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Full Length Research Paper

A nonlocal integral boundary value problem of nonlinear integro-differential equations of fractional order

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This study discussed the existence of solutions for a nonlinear fractional integro-differential equation of order $q \in (1, 2]$ with four-point nonlocal integral boundary conditions. The given problem is transformed to an equivalent fixed point problem in terms of an operator equation. Then, by means of Banach contraction principle and a fixed point theorem due to Krasnoselskii, the existence results are obtained. The last existence result is based on nonlinear alternative of Leray-Schauder type. An illustrative example is also presented.

Key words: Integro-differential equations, fractional order, four-point integral boundary conditions, fixed point theorems, Leray-Schauder degree.

INTRODUCTION

The subject of fractional calculus has recently emerged as an important and popular field of research. Fractional derivatives are found to be quite effective in describing memory and hereditary properties of various materials and processes. The mathematical modeling of various physical and engineering problems achieved through fractional calculus has turned out to be more realistic and practical than the classical calculus. In fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006; Sabatier et al., 2007). For recent development of the subject, see Agarwal et al. (2010), Ahmad and Sivasundaram (2012), Bai (2010), Baleanu et al. (2010), Bhalekar et al. (2011), Liang et al. (2009), Zhang (2010) and Zhao et al. (2011).

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. Various problems in heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics give rise to the nonlocal problems with integral boundary conditions. Integral boundary conditions for unsteady biomedical CFD applications are also taking much importance these days. For a detailed description of the integral boundary conditions, we refer the reader to the papers (Ahmad et al., 2008) and references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are normally taken to be fixed on the given interval (for instance, [0, 1]).

In the present work, we consider a nonlocal type of integral boundary conditions with limits of integration involving the parameters $0 < \xi, \eta < 1.$ These boundary conditions correspond to the situation when the controllers at the end-points of the interval dissipate/absorb energy due to the sensors of arbitrary finite lengths (continuous distribution of intermediate points of arbitrary length: segments of the interval). Precisely, we study a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with four-point integral boundary conditions

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given by:

$$\begin{aligned}&{}^cD^q x(t) = f(t,\ x(t),\ (\phi_1 x)(t),\ (\phi_2 x)(t)), \quad 0<t<1, \quad 1<q\le 2,\\ &\delta_1 x(0)+\delta_2 x'(0)=\alpha\int_0^{\xi} x(s)ds, \quad \delta_1 x(1)+\delta_2 x'(1)=\beta\int_0^{\eta} x(s)ds,\ 0<\xi,\ \eta<1,\end{aligned} \tag{1}$$

Where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f:[0,1]\times R\times R\times R\to R$ is continuous and $\delta_1,\ \delta_2,\ \alpha,\beta\in R$, and for $\mu,\ \upsilon:[0,\ 1]\times[0,1]\to[0,\ \infty)$,

$$(\phi_1 x)(t)=\int_0^t \mu(t,\ s)x(s)ds, \qquad (\phi_2 x)(t)=\int_0^t \upsilon(t,\ s)x(s)ds.$$

Integro-differential equations naturally occur in numerous applied fields such as transport theory, acoustic scattering theory, nonlinear viscoelastic bodies, probability theory, and biological population models and systems with substantially distributed parameters. Integro-differential equations are also regarded as a “continuous analogue” to countable systems of ordinary differential equations.

The objective of this paper is to present some existence results for the Problem 1. The first result is obtained by applying Banach contraction principle, the second result is based on a fixed point theorem due to Krasnoselskii, while the third result relies on nonlinear alternative of Leray-Schauder type.

PRELIMINARIES

Let us recall some basic definitions of fractional calculus (Samko et al., 1993; Kilbas et al., 2006).

Definition 1

For $(n-1)-$times absolutely continuous function $g:[0,\infty)\to R,$ the Caputo derivative of fractional order q is defined as:

$${}^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds, \quad n-1<q<n,\ \ n=[q]+1,$$

Where $[q]$ denotes the integer part of the real number $q.$

Definition 2

The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds, \quad q>0,$$

provided the integral exists.

For the sequel, we need the following known lemma. The proof of this lemma is given in Ahmad and Sivasundaram (2012). However, for the reader’s convenience, we outline it here.

