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Review

Characterizations of fuzzy intra-regular Abel-Grassmann's groupoids

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Mostly, models of real life problems are consist of uncertainties. Fuzzy set theory is one of the tool for handling such uncertainties and for applications of fuzzy set theory we have used non-associative algebraic structure namely AG-groupoids. Specifically, we have discussed the $(\in,\in\vee q_k)$-fuzzy left ideals, $(\in,\in\vee q_k)$-fuzzy ideals, $(\in,\in\vee q_k)$-fuzzy semiprime ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals in an AG-groupoid. We have characterized intra-regular AG-groupoids by the properties of their ideals and $(\in,\in\vee q_k)$-fuzzy ideals.

Key words: AG-groupoid, left invertive law, medial law, paramedial law, $(\in,\in\vee q_k)$-fuzzy ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals.

INTRODUCTION

The real world has a lot of different aspects which are not usually been specified. In different fields of knowledge like engineering, medical science, mathematics, physics, computer science and artificial intelligence, many problems are simplified by constructing their "models". These models are very complicated and it is impossible to find the exact solutions in many occasions. Therefore, the classical set theory, which is precise and exact, may not be suitable for such problems of uncertainty.

In today's world, many theories have been developed to deal with such uncertainties like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets and theory of rough sets. The theory of soft sets has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc. The basic concept of fuzzy set theory was first given by Zadeh (1965). Zadeh discussed the relationships between fuzzy set theory and probability theory. Rosenfeld (1971) initiated the fuzzy groups in fuzzy set theory. (Moderson et al. (2003) have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines.

The idea of belongingness of a fuzzy point to a fuzzy subset under the natural equivalence on a fuzzy subset has been defined by (Murali, 2004). Bhakat and Das (1992) gave the concept of (α,β)-fuzzy subgroups where $\alpha,\beta\in\{\in,q,\in\vee q,\in\wedge q\}$ and $\alpha\neq\in\wedge q$. The idea of $(\in,\in\vee q)$-fuzzy subgroups is a generalization of fuzzy subgroupoid defined by Rosenfeld. An $(\in,\in\vee q_k)$-fuzzy bi-ideals and $(\in,\in\vee q_k)$- fuzzy quasi-ideals, and $(\in,\in\vee q_k)$-fuzzy ideals of a semigroup are defined in Shabir et al. (2010a).

In this paper, we discussed the $(\in,\in\vee q_k)$-fuzzy ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals in a new non-associative

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algebraic structure, that is, in AG-groupoids and developed some new results. We characterized intra-regular AG-groupoids by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals. A groupoid S is called an AG-groupoid if it satisfies the left invertive law, that is,

$$(ab)c = (cb)a, \text{for all } a, b, c \in S.$$

Every AG-groupoid satisfies the medial law

$$(ab)(cd) = (ac)(bd), \text{for all } a, b, c, d \in S.$$

It is basically a non-associative algebraic structure in between a groupoid and a commutative semigroup. It is important to mention here that, if an AG-groupoid contains identity or even right identity, then it becomes a commutative monoid. It not necessarily that AG-groupoid should have a left identity, but if it has a left identity, then it is unique (Mushtaq and Yusuf, 1978). An AG-groupoid S with left identity satisfies the paramedial law, that is,

$$(ab)(cd) = (db)(ca), \text{for all } a, b, c, d \in S.$$

Also, S satisfies the following law

$$a(bc) = b(ac), \text{for all } a, b, c, d \in S.$$

Let S be an AG-groupoid. By an AG-subgroupoid of S, we means a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$ and it is called a two-sided ideal if it is both left and a right ideal of S. A non-empty subset A of an AG-groupoid S is called a generalized bi-ideal of S if $(AS)A \subseteq A$ and an AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A \subseteq A$. A subset A of an AG-groupoid S is called semiprime if for all $a \in S$, $a^2 \in A$ implies $a \in A$.

If S is an AG-groupoid with left identity e then $Sa = \{sa : s \in S\}$ is both left and bi-ideal of S containing a. Moreover $S = eS \subseteq S^2$. Therefore $S = S^2$.

The following definitions are available in Moderson et al. (2003).

(1) A fuzzy subset f of an LA-semigroup S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all x, $y \in S$.
(2) A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all x, $y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy two-sided ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.
(3) A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$.
(4) A fuzzy subset f of an AG-groupoid S is called fuzzy semiprime if $f(a) \geq f(a^2)$, for all $a \in S$.

For a subset A of S the characteristics function, C_A is defined by

$$C_A = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}.$$

It is important to note that an AG-groupoid can also be considered as a fuzzy subset of itself and we can write $S = C_S$, that is, $S(x) = 1$, for all x in S.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} f(b) \wedge g(c), \text{if there exist } b, c \in S, \text{such that } a = bc. \\ 0, \text{ otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of S

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } S$$

and

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } S.$$

The following definitions for AG-groupoids are same as for semigroups in Shabir et al. (2010b).

Definition 1

A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies, $x_t \in f$, $y_r \in f$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k f$.

