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A novel technique for solving Cauchy problem for the third-order linear dispersive partial differential equation

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To solve Cauchy problem for the third-order linear dispersive partial differential equation, the reduced differential transform (RDT) method is used. Without using any discretization or perturbation the technique gives analytical approximation in the form of rapidly convergent sequence with well-structured terms that can usually be expressed in a compact form. The efficiency and reliability of the method is demonstrated through a variety of homogeneous/ non-homogeneous, one-, two-, and three-dimensional model problems. Comparison of the results by RDT method and those obtained through variational iteration method (VIM) signifies that RDT method is more efficient and rapidly convergent. Moreover it is computationally an inexpensive method.

Key words: Cauchy problem, dispersive partial differential equation, reduced differential transformation.

INTRODUCTION

The classical Taylor series method has been one of the earlier methods for solving the differential equations. With an advent of high-speed computers there has been an increasing trend towards exploring new ideas out of traditional techniques for the last couple of decades. An updated version of Taylor series method, called the differential transform method (DTM) was introduced by Zhou (1986). Another improved approach for solving initial-value problem for partial differential equation, known as the reduced differential transform (RDT) method, has recently been used by Keskin and Oturance (2009), Taha (2011), Hesam et al (2012), Abazari and Abazari (2012), and Secer (2012).

In our present work we are interested in the third-order dispersive partial differential equations. In an analytic study of the third-order dispersive partial differential equations Wazwaz (2003) demonstrated how exact solutions to third-order dispersive partial differential equations are derived through the Adomian decomposition method.

Djidjeli and Twizell (1991) developed a family of numerical methods to solve the third-order dispersive equations in single space-variable with time-dependent boundary conditions. Batiha (2009) found an approximate solution of the dispersive equations by variational iteration method (He 1999, 2007). In this paper, we use RDT method for solving the linear third-order dispersive partial differential equations. Reliability and efficiency of the method are expressed through test problems.

DIFFERENTIAL AND REDUCED DIFFERENTIAL TRANSFORMS – BASIC TERMINOLOGY

The differential transform of an analytic function $u(x)$ is defined by Chen and Ho (1996):

$$U(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}u(x)\right]_{x=0}, \tag{1}$$

where $u(x)$ is the original analytic function and $U(k)$ is the transformed function. Further the inverse differential transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k. \tag{2}$$

From the expressions (1) and (2) it immediately follows

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that;

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dx^k} u(x) \right]_{x=0} x^k. \qquad (3)$$

Similarly the two-dimensional differential transform of an analytic function $u(x,t)$ is defined as by Jang et al (2006):

$$U(k,h) = \frac{1}{k!\,h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t) \right]_{(0,0)}, \qquad (4)$$

where $u(x,t)$ is the original function and $U(k,h)$ is the transformed function. The inverse differential transform of $U(k,h)$ is given by

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h) x^k t^h. \qquad (5)$$

From (4) and (5) the following expression is obtained

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!\,h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t) \right]_{(0,0)} x^k t^h. \qquad (6)$$

Now suppose that $u(x,t)$ can be represented as a product of two functions $f(x)$ and $g(t)$ so that

$$u(x,t) = f(x)\,g(t), \qquad (7)$$

which can further be expressed as

$$u(x,t) = \sum_{i=0}^{\infty} F(i) x^i \sum_{j=0}^{\infty} G(j) t^j = \sum_{k=0}^{\infty} U_k(x) t^k, \qquad (8)$$

where $U_k(x)$ is referred to as *t*-dimensional spectrum function of $u(x,t)$. The reduced differential transform (RDT) therefore can be defined as below.

Definition

Let $u(x,t)$ be an analytic function (obviously sufficiently smooth with respect to x and t in the domain of definition). The reduced differential transform $U_k(x)$ of $u(x,t)$ is defined as (Keskin, Oturance 2009)

$$U_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0}, \qquad (9)$$

and the inverse-RDT of $U_k(x)$ takes the form

$$u = \sum_{k=0}^{\infty} U_k(x) t^k. \qquad (10)$$

Table 1. Basic reduced differential transforms.

Function	Reduced differential transform (RDT)
$u(x,t)$	$U_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0}$
$u(x,t) \pm v(x,t)$	$U_k(x) \pm V_k(x)$
$\alpha\, u(x,t)$	$\alpha\, U_k(x)$, α being a constant
$u(x,t)\, v(x,t)$	$\sum_{r=0}^{k} U_r(x) V_{k-r}(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x)$
$x^m t^n$	$x^m \delta(k-n)$, $\delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
$\frac{\partial^n}{\partial x^n} u(x,t)$	$\frac{\partial^n}{\partial x^n} U_k(x)$
$\frac{\partial^r}{\partial t^r} u(x,t)$	$(k+1)(k+2)\dots(k+r) U_{k+r}(x)$

The expressions (9) and (10) then yield the following result.