Lemma 1

Let $g:[0,\infty)\to R$ be a given continuous function. Then a unique solution of the boundary value problem

$$\begin{cases}{}^cD^q x(t)=g(t), \quad 0<t<1, \quad 1<q\le 2,\\ \delta_1 x(0)+\delta_2 x'(0)=\alpha\int_0^{\xi} x(s)ds, \quad \delta_1 x(1)+\delta_2 x'(1)=\beta\int_0^{\eta} x(s)ds,\ 0<\xi,\ \eta<1,\end{cases} \tag{2}$$

is given by

$$\begin{aligned}x(t)=&\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}g(s)ds - \alpha\, a_1(t)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds\\ &+a_2(t)\left[\beta\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds-\delta_1\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds-\delta_2\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}g(s)ds\right],\end{aligned} \tag{3}$$

Where

$$a_1(t)=\frac{1}{\Delta}\left(\delta_1+\delta_2-\frac{\beta\eta^2}{2}-(\delta_1-\beta\eta)t\right), \quad a_2(t)=\frac{1}{\Delta}\left(\delta_2-\frac{\alpha\xi^2}{2}-(\delta_1-\alpha\xi)t\right), \tag{4}$$

$$\Delta=\left[(\delta_1-\beta\eta)\left(\delta_2-\frac{\alpha\xi^2}{2}\right)-(\delta_1-\alpha\xi)\left(\delta_1+\delta_2-\frac{\beta\eta^2}{2}\right)\right]\neq 0. \tag{5}$$

Proof: For some constants $c_0,\ c_1\in R,$ it is well known that the solution of fractional differential equation in Equation 2 can be written as:

$$x(t)=I^q g(t)-c_0-c_1 t=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}g(s)ds-c_0-c_1 t. \tag{6}$$

Using the boundary conditions for Equation 2, we find that

$$(\delta_1-\alpha\xi)c_0+\left(\delta_2-\frac{\alpha\xi^2}{2}\right)c_1=-\frac{\alpha}{\Gamma(q)}\int_0^{\xi}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds, \tag{7}$$

$$(\delta_1-\beta\eta)c_0-\left(\delta_1+\delta_2-\frac{\beta\eta^2}{2}\right)c_1=-\frac{\beta}{\Gamma(q)}\int_0^{\eta}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}g(m)dm\right)ds$$
$$+\frac{\delta_1}{\Gamma(q)}\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds+\frac{\delta_2}{\Gamma(q-1)}\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}g(s)ds. \tag{8}$$

Solving Equations 7 and 8 for c_0, c_1 and substituting these values in Equation 6, we obtain Equation 3.

Let $E=C([0,1],R)$ denotes the Banach space of all continuous functions from $[0,1]$ to R endowed with the norm defined by $\|x\|=\sup\{|x(t)|,\ t\in[0,1]\}$.

In view of Lemma 1, we define an operator $\mathrm{T}:E\to E$ by

$$(\mathrm{T}x)(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))ds$$
$$-\alpha\,a_1(t)\int_0^{\xi}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))dm\right)ds$$
$$+a_2(t)\left\{\beta\int_0^{\eta}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))dm\right)ds\right.$$
$$-\delta_1\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))ds$$
$$\left.-\delta_2\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))ds\right\},\quad t\in[0,\,1]. \tag{9}$$

Observe that Problem 1 has solutions if the operator equation $x=\mathrm{T}x$ has fixed points.

For the sequel, we need the following assumptions:

$(\mathbf{A_1})$ There exist positive functions $L_1(t),\,L_2(t),\,L_3(t)$ such that

$$|f(t,\,x(t),\,(\phi_1x)(t),\,(\phi_2x)(t))-f(t,\,y(t),\,(\phi_1y)(t),\,(\phi_2y)(t))|$$
$$\le L_1(t)|x-y|+L_2(t)|\phi_1x-\phi_1y|+L_3(t)|\phi_2x-\phi_2y|,\ \ t\in[0,1],\ x,y\in R;$$

$(\mathbf{A_2})$ $\quad\Lambda=(1+\mu_0+\upsilon_0)\{p_1+\xi|\alpha\bar a_1|\,p_2+\eta|\beta\bar a_2|\,p_3+(\delta_1+\delta_2)|\bar a_2|\,p_4+\}<1,$

Where

$$\mu_0=\sup_{t\in[0,1]}\left|\int_0^t\mu(t,s)x(s)ds\right|,\ \ \upsilon_0=\sup_{t\in[0,1]}\left|\int_0^t\upsilon(t,s)x(s)ds\right|,\ \ \bar a_1=\sup_{t\in[0,1]}|a_1|,\ \ \bar a_2=\sup_{t\in[0,1]}|a_2|,$$

$$p_1=\sup_{t\in[0,1]}\{|I^qL_1(t)|,\ |I^qL_2(t)|,|I^qL_3(t)|\},\ \ p_2=\max\{|I^qL_1(\xi)|,\ |I^qL_2(\xi)|,|I^qL_3(\xi)|\},$$

$$p_3=\max\{|I^qL_1(\mu)|,\ |I^qL_2(\eta)|,|I^qL_3(\eta)|\},\ \ p_4=\max\{|I^qL_1(1)|,\ |I^qL_2(1)|,|I^qL_3(1)|\};$$