Definition 2

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies, $x_t \in f$ implies $(yx)_t \in \vee q_k f$ ($x_t \in f$ implies $(xy)_t \in \vee q_k f$).

Definition 3

A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$, for all $x, y, z \in S$ and $t, r \in (0,1]$.

Definition 4

A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following conditions hold

(i) If $x_t \in f$ and $y_r \in S$ implies $(xy)_{\min\{t,r\}} \in \vee q_k f$,
(ii) If $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$.

Definition 5

A fuzzy subset f of an AG-groupoid S is said to be $(\in, \in \vee q)$-fuzzy semiprime if it satisfies

$$x_t^2 \in f \Rightarrow x_t \in \vee qf$$

for all $x \in S$ and $t \in (0,1]$.

Definition 6

Let A be any subset of an AG-groupoid S, then the characteristic function $(C_A)_k$ is defined as,

$$(C_A)_k(a) = \begin{cases} \geq \frac{1-k}{2} & \text{if } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

The proofs of the following four theorems are same as in (Shabir et al., 2010a, b).

Theorem 1

Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$.

Theorem 2

A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if $f(xy) \geq \min\{f(y), \frac{1-k}{2}\}(f(xy) \geq \min\{f(x), \frac{1-k}{2}\})$.

Theorem 3

Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if

(i) $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ for all $x, y \in S$,
(ii) $f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$ for all $x, y, z \in S$.

Theorem 4

A fuzzy subset f of an AG-groupoid S is $(\in, \in \vee q_k)$-fuzzy semiprime if and only if $f(x) \geq f(x^2) \wedge \frac{1-k}{2}$ for all $x \in S$.

Proof

Let f be a fuzzy subset of an AG-groupoid S which is $(\in, \in \vee q_k)$-fuzzy semiprime. If there exists some $x_0 \in S$ such that $f(x_0) < t_0 = f(x_0^2) \wedge \frac{1-k}{2}$. Then $(x_0^2)_{t_0} \in f$, but $(x_0)_{t_0} \overline{\in} f$. In addition, we have $(x_0)_{t_0} \in \vee q_k f$ since f is $(\in, \in \vee q_k)$-fuzzy semiprime. On the other hand, we have $f(x_0) + t_0 \leq t_0 + t_0 \leq 1$. Thus $(x_0)_{t_0} \overline{q_k} f$, and so $(x_0)_{t_0} \overline{\in \vee q_k} f$. This is a contradiction. Hence $f(x) \geq f(x^2) \wedge \frac{1-k}{2}$ for all $x \in S$. Conversely, assuming that f is a fuzzy subset of an AG-groupoid S such that $f(x) \geq f(x^2) \wedge \frac{1-k}{2}$ for all $x \in S$. Let $x_t^2 \in f$. Then $f(x^2) \geq t$, and so $f(x) \geq f(x^2) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2}$. Now, we consider the following two cases:

(i) If $t \leq \frac{1-k}{2}$, then $f(x) \geq t$. That is, $x_t \in f$. Thus we have $x_t \in \vee q_k f$.

(ii) If $t > \frac{1-k}{2}$, then $f(x) \geq \frac{1-k}{2}$. It follows that $f(x)+t \geq \frac{1-k}{2}+t > 1$. That is, $x_t q_k f$, and so $x_t \in \vee q_k f$ also holds. Therefore, we conclude that f is $(\in, \in \vee q)$-fuzzy semiprime as required.

Example 1

Let $S=\{1,2,3\}$, then from the following multiplication table, one can easily verify that S is an AG- groupoid.

.	1	2	3
1	1	1	1
2	3	3	1
3	1	1	1

Let us define fuzzy subset f of S as: $f(1)=0.9, f(2)=0.6, \quad f(3)=0.8.$ Then f is clearly an $(\in, \in \vee q_k)$-fuzzy ideal.

Definition 7

An AG groupoid S is called intra-regular AG-groupoid if for each a in S there exists x, y in S such that $a=(xa^2)y$.

Example 2

Let $S=\{1,2,3,4,5,6\}$, the following table shows that S is an intra-regular AG-groupoid.

*	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	4	5	6	3
4	1	2	3	4	5	6
5	1	2	6	3	4	5
6	1	2	5	6	3	4

It is easy to see that $(S,*)$ is an AG-groupoid and is non-commutative and non-associative structure because $(3*4) \neq (4*3)$ and $(3*6)*4 \neq 3*(6*4)$. Also, $1=(3*1^2)*1,\ 2=(2*2^2)*2,\ 3=(4*3^2)*6,\ 4=(4*4^2)*4,\ 5=(6*5^2)*3,\ 6=(5*6^2)*5.$

Therefore, $(S,*)$ is an intra-regular AG-groupoid. Clearly $\{1\}$ and $\{1,2\}$ are ideals of S. A fuzzy subset $f : S \rightarrow [0,1]$ is defined as

$$f(x)=\begin{cases} 0.9 \ \ for \ x=1 \\ 0.8 \ \ for \ x=2 \\ 0.7 \ \ for \ x=3 \\ 0.6 \ \ for \ x=4 \\ 0.5 \ \ for \ x=5 \\ 0.5 \ \ for \ x=6 \end{cases}$$

Then clearly f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Also, f is $(\in, \in \vee q_k)$-fuzzy semiprime.