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k. \qquad (11)$$

The basic RDTs are given in Table 1 and can be proved using definitions (9) and (10), refer to Keskin (2010) for more details.

SOLVING THIRD-ORDER LINEAR DISPERSIVE PARTIAL DIFFERENTIAL EQUATION BY RDT METHOD

Consider the Cauchy problem for the third-order linear dispersive partial differential equation of the form

$$L[u(x,t)] + R[u(x,t)] = g(x,t), \quad x \in \mathbb{R}, t > 0, \qquad (12)$$

subject to Cauchy condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}, \qquad (13)$$

where the linear differential operators L and R are of the type

$$L \equiv \partial_t, \quad R \equiv \sum_{|\alpha| \leq 3} c_\alpha \partial_x^\alpha,$$

and $g(x,t)$ is an inhomogeneous term. The reduced differential transformation on the problem (12)-(13) yields

$$(k + 1)U_{k+1}(x) = G_k(x) - R[U_k(x)], \tag{14}$$

$$U_0(x) = f(x), \tag{15}$$

where $R[U_k(x)]$ and $G_k(x)$ represent the RDTs of $R[u(x,t)]$ and $g(x,t)$, respectively. The equations (14)-(15) straight away produce all values of $U_k(x)$ iteratively. The inverse-RDTs of the set of values $\{U_k(x)\}_{k=0}^n$ then give the approximate solution in the form

$$\tilde{u}_n(x,t) = \sum_{k=0}^n U_k(x)t^k, \tag{16}$$

where n is the order of approximation for the solution. The exact solution of the problem (12)-(13) is therefore given by

$$u(x,t) = \lim_{n \to \infty} \tilde{u}_n(x,t). \tag{17}$$

IMPLEMENTATION OF RDT METHOD

We now apply RDT method to solve Cauchy problem for the third-order linear dispersive partial differential equation in one and the higher dimensions. We consider the following test problems.

Test Problem 1. (One-dimensional homogeneous case)

We consider the following partial differential equation

$$u_t + 2u_x + u_{xxx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{18}$$

subject to the Cauchy condition

$$u(x,0) = \sin x, \tag{19}$$

whose exact solution is known to be

$$u(x,t) = \sin(x - t). \tag{20}$$

The reduced differential transformation on (18)-(19) yields

$$\left.\begin{array}{c} (k + 1)U_{k+1}(x) = -2\dfrac{\partial}{\partial x}U_k(x) - \dfrac{\partial^3}{\partial x^3}U_k(x), \\ U_0(x) = \sin x. \end{array}\right\} \tag{21}$$

From iterative scheme (21), we obtain

$$U_1(x) = -\cos x, \ U_2(x) = -\tfrac{1}{2}\sin x, \ U_3(x) = \tfrac{1}{6}\cos x, \ \dots$$

The inverse-RDTs of $U_k(x)$ give

$$u(x,t) = \sum_{k=0}^\infty U_k(x)t^k$$

$$= \sin x \left\{1 - \frac{t^2}{2!} + \cdots\right\} - \cos x \left\{t - \frac{t^3}{3!} + \cdots\right\} \tag{22}$$

and finally we obtain

$$u(x,t) = \sin x \cos t - \cos x \sin t = \sin(x - t).$$

Test Problem 2. (One-dimensional inhomogeneous case)

We consider the following one-dimensional inhomogeneous partial differential equation

$$u_t + u_{xxx} = -\sin(\pi x)\sin t - \pi^3 \cos(\pi x)\cos t, \quad 0 < x < 1, t > 0 \tag{23}$$

subject to the Cauchy condition

$$u(x,0) = \sin(\pi x), \quad 0 < x < 1, \tag{24}$$

whose exact solution is known to be

$$u(x,t) = \sin(\pi x)\cos t. \tag{25}$$

The reduced differential transformation on (23)-(24) yields

$$(k + 1)U_{k+1}(x) = -\frac{\partial^3}{\partial x^3}U_k(x)$$

$$- \sin(\pi x)\left\{\delta(k - 1) - \tfrac{1}{3!}\delta(k - 3) + \cdots\right\}$$

$$- \pi^3 \cos(\pi x)\left\{\delta(k) - \tfrac{1}{2!}\delta(k - 2) + \cdots\right\}$$

$$U_0(x) = \sin(\pi x) \tag{26}$$

The above iterative scheme gives the following set of values

$$U_1(x) = 0, U_3(x) = 0, U_5(x) = 0, \dots$$

$$U_2(x) = -\tfrac{1}{2!}\sin(\pi x), U_4(x) = \tfrac{1}{4!}\sin(\pi x),$$

$$U_6(x) = -\tfrac{1}{6!}\sin(\pi x), \dots \tag{27}$$

The inverse-RDTs of $U_k(x)$ give us

$$u(x,t) = \sum_{k=0}^\infty U_k(x)t^k = \sin(\pi x)\left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right\} \tag{28}$$

and we finally obtain

$$u(x,t) = \sin(\pi x)\cos t.$$

Test Problem 3. (Two-dimensional homogeneous case)

