_{43}) + a_{33}(\alpha a_{43} + .a_{42})$ $W' = a_{33}(a_{43} - \alpha a_{42}) + a_{32}(\alpha a_{43} + a_{42})$	$\begin{bmatrix} a_6^2+a_7^2 & -W/(1+\alpha^2) & -W'/(1+\alpha^2) & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $W = a_6(\alpha a_{12} + a_8) + a_7(\alpha a_8 - a_{12})$ $W' = a_6(a_{12} - \alpha a_8) + a_7(\alpha a_{12} + a_8)$
27	$A_{4,12}$	$\begin{bmatrix} a_{22} & -a_{21} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -a_{42} & a_{32} & 1 & 0 \\ a_{32} & a_{42} & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -a_{22} & a_{21} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{42} & a_{32} & 1 & 0 \\ a_{32} & a_{42} & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & -a_4 & a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

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