Lemma 1

Let A be a non-empty subset of an AG-groupoid S, then

(i) A is a left (right, two-sided) ideal of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

(ii) A of an AG-groupoid S with left identity is bi-ideal if and only if $(C_A)_k$ is $(\in, \in \vee q_k)$ -fuzzy bi-ideal.

Proof

This is the same as in (Shabir et al., 2010b).

Lemma 2

Let A and B be non-empty subsets of an AG-groupoid S, then the following properties hold.

(i) $(C_{A \cap B})_k = (C_A \wedge_k C_B)$.

(ii) $(C_{A \cup B})_k = (C_A \vee_k C_B)$.

(iii) $(C_{AB})_k = (C_A \circ_k C_B)$.

Proof

This is the same as in Shabir et al. (2010a). Let f and g be any two fuzzy subsets of an AG-groupoid S, then for $k \in [0,1)$, the product $f \circ_k g$ and $f \wedge_k g$ are defined by,

$$(f\circ_k g)(a)=\begin{cases}\bigvee_{a=bc}\left\{f(b)\wedge g(c)\wedge\frac{1-k}{2}\right\}, & \text{if there exist } b,c\in S, \text{ such that } a=bc.\\ 0, & \text{otherwise.}\end{cases}$$

$$f\wedge_k g(a)=(f\wedge g)(a)\wedge\frac{1-k}{2}.$$

Theorem 5

Let S be an AG-groupoid with left identity then the following conditions are equivalent

(i) S is intra-regular.

(ii) For every left ideal L and for every ideal I, $L\cap I=IL.$

(iii) For every $(\in,\in\vee q_k)$-fuzzy left ideal f and $(\in,\in\vee q_k)$-fuzzy ideal g, $f\wedge_k g=g\circ_k f$.

Proof

$(i)\Rightarrow(iii)$ assume that S is an intra-regular AG-groupoid and f and g are $(\in,\in\vee q_k)$-fuzzy left and $(\in,\in\vee q_k)$-fuzzy ideal of S. Since S is intra-regular therefore for any a in S there exist x,y in S such that $a=(xa^2)y.$ By using Moderson et al. (2003) definition (4) and (1)

$$a=(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a.$$

So for any a in S there exist u and v in S such that $a=uv,$ then

$$\begin{aligned}(g\circ_k f)(a)&=\bigvee_{a=uv} g(u)\wedge f(v)\wedge\frac{1-k}{2}\\ &\geq g(y(xa))\wedge f(a)\wedge\frac{1-k}{2}\\ &\geq g(xa)\wedge f(a)\wedge\frac{1-k}{2}\\ &\geq g(a)\wedge f(a)\wedge\frac{1-k}{2}\\ &=(g\wedge f)(a)\wedge\frac{1-k}{2}\\ &=(f\wedge g)(a)\wedge\frac{1-k}{2}\\ &=(f\wedge_k g)(a).\end{aligned}$$

This implies that

$$f\wedge_k g\leq g\circ_k f.$$

Now

$$\begin{aligned}(g\circ_k f)(a)&=\bigvee_{a=bc} g(b)\wedge f(c)\wedge\frac{1-k}{2}\\ &\leq\bigvee_{a=bc} g(bc)\wedge f(bc)\wedge\frac{1-k}{2}\\ &=g(a)\wedge f(a)\wedge\frac{1-k}{2}\\ &=(g\wedge f)(a)\wedge\frac{1-k}{2}\\ &=(f\wedge_k g)(a).\end{aligned}$$

This implies that

$$g\circ_k f\leq f\wedge_k g.$$

Therefore

$$g\circ_k f=f\wedge_k g.$$

$(iii)\Rightarrow(ii)$ Let L be the left and I be an ideal of S. Then by Lemma 1, $(C_L)_k$ and $(C_I)_k$ are the $(\in,\in\vee q_k)$-fuzzy left and $(\in,\in\vee q_k)$ fuzzy ideal of S, respectively. Therefore, by using Lemma 2 and $(iii),$ we get

$$(C_{L\cap I})_k=(C_L\wedge_k C_I)\leq C_I\circ_k C_L=(C_{IL})_k.$$

This implies

$$(C_{L\cap I})_k\leq(C_{IL})_k.$$

Then by Lemma 1, we have $L\cap I\subseteq IL.$

$ii)\Rightarrow(i)$ By using $(2),(3)$ and $(4),$ we get, $(Sa^2)S\subseteq Sa^2.$ Also,

$$S(Sa^2)=(SS)(Sa^2)=(a^2S)(SS)\subseteq(a^2S)S=(SS)a^2=Sa^2.$$

This implies $S(Sa^2)\subseteq Sa^2.$ Since Sa^2 is both left and right ideal, therefore it is an ideal containing $a^2.$ Since Sa^2 is semiprime, therefore $a\in Sa^2$. Now by using (ii)

$$a\in Sa\cap Sa^2=(Sa^2)(Sa)\subseteq(Sa^2)S.$$

Hence S is intra-regular.