$(\mathbf{A_3})$ $\ |f(t,\,x(t),\,(\phi_1x)(t),\,(\phi_2x)(t))|\le\zeta(t),\ \ \forall(t,\,x(t),\,(\phi_1x)(t),\,(\phi_2x)(t))\in[0,\,1]\times R\times R\times R,$
$\zeta\in C([0,1],R^+).$

MAIN RESULTS

Theorem 1

Suppose that the assumptions $(\mathbf{A_1})$ and $(\mathbf{A}_2)$ hold, then the boundary value Problem 1 have a unique solution.

Proof

Selecting $\Lambda\le\kappa\le1$ (Λ is given by $(\mathbf{A}_2)$), we fix

$$r\ge\frac{M}{(1-\kappa)\Gamma(q+1)}\left\{1+\frac{|\alpha\,\bar a_1|\,\xi^{q+1}}{(q+1)}+\frac{|\bar a_2|\,\beta\eta^{q+1}}{(q+1)}+|\delta_1\bar a_2|+q|\delta_2\bar a_2|\right\},$$

Where $\sup\{|f(t,\,0,\,0,\,0)|:\,t\in[0,1]\}=M<\infty.$ Let us define $\mathrm{B}_r=\{x\in E:\|x\|\le r\}$ and show that $\mathrm{TB}_r\subset\mathrm{B}_r$. For $x\in\mathrm{B}_r$, we have

$$\|(\mathrm{T}x)(t)\|\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))\|ds$$
$$+|\alpha\,a_1(t)|\int_0^{\xi}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}\|f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))\|dm\right)ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\eta}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}\|f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))\|dm\right)d\right.$$
$$+|\delta_1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))\|ds$$
$$\left.+|\delta_2|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))\|ds\right\}$$

$$\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))-f(s,0,0,0)\|+\|f(s,0,0,0)\|)ds$$
$$+|\alpha\,a_1(t)|\int_0^{\xi}\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}(\|f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))-f(m,0,0,0)\|$$
$$+\|f(m,0,0,0)\|)dm\,ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\xi}\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}(\|f(m,\,x(m),\,(\phi_1x)(m),\,(\phi_2x)(m))-f(m,0,0,0)\|\right.$$
$$+\|f(m,0,0,0)\|)dm\,ds$$
$$+|\delta_1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}(\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))-f(s,0,0,0)\|+\|f(s,0,0,0)\|)ds$$
$$\left.+|\delta_2|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}(\|f(s,\,x(s),\,(\phi_1x)(s),\,(\phi_2x)(s))-f(s,0,0,0)\|+\|f(s,0,0,0)\|)ds\right\}$$

$$\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+L_2(s)\|(\phi_1x)(s)\|+L_3(s)\|(\phi_2x)(s)\|+M\,|)ds$$
$$+|\alpha\,a_1(t)|\int_0^{\xi}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}(L_1(m)\|x(m)\|+L_2(m)\|(\phi_1x)(m)\|+L_3(m)\|(\phi_2x)(m)\|+M\,|)dm\right)ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\eta}\left(\int_0^s\frac{(s-m)^{q-1}}{\Gamma(q)}(L_1(m)\|x(m)\|+L_2(m)\|(\phi_1x)(m)\|+L_3(m)\|(\phi_2x)(m)\|+M\,|)dm\right)ds\right.$$
$$+|\delta_1|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}(L_1(s)\|x(s)\|+L_2(s)\|(\phi_1x)(s)\|+L_3(s)\|(\phi_2x)(s)\|+M\,|)ds$$
$$\left.+|\delta_2|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}(L_1(s)\|x(s)\|+L_2(s)\|(\phi_1x)(s)\|+L_3(s)\|(\phi_2x)(s)\|+M\,|)ds\right\}$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(L_1(s)\|x(s)\|+L_2(s)\mu_0\|x(s)\|+L_3(s)\upsilon_0\|x(s)\|+M\,|\right)ds$$
$$+|\alpha\, \mathrm{a}_1(t)|\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\left(L_1(m)\|x(m)\|+L_2(m)\mu_0\|x(m)\|+L_3(m)\upsilon_0\|x(m)\|+M\,|\right)dm\right)ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\left(L_1(m)\|x(m)\|+L_2(m)\mu_0\|x(m)\|+L_3(m)\upsilon_0\|x(m)\|+M\,|\right)dm\right)\right.$$
$$+|\delta_1|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\left(L_1(s)\|x(s)\|+L_2(s)\mu_0\|x(s)\|+L_3(s)\upsilon_0\|x(s)\|+M\,|\right)ds$$
$$\left.+|\delta_2|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(L_1(s)\|x(s)\|+L_2(s)\mu_0\|x(s)\|+L_3(s)\upsilon_0\|x(s)\|+M\,|\right)ds\right\}$$