We consider the following partial differential equation

$$u_t + u_{xxx} + u_{yyy} = 0, \qquad t > 0, \tag{29}$$

subject to the Cauchy condition

$$u(x,y,0) = \cos(x+y), \tag{30}$$

whose exact solution is known to be

$$u(x,y,t) = \cos(x+y+2t). \tag{31}$$

The reduced differential transformation on (29)-(30) yields

$$\left. \begin{aligned} (k+1)U_{k+1}(x,y) = -\frac{\partial^3}{\partial x^3}U_k(x,y) - \frac{\partial^3}{\partial y^3}U_k(x,y), \\ U_0(x,y) = \cos(x+y). \end{aligned} \right\} \tag{32}$$

The iterative scheme (32) gives the following values of $U_k(x,y)$.

$$U_1(x,y) = -2\sin(x+y), \qquad U_2(x,y) = -2\cos(x+y),$$

$$U_3(x,y) = \tfrac{4}{3}\sin(x+y),\ U_4(x,y) = \tfrac{2}{3}\cos(x+y),$$

$$U_5(x,y) = -\tfrac{4}{15}\sin(x+y),\ U_6(x,y) = -\tfrac{4}{45}\cos(x+y),$$

$$U_7(x,y) = \tfrac{8}{315}\sin(x+y),\ ...$$

The inverse-RDTs of $U_k(x,y)$ yield

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k$$

$$= \cos(x+y)\left\{1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \frac{(2t)^6}{6!} + \cdots\right\}$$

$$-\sin(x+y)\left\{2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right\},$$

and we finally obtain

$$u(x,y,t) = \cos(x+y+2t).$$

Test Problem 4. (Three-dimensional inhomogeneous case)

Finally, we consider the following three-dimensional partial differential equation

$$u_t + u_{xxx} + \tfrac{1}{8}u_{yyy} + \tfrac{1}{27}u_{zzz} = -3\cos(x+2y+3z)\sin t + \sin(x+2y+3z)\cos t, \quad t>0, \tag{33}$$

subject to the Cauchy condition

$$u(x,y,z,0) = 0, \tag{34}$$

whose exact solution is known to be

$$u(x,y,z,t) = \sin(x+2y+3z)\sin t. \tag{35}$$

The reduced differential transformation on (33)-(34) gives the following iterative scheme

$$(k+1)U_{k+1}(x,y,z) = -\frac{\partial^3}{\partial x^3}U_k(x,y,z)$$

$$-\frac{1}{8}\frac{\partial^3}{\partial y^3}U_k(x,y,z) - \frac{1}{27}\frac{\partial^3}{\partial z^3}U_k(x,y,z)$$

$$-3\cos(x+2y+3z)\left\{\delta(k-1) - \tfrac{1}{3!}\delta(k-3) + \cdots\right\}$$

$$+\sin(x+2y+3z)\left\{\delta(k) - \tfrac{1}{2!}\delta(k-2) + \cdots\right\} \tag{36}$$

$$U_0(x,y,z) = 0. \tag{37}$$

From (36)-(37) we obtain the following values for $U_k(x,y,z)$.

$$U_1(x,y,z) = \sin(x+2y+3z),$$

$$U_3(x,y,z) = \tfrac{1}{3!}\sin(x+2y+3z)$$

$$U_5(x,y,z) = -\tfrac{1}{5!}\sin(x+2y+3z),\ ...$$

$$U_2(x,y,z) = 0 = U_4(x,y,z) = U_6(x,y,z) = \cdots$$

The inverse-RDTs of $U_k(x,y,z)$ yields

$$u(x,y,z,t) = \sum_{k=0}^{\infty} U_k(x,y,z)t^k = \sin(x+2y+3z)\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right\} = \sin(x+2y+3z)\sin t,$$

which is the exact solution.

To compute absolute errors we take into account the first five iterates of RDT method for each model problem. The comparison of these absolute errors with those computed by Batiha (2009), at the fifth iterates using variational iteration method (VIM), is shown in the Tables 2 below. The results signify that RDT method is more efficient and rapidly convergent as compared to VIM.