Lemma 3

Every $(\in,\in\vee q_k)$-fuzzy left ideal of an AG-groupoid S is $(\in,\in\vee q_k)$-fuzzy bi-ideal.

Proof

Let S be an AG-groupoid and f be an $(\in,\in\vee q_k)$-fuzzy left ideal of S. Then for any x in S, there exist a and b in S such that

$$\begin{aligned} f((ax)b) &\geq f(b)\wedge\frac{1-k}{2} \\ &\geq f(a)\wedge f(b)\wedge\frac{1-k}{2}. \end{aligned}$$

Hence, f is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

Theorem 6

Let S be an AG-groupoid with left identity, then will the following conditions equivalent.

(i) S is intra-regular.
(ii) For every bi-ideal B and left ideal L, $B\cap L\subseteq BL.$
(iii) For every $(\in,\in\vee q_k)$-fuzzy bi-ideal f and $(\in,\in\vee q_k)$-fuzzy left ideal g, $f\wedge_k g\leq f\circ_k g.$
(iv) For every $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal f and every $(\in,\in\vee q_k)$-fuzzy left ideal g, $f\wedge_k g\leq f\circ_k g.$
(v) For all $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal f and g , $f\wedge_k g\leq f\circ_k g.$

Proof

$(i)\Rightarrow(v)$ Assume that S is an intra-regular AG-groupoid with left identity and f and g are $(\in,\in\vee q_k)$-generalized fuzzy bi-ideal of S respectively. Thus, for any a in S, there exist u and v in S such that $a=uv,$ then

$$(f\circ_k g)(a)=\bigvee_{a=uv} f(u)\wedge g(v)\wedge\frac{1-k}{2}.$$

Since S is intra-regular, so for any a in S, there exist $x,y\in S$ such that $a=(xa^2)y.$ By using Moderson et al. (2003) definition (4) and (1), (3) and (2), we get

$$\begin{aligned} a &= (xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a \\ &= (y(x((xa^2)y)))a=(y((xa^2)(xy)))a=((xa^2)(y(xy)))a \\ &= ((x(aa))(y(xy)))a=((a(xa))(y(xy)))a \\ &= (((y(xy))(xa))a)a=(((y(xy))(x((xa^2)y)))a)a \\ &= (((y(xy))((xa^2)(xy)))a)a=(((xa^2)((y(xy))(xy)))a)a \\ &= ((((xy)a^2)((y(xy))x))a)a=((((xy)(y(xy)))(a^2x))a)a \\ &= ((a^2(((xy)(y(xy)))x))a)a=(((aa)(((xy)(y(xy)))x))a)a \\ &= (((x((xy)(y(xy))))(aa))a)a=((a((x(xy)(y(xy)))a))a)a \end{aligned}$$

Thus, we have

$$\begin{aligned} (f\circ_k g)(a) &= \bigvee_{a=uv} f(u)\wedge g(v)\wedge\frac{1-k}{2} \\ &\geq f((a(((x(xy)(y(xy)))a))a)\wedge g(a)\wedge\frac{1-k}{2} \\ &\geq (f(a)\wedge f(a))\wedge g(a)\wedge\frac{1-k}{2} \\ &= (f\wedge g)(a)\wedge\frac{1-k}{2}. \\ &= (f\wedge_k g)(a). \end{aligned}$$

This implies that $f\wedge_k g\leq f\circ_k g.$

$(v)\Rightarrow(iv)$ Let f be an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal and g be an $(\in,\in\vee q_k)$-fuzzy left ideal. Then by Lemma 3, g is also an $(\in,\in\vee q_k)$-fuzzy bi-ideal, therefore (iv) is obvious.

$(iv)\Rightarrow(iii)$ is obvious.

$(iii)\Rightarrow(ii)$ Let B be a bi-ideal and L be a left ideal of S. Then, by Lemma 1, $(C_B)_k$ and $(C_L)_k$ are $(\in,\in\vee q_k)$-fuzzy bi-ideal and $(\in,\in\vee q_k)$-fuzzy left ideal of S. Then, by using Lemma 2 and (ii), we get

$$(C_{B\cap L})_k=C_B\wedge_k C_L\leq C_B\circ_k C_L=(C_{BL})_k.$$

By using Lemma 1, this implies that $B\cap L\subseteq BL.$

$(ii) \Rightarrow (i)$ Since Sa is both bi-ideal and left ideal containing a. Therefore by (ii) and using $(3),(2)$ and $(4),$ we obtain

$$a \in Sa \cap Sa \subseteq (Sa)(Sa) = (aa)(SS) = (aS)(aS) = a^2(SS)$$
$$= (aa)(SS) = S(a^2S) = (SS)(a^2S) = (Sa^2)(SS) = (Sa^2)S$$

Hence, S is an intra-regular AG-groupoid.