$$\leq \int_0^t I^qL_1(t)+\mu_0\, I^qL_2(t)+\upsilon_0 I^qL_3(t)+\frac{Mt^q}{\Gamma(q+1)}$$
$$+|\alpha\, \mathrm{a}_1(t)|\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\left(L_1(m)\|x(m)\|+L_2(m)\mu_0\|x(m)\|+L_3(m)\upsilon_0\|x(m)\|+M\,|\right)dm\right)ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\left(L_1(m)\|x(m)\|+L_2(m)\mu_0\|x(m)\|+L_3(m)\upsilon_0\|x(m)\|+M\,|\right)dm\right)\right.$$
$$+|\delta_1|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\left(L_1(s)\|x(s)\|+L_2(s)\mu_0\|x(s)\|+L_3(s)\upsilon_0\|x(s)\|+M\,|\right)ds$$
$$\left.+|\delta_2|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(L_1(s)\|x(s)\|+L_2(s)\mu_0\|x(s)\|+L_3(s)\upsilon_0\|x(s)\|+M\,|\right)ds\right\}$$

$$\leq \left(I^qL_1(t)+\mu_0\, I^qL_2(t)+\upsilon_0 I^qL_3(t)\right)r+\frac{Mt^q}{\Gamma(q+1)}$$
$$+|\alpha\, \mathrm{a}_1(t)|\left(\left(I^qL_1(\xi)+\mu_0\, I^qL_2(\xi)+\upsilon_0 I^qL_3(\xi)\right)\xi\, r+\frac{M\xi^{q+1}}{\Gamma(q+2)}\right)$$
$$+|\beta\, \mathrm{a}_2(t)|\left(\left(I^qL_1(\eta)+\mu_0\, I^qL_2(\eta)+\upsilon_0 I^qL_3(\eta)\right)\eta\, r+\frac{M\eta^{q+1}}{\Gamma(q+2)}\right)$$
$$+|\delta_1\, \mathrm{a}_2(t)|\left(\left(I^qL_1(1)+\mu_0\, I^qL_2(1)+\upsilon_0 I^qL_3(1)\right)r+\frac{M}{\Gamma(q+1)}\right)$$
$$+|\delta_2\, \mathrm{a}_2(t)|\left(\left(I^qL_1(1)+\mu_0\, I^qL_2(1)+\upsilon_0 I^qL_3(1)\right)r+\frac{M}{\Gamma(q)}\right)$$

$$\leq p_1(1+\mu_0+\upsilon_0)r+\frac{M}{\Gamma(q+1)}+|\alpha\, \bar{\mathrm{a}}_1|\left(p_2(1+\mu_0+\upsilon_0)\xi\, r+\frac{M\xi^{q+1}}{\Gamma(q+2)}\right)$$
$$+|\beta\, \bar{\mathrm{a}}_2|\left(p_3(1+\mu_0+\upsilon_0)\eta\, r+\frac{M\eta^{q+1}}{\Gamma(q+2)}\right)+|\delta_1\, \mathrm{a}_2(t)|\left(p_4(1+\mu_0+\upsilon_0)r+\frac{M}{\Gamma(q+1)}\right)$$
$$+|\delta_2\, \mathrm{a}_2(t)|\left(p_4(1+\mu_0+\upsilon_0)r+\frac{M}{\Gamma(q)}\right)$$

$$\leq (1+\mu_0+\upsilon_0)\{p_1+\xi\,|\alpha\,\bar{a}_1|\,p_2+\eta\,|\beta\,\bar{a}_2|\,p_3+(\delta_1+\delta_2)\,|\bar{a}_2|\,p_4\}r$$
$$+\frac{M}{\Gamma(q+1)}\left\{1+\frac{|\alpha\,\bar{a}_1|\,\xi^{q+1}}{(q+1)}+\frac{|\bar{a}_2|\,\beta\eta^{q+1}}{(q+1)}+|\delta_1\bar{a}_2|+q\,|\delta_2\bar{a}_2|\right\}$$