Tables 2. Comparison of RDT method with VIM.

	x	t	Absolute error with	
			VIM	RDT method
Test Problem 1	0.1	0.2	6.290E-09	6.300E-09
		0.3	5.780E-08	5.780E-08
		0.4	2.438E-07	2.436E-07
	0.6	0.2	4.800E-08	4.830E-08
		0.3	5.350E-07	5.351E-07
		0.4	2.935E-06	2.935E-06
	0.9	0.2	6.800E-08	6.830E-08
		0.3	7.649E-07	7.650E-07
		0.4	4.242E-06	4.242E-06

	x	t	Absolute error with	
			VIM	RDT method
Test Problem 2	0.1	0.2	4.783E-03	2.750E-08
		0.3	1.870E-02	3.124E-07
		0.4	4.519E-02	1.753E-06
	0.6	0.2	1.679E-02	8.490E-08
		0.3	4.224E-02	9.614E-07
		0.4	7.799E-02	5.395E-06
	0.9	0.2	2.217E-02	2.760E-08
		0.3	5.517E-02	3.124E-07
		0.4	8.884E-02	1.753E-06

	x	y	t	Absolute error with	
				VIM	RDT method
Test Problem 3	0.1	0.1	0.2	2.245E-05	5.956E-06
			0.3	1.907E-04	6.200E-05
			0.4	8.871E-04	3.446E-04
	0.6	0.6	0.2	1.753E-05	1.753E-06
			0.3	1.818E-05	1.818E-05
			0.4	2.637E-03	9.199E-05
	0.9	0.9	0.2	8.150E-05	1.605E-06
			0.3	6.110E-04	2.001E-05
			0.4	2.537E-03	1.220E-04

	x	y	z	t	Absolute error with	
					VIM	RDT method
Test Problem 4	0.1	0.1	0.1	0.2	1.675E-02	1.500E-09
				0.3	3.098E-02	2.440E-08
				0.4	1.951E-02	1.831E-07
	0.6	0.6	0.6	0.2	3.524E-02	1.150E-09
				0.3	4.287E-02	1.910E-08
				0.4	2.637E-03	1.435E-07
	0.9	0.9	0.9	0.2	2.217E-02	2.600E-09
				0.3	5.168E-02	4.310E-08
				0.4	1.503E-01	3.231E-07

Conclusion

Reduced differential transform method has been used to solve Cauchy problem for the third-order linear dispersive partial differential equation. The method gives the analytical approximation in the form of rapidly convergent sequence with well-patterned terms that usually can thereafter be represented in compact form – the exact solution to the partial differential equation. The efficiency of this novel technique has been illustrated through a variety of test problems.

REFERENCES

Abazari R, Abazari M (2012). Numerical simulation of generalized Hirota-Satsuma coupled KdV equation by RDTM and comparison with DTM. Commun. Nonlinear Sci. Numer. Simul. 17(2):619-629.
Batiha B (2009). Application of variational iteration method to linear partial differential equations. Appl. Math. Sci. 3(50): 2491-2498.
Chen CK, Ho SH (1996). Application of differential transformation to eigenvalue problems. Appl. Math. Comput. 79:173-188.
Djidjeli K, Twizell EH (1991). Global extrapolations of numerical methods for solving a third-order dispersive partial differential equations, Int. J. Comput. Math. 41: 81-89.
He JH (1999). Variational iteration method – a kind of non-linear analytical technique: some examples. Int. J. Non-Linear Mech. 34: 699-708.
He JH (2007). Variational iteration method – some recent results and new interpretations. J. Comput. Appl. Mat. 207: 3-17.
Hesam S, Nazemi A, Haghbin A (2012). Reduced differential transform method for solving the Fornberg-Whithman type equation. Int. J. Nonlinear Sci. 13(2):158-162.
Jang M-J, Chen C-L, Liu Y-C (2006). Two-dimensional differential transform for partial differential equations. Appl. Math. Comput. 181(1):767-774.
Keskin Y, Oturanc G (2009). Reduced differential transform method for partial differential equations. Int. J. Nonlinear Sci. Numer. Simul. 10(6): 741-749.
Keskin Y (2010). PhD thesis, Selcuk University Konya, Turkey (in Turkish).
Secer A (2012). Solving time-fractional reaction diffusion equation by reduced differential transform method. Int. J. Comput. Sci. Emerging Tech. 3(1):19-22.
Taha BA (2011). The use of reduced differential transform method for solving partial differential equations with variable coefficients. J. Basrah Res. Sci. 37(4C):226-233.
Wazwaz AM (2003). An analytic study on the third-order dispersive partial differential equations. Appl. Math. Comput. 142: 511-520.
Zhou JK (1986). Differential transformation and its application for electric circuits, *Huazhong University Press China* (in Chinese).