Theorem 7

Let S be an AG-groupoid with left identity, then will the following conditions equivalent.

(i) S is intra-regular.

(ii) For all left ideals $A,B,$ $A \cap B \subseteq BA.$

(iii) For all $(\in, \in \vee q_k)$-fuzzy left ideals f and g, $f \wedge_k g \leq g \circ_k f.$

(iv) For all $(\in, \in \vee q_k)$-fuzzy bi-ideals f and g, $f \wedge_k g \leq g \circ_k f.$

(v) For all $(\in, \in \vee q_k)$- generalized fuzzy bi-ideals f and g, $f \wedge_k g \leq g \circ_k f.$

Proof

$(i) \Rightarrow (v)$ Let S be an intra-regular AG-groupoid and f and g are both $(\in, \in \vee q_k)$-fuzzy bi-ideals. For any $a \in S,$ there exist u and v in S such that $a = uv,$ then by using Theorem 6, we get

$$\begin{aligned}(g \circ_k f)(a) &= \bigvee_{a=uv} g(u) \wedge f(v) \wedge \frac{1-k}{2} \\ &\geq g((a(((x(xy)(y(xy)))a))a) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq (g(a) \wedge g(a)) \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq g(a) \wedge f(a) \wedge \frac{1-k}{2} \\ &= (f \wedge_k g)(a).\end{aligned}$$

This implies $f \wedge_k g \leq g \circ_k f.$

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$ Let f and g be $(\in, \in \vee q_k)$-fuzzy left ideals. Then by Lemma 3, f and g are $(\in, \in \vee q_k)$-fuzzy bi-ideals. Then (iii) is obvious.

$(iii) \Rightarrow (ii)$ Assume that A and B are the left ideals of S then by Lemma 1, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideals of $S.$ Then by using Lemma 2 and $(ii),$ we get

$$(C_{A \cap B})_k = (C_A \wedge_k C_B) \leq C_B \circ_k C_A = (C_{BA})_k.$$

Thus by using Lemma 1, we get $A \cap B \subseteq BA.$

$(ii) \Rightarrow (i)$ Since Sa is both bi-ideal and left ideal containing a. Using $(ii),$ we get

$$a \in Sa \cap Sa \subseteq (Sa)(Sa) = Sa^2 = (Sa^2)S.$$

Hence, S is intra-regular.

Theorem 8

Let S be an AG-groupoid with left identity, then will the following conditions equivalent

(i) S is intra-regular.

(ii) For every $(\in, \in \vee q_k)$-fuzzy left ideals f, g and $(\in, \in \vee q_k)$-fuzzy right ideal h, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h.$

(iii) For every $(\in, \in \vee q_k)$-fuzzy left ideals f, g and $(\in, \in \vee q_k)$-fuzzy bi ideal h, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h.$

(iv) For all $(\in, \in \vee q_k)$-fuzzy bi-ideals f, g and $h,$ $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h.$

(v) For all $(\in, \in \vee q_k)$-generalized fuzzy bi-ideals f, g and h, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h.$

Proof

$(i) \Rightarrow (iii)$ Assume that S is an intra-regular AG-groupoid and f and g are $(\in, \in \vee q_k)$-left ideals and h is an $(\in, \in \vee q_k)$-bi-ideal of $S.$ Since S is an intra-regular

AG-groupoid, therefore for all $a \in S$, there exist x, y in S such that $a = (xa^2)y.$ By using Moderson et al. (2003) definition (4) and (1), we get

$$
\begin{aligned}
a &= (x(aa))y = (a(xa))y = (y(xa))a. \\
&= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\
&= ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\
&= ((a(xa))(y(xy)))a = (((y(xy))(xa))a)a \\
&= (((ax)((xy)y))a)a = (((((xy)y)x)a)a)a \\
&= (((((xy)y)x)((xa^2)y)))a)a = (((xa^2)(((xy)y)x)y))a)a \\
&= (((a(xa))(((xy)y)x)y))a)a = ((((((((xy)y)x)y)(xa))a)a)a \\
&= ((((y(((xy)y)x))(ax))a)a)a = (((a(y(((xy)y)x))x)a)a)a.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=uv} (f \circ_k g)(u) \wedge h(v) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=uv} \left(\left\{ \bigvee_{u=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} \right\} \wedge h(v) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(pq)v} \left(f(p) \wedge g(q) \wedge h(v) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(((a(y(((xy)y)x))x)a)a)a=(pq)v} \left(f(p) \wedge g(q) \wedge h(v) \wedge \frac{1-k}{2} \right) \\
&\geq (f(a) \wedge f(a)) \wedge g(a)) \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq ((f(a) \wedge \frac{1-k}{2}) \wedge g(a)) \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f(a) \wedge g(a) \wedge \frac{1-k}{2}) \wedge h(a)) \wedge \frac{1-k}{2} \\
&= (f \wedge_k g) \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}
$$

This implies $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h.$

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$ Assume that f and g are $(\in, \in \vee q_k)$-fuzzy left ideals and h is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of $S,$ then by Lemma 3, f and g are $(\in, \in \vee q_k)$-fuzzy bi-ideal, therefore (iii) is obvious.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that f, g are left ideals and S is a right ideal. Then by using (ii) we get

$$f \wedge_k g = f \wedge_k g \wedge_k S \leq (g \circ_k f) \circ_k S \leq g \circ_k f.$$

Therefore, $f \wedge_k g \leq g \circ_k f$. Hence, by Theorem 5, S is intra-regular.

Theorem 9

Let S be an AG-groupoid with left identity, then will the following conditions equivalent

(i) S is intra-regular.

(ii) Every ideal of S is semiprime.

(iii) Every bi-ideal of S is semiprime.

(iv) Every $(\in, \in \vee q_k)$-fuzzy bi-ideal of S is fuzzy semiprime.

(v) Every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S is fuzzy semiprime.

Proof

$(i) \Rightarrow (v)$ Let S be an intra-regular and f be an $(\in, \in \vee q_k)$-generalized bi-ideal of S. Then for all $a \in S$, there exists x, y in S such that $a = (xa^2)y.$ By using Moderson et al. (2003) definition (4), (1), (2) and (3), we get

$$
\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
&= (y(x((xa^2)y))a = (x(y((xa^2)y)))a \\
&= (x((xa^2)y^2))a = ((xa^2)(xy^2))a \\
&= (x^2(a^2y^2))a = (a^2(x^2y^2))a = (a(x^2y^2))a^2 \\
&= (((xa^2)y)(x^2y^2))a^2 = ((y^2.y)(x^2(xa^2)))a^2 \\
&= ((y^2x^2)(y(xa^2)))a^2 = ((y^2x^2)((uv)(xa^2)))a^2 \\
&= ((y^2x^2)((a^2v)(xu)))a^2 = ((y^2x^2)((a^2x)(vu)))a^2 \\
&= ((((y^2x^2)(vu))x)(aa)))a^2 = ((ax)(a((y^2x^2)(vu))))a^2 \\
&= ((aa)(x((x^2y^2)(vu))))a^2 = (a^2((x((x^2y^2)(vu))))a^2.
\end{aligned}
$$

we have

$$f(a) = f((a^2((x((x^2y^2)(vu))))a^2) \geq f(a^2) \wedge f(a^2) = f(a^2).$$

Therefore $f(a) \geq f(a^2).$

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$ Let B be a bi-ideal of S, then by Lemma 1, $(C_B)_k$ is an $(\in, \in \vee q_k)$ fuzzy bi-ideal of S. Let $a^2 \in B$, then since $(C_B)_k$ is an $(\in,\in\vee q_k)$ fuzzy bi-ideal, therefore by $(iv), (C_B(a))_k \geq (C_B(a^2))_k$, as $a^2 \in B$ so, $(C_B(a^2))_k = 1 \leq (C_B(a))_k$ this implies $(C_B(a))_k = 1$. Thus $a \in B$. Hence, B is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (i)$ Assume that every ideal is semiprime and since Sa^2 is an ideal containing a^2. Thus $a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S$. Hence S is an intra-regular AG-groupoid.

Theorem 10

Let S be an AG-groupoid with left identity, then will the following conditions equivalent.
(i) S is intra-regular.
(ii) For every left ideal L and bi-ideal B, $L \cap B \subseteq (LB)L$.
(iii) For every $(\in,\in\vee q_k)$-fuzzy left ideal f and $(\in,\in\vee q_k)$-fuzzy bi-ideal g, $f \wedge_k g \leq (f \circ_k g) \circ_k f$.

Proof

$(i) \Rightarrow (iii)$ Let S be an intra-regular AG-groupoid and f be an $(\in,\in\vee q_k)$-fuzzy left ideal and g be an ($\in,\in\vee q_k$)-fuzzy bi-ideal of S. Since S is an intra-regular AG-groupoid then for any $a \in S$, there exists $x, y \in S$ such that $a = (xa^2)y$. Then by using Moderson et al. (2003) definition (4) and (1), we get

$$a = (xa^2)y = (a(xa))y = (y(xa))a = (y(x((xa^2)y))a = (x(y((a(xa)y)))a$$
$$= (x(a(xa))y^2)a = ((a(xa))(xy^2))a = (((xy^2)(xa))a)a.$$

Therefore

$$((f \circ_k g) \circ_k f)(a) = \bigvee_{a=uv} (f \circ_k g)(u) \wedge f(v) \wedge \frac{1-k}{2}$$
$$\geq (f \circ_k g)(((xy^2)(xa))a) \wedge f(a).$$

Since f is an $(\in,\in\vee q_k)$-fuzzy left ideal, therefore

$$(f \circ_k g)(((xy^2)(xa))a) = \bigvee_{((xy^2)(xa))a=rs} f(r) \wedge g(s) \wedge \frac{1-k}{2}$$
$$\geq f((xy^2)(xa)) \wedge g(a) \wedge \frac{1-k}{2}$$
$$\geq f(xa) \wedge g(a) \wedge \frac{1-k}{2}$$
$$\geq f(a) \wedge g(a) \wedge \frac{1-k}{2}.$$

Thus

$$((f \circ_k g) \circ_k f)(a) \geq f(a) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge g(a) \wedge \frac{1-k}{2}.$$

Hence, $f \wedge_k g \leq ((f \circ_k g) \circ_k f)$.
$(iii) \Rightarrow (ii)$ Let L be a left ideal and B be a bi-ideal of S. Then by Lemma 1 and (iii), $(C_L)_k$ and $(C_B)_k$ are $(\in,\in\vee q_k)$-fuzzy left and $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

Then,

$$(C_{L\cap B})_k = C_L \wedge_k C_B \leq (C_L \circ_k C_B) \circ_k C_L \leq ((C_{LB})_k \circ_k C_L) = (C_{(LB)L})_k.$$

Therefore by using Lemma 1, we get $L \cap B \subseteq (LB)L$.

$(ii) \Rightarrow (i)$ Since Sa is both left and bi-ideal. Let $a \in S$. So By using (ii),

$$a \in Sa \cap Sa = ((Sa)(Sa))Sa = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

Therefore, S is an intra-regular AG-groupoid.

Theorem 11

Let S be an AG-groupoid with left identity then the following conditions are equivalent.

(i) S is intra-regular.
(ii) $f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k (f \circ_k h)$, where f is an $(\in,\in\vee q_k)$-fuzzy left ideal, h is an $(\in,\in\vee q_k)$-fuzzy right ideal and g is an $(\in,\in\vee q_k)$ -fuzzy bi-ideal.

Proof

$(i) \Rightarrow (ii)$ Let S be an intra-regular AG-groupoid. For any a in S, there exist u and v in S such that $a = uv$, then

$$((f \circ_k g) \circ_k (f \circ_k h))(a) = \bigvee_{a=uv} (f \circ_k g)(u) \wedge (f \circ_k h)(v) \wedge \frac{1-k}{2}.$$

Since S is intra-regular, so for any $a \in S$, there exist x, y in S such that $a = (xa^2)y$. Then by using Moderson et al. (2003) definition (4) and (1), we get

$$a = (xa^2)y = (a(xa))y = (y(xa))a.$$

Thus

$$(f \circ_k g)(y(xa)) = \bigvee_{y(xa)=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2}.$$

Therefore

$$y(xa) = y(x((xa^2)y)) = y(x(a(xa))y) = y((a(xa))(xy))$$
$$= (a(xa))(y(xy)) = ((y(xy))(xa))a.$$

Thus

$$\begin{aligned}(f \circ_k g)(y(xa)) &= \bigvee_{y(xa)=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} \\ &\geq f((y(xy))(xa)) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(xa) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2}.\end{aligned}$$

And

$$(f \circ_k h)(a) = \bigvee_{a=rs} f(r) \wedge h(s) \wedge \frac{1-k}{2}.$$

Since $a \in S$ so for any a in S there exist x, y in S such that $a = (xa^2)y$. Thus

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

So

$$\begin{aligned}(f \circ_k h)(a) &= \bigvee_{a=rs} f(r) \wedge h(s) \wedge \frac{1-k}{2} \\ &\geq f(y(xa)) \wedge h(a) \wedge \frac{1-k}{2} \\ &\geq f(xa) \wedge h(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge h(a) \wedge \frac{1-k}{2}.\end{aligned}$$

Therefore this implies that

$$f \wedge_k g \wedge_k h \leq (f \circ_k g) \circ_k (f \circ_k h) .$$

$(ii) \Rightarrow (i)$ Since S is a right ideal, so by using (ii)

$$f \wedge_k g = f \wedge_k g \wedge_k S \leq (f \circ_k g) \circ_k (f \circ_k S) \leq (f \circ_k g) \circ_k f.$$

Thus $f \wedge_k g \leq (f \circ_k g) \circ_k f$. Hence, by Theorem 10, S is an intra-regular AG-groupoid. The proofs of following two lemmas are easy and therefore omitted.

Lemma 4

For any fuzzy subset f of an AG-groupoid S, $S \circ_k f \leq f$ and for any fuzzy right ideal g, $g \circ_k S \leq g$.

Lemma 5

Let S be an intra-regular AG-groupoid, then for any $(\in, \in \vee q_k)$-fuzzy subsets f and g, $f \wedge_k g \wedge_k S = f \wedge_k g$.

Theorem 12

Let S be an intra-regular AG-groupoid then for any $(\in, \in \vee q_k)$-fuzzy subsets f, g and h, $(f \circ_k g) \circ_k h = (h \circ_k g) \circ_k f$.

Proof

Let S be an intra-regular AG-groupoid. For any a in S, there exist p and q in S such that $a = pq$, then

$$((f\circ_k g)\circ_k h)(a) = \bigvee_{a=pq}\left\{(f\circ_k g)\wedge h(q)\wedge \frac{1-k}{2}\right\}$$
$$= \bigvee_{a=pq}\left(\bigvee_{p=rs}\left\{f(r)\wedge g(s)\wedge \frac{1-k}{2}\right\}\wedge h(q)\wedge \frac{1-k}{2}\right)$$
$$= \bigvee_{a=(rs)q}\left\{f(r)\wedge g(s)\wedge h(q)\wedge \frac{1-k}{2}\right\}$$
$$= \bigvee_{a=(qs)r}\left\{h(q)\wedge g(s)\wedge f(r)\wedge \frac{1-k}{2}\right\}$$
$$= \bigvee_{a=ur}\left(\bigvee_{u=qs}\left\{h(q)\wedge g(s)\wedge \frac{1-k}{2}\right\}\wedge f(r)\wedge \frac{1-k}{2}\right)$$
$$= \bigvee_{a=ur}\left\{(h\circ_k g)\wedge f(r)\wedge \frac{1-k}{2}\right\}$$
$$= ((h\circ_k g)\circ_k f)(a).$$

This implies that $(f\circ_k g)\circ_k h = (h\circ_k g)\circ_k f.$

Lemma 6

Let S be an intra-regular AG-groupoid, then for any $(\in,\in\vee q_k)$- fuzzy - subsets f and g, $(g\circ_k f)\circ_k S \leq g\circ_k f.$

Proof

Let S be an intra-regular AG-groupoid and f and g are any $(\in,\in\vee q_k)$-fuzzy-subsets then by using Lemma 4 and Theorem 12, we get

$$(g\circ_k f)\circ_k S = (S\circ_k f)\circ_k g \leq f\circ_k g = g\circ_k f.$$

Hence, $(g\circ_k f)\circ_k S \leq g\circ_k f.$

Theorem 13

Let S be an intra-regular AG-groupoid and f and g are $(\in,\in\vee q_k)$-fuzzy ideals of S, then

$$f\circ_k g = f\wedge_k g.$$

Proof

Let S be an intra-regular AG-groupoid and f and g are $(\in,\in\vee q_k)$-fuzzy ideals of S. Then for any a in S there exist x and y in S such $a=(xa^2)y,$ then

$$a=(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a.$$

Now

$$(f\circ_k g)(a) = \bigvee_{a=uv}\left\{f(u)\wedge g(v)\wedge\frac{1-k}{2}\right\}$$
$$\geq f(y(xa))\wedge g(a)\wedge\frac{1-k}{2}$$
$$\geq f(xa)\wedge g(a)\wedge\frac{1-k}{2}$$
$$\geq f(a)\wedge g(a)\wedge\frac{1-k}{2}$$
$$=(f\wedge g)(a)\wedge\frac{1-k}{2}$$
$$=(f\wedge_k g)(a).$$

This implies that

$$f\circ_k g \geq f\wedge_k g.$$

Now

$$(f\circ_k g)(a) = \bigvee_{a=uv} f(u)\wedge g(v)\wedge\frac{1-k}{2}$$
$$\leq \bigvee_{a=uv} f(uv)\wedge g(uv)\wedge\frac{1-k}{2}$$
$$= f(a)\wedge g(a)\wedge\frac{1-k}{2}$$
$$=(f\wedge g)(a)\wedge\frac{1-k}{2}$$
$$=(f\wedge_k g)(a).$$

This implies that $f\circ_k g\leq f\wedge_k g.$ Hence $f\circ_k g = f\wedge_k g.$

Theorem 14

Let S be an AG-groupoid with left identity, then the following conditions are equivalent.

(i) S is intra-regular.

(ii) For every ideals A and B, $AB \subseteq BA$, and A and B are semiprime.

(iii) For every $(\in, \in \vee q_k)$-fuzzy ideals f and g, $f \circ_k g \leq g \circ_k f$, and f and g are semiprime ideals.

Proof

$(i) \Rightarrow (iii)$ Let S be an intra-regular AG-groupoid and f and g are $(\in, \in \vee q_k)$-fuzzy ideals of S. Then by Theorems 13, 4 and 6, we get

$$f \circ_k g = f \wedge_k g = (f \wedge_k g) \wedge_k S = (g \circ_k f) \wedge_k S = S \circ_k (g \circ_k f) \leq g \circ_k f.$$

This implies $f \circ_k g \leq g \circ_k f$. Also we will show that f and g are semiprime ideals. So,

$$f(a) = f((xa^2)y) \geq f(a^2).$$

Thus $f(a) \geq f(a^2)$. Similarly $g(a) \geq g(a^2)$.

$(iii) \Rightarrow (ii)$ Let A and B be ideals then by Lemma 1, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy ideals, therefore by using Lemma 2 and (iii),

$$(C_{AB})_k = C_A \circ_k C_B \leq C_B \circ_k C_A = (C_{BA})_k.$$

Therefore by using Lemma 1, we get $AB \subseteq BA$.

$(ii) \Rightarrow (i)$ Let $a^2 \in Sa^2$. Since Sa^2 is semiprime, therefore $a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S$. Hence, S is an intra-regular AG-groupoid.

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