$$\leq (\Lambda+1-\kappa)r\leq r.$$

For each $x,\ y\in E$ and each $t\in[0,1],$ we obtain

$$\|(\mathrm{T}x)(t)-(\mathrm{T}y)(t)\|$$
$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,\ x(s),\ (\phi_1 x)(s),\ (\phi_2 x)(s))-f(s,\ y(s),\ (\phi_1 y)(s),\ (\phi_2 y)(s))\|\,ds$$
$$+|\alpha\, \mathrm{a}_1(t)|\int_0^{\xi}\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\|f(m,\ x(m),\ (\phi_1 x)(m),\ (\phi_2 x)(m))$$
$$-f(m,\ y(m),\ (\phi_1 y)(m),\ (\phi_2 y)(m))\|\,dm\,ds$$
$$+|a_2(t)|\left\{|\beta|\int_0^{\eta}\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\|f(m,\ x(m),\ (\phi_1 x)(m),\ (\phi_2 x)(m))\right.$$
$$-f(m,\ y(m),\ (\phi_1 y)(m),\ (\phi_2 y)(m))\|\,dmds$$
$$+|\delta_1|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,\ x(s),\ (\phi_1 x)(s),\ (\phi_2 x)(s))-f(s,\ y(s),\ (\phi_1 y)(s),\ (\phi_2 y)(s))\|\,ds$$
$$\left.+|\delta_2|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\|f(s,\ x(s),\ (\phi_1 x)(s),\ (\phi_2 x)(s))-f(s,\ y(s),\ (\phi_1 y)(s),\ (\phi_2 y)(s))\|\,ds\right\}$$

$$\leq \left(I^qL_1(t)+\mu_0\, I^qL_2(t)+\upsilon_0 I^qL_3(t)\right)\|x-y\|$$
$$+|\alpha\, \mathrm{a}_1(t)|\left(I^qL_1(\xi)+\mu_0\, I^qL_2(\xi)+\upsilon_0 I^qL_3(\xi)\right)\|x-y\|$$
$$+|\beta\, \mathrm{a}_2(t)|\left(I^qL_1(\eta)+\mu_0\, I^qL_2(\eta)+\upsilon_0 I^qL_3(\eta)\right)\|x-y\|$$
$$+|\delta_1\, \mathrm{a}_2(t)|\left(I^qL_1(1)+\mu_0\, I^qL_2(1)+\upsilon_0 I^qL_3(1)\right)\|x-y\|$$
$$+|\delta_2\, \mathrm{a}_2(t)|\left(I^qL_1(1)+\mu_0\, I^qL_2(1)+\upsilon_0 I^qL_3(1)\right)\|x-y\|$$

$$\leq (1+\mu_0+\upsilon_0)\{p_1+\xi\,|\alpha\,\bar{a}_1|\,p_2+\eta\,|\beta\,\bar{a}_2|\,p_3+(\delta_1+\delta_2)\,|\bar{a}_2|\,p_4\}\|x-y\|$$
$$\leq \|x-y\|,$$

Here, we have used the assumption $(\mathbf{A}_2)$ in the last inequality. Therefore, T is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem).

Our next existence result is based on Krasnoselskii's fixed point theorem (Krasnoselskii, 1955).

Theorem 2 (Krasnoselskii's fixed point theorem)

Let X_1 be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that

(i) $Ax+By\in X_1$ whenever $x,y\in X_1$;

(ii) A is compact and continuous;

(iii) B *is a contraction mapping*.

Then there exists $z\in X_1$ such that $z=Az+Bz.$

Theorem 3

Let $f:[0,1]\times R\times R\times R\to R$ be a jointly continuous function and the assumptions $(\mathbf{A}_1)$ and $(\mathbf{A}_3)$ hold with

$$(1+\mu_0+\upsilon_0)\{\xi\,|\alpha\,\bar{a}_1|\,p_2+\eta\,|\beta\,\bar{a}_2|\,p_3+(\delta_1+\delta_2)\,|\bar{a}_2|\,p_4+\}<1. \tag{10}$$

Then the boundary value Problem 1 has at least one solution on $[0, 1]$.

Proof

Fixing

$$\bar{r}\geq \frac{\|\zeta\|}{(1-\kappa)\Gamma(q+1)}\left\{1+\frac{|\alpha\,\bar{a}_1|\,\xi^{q+1}}{(q+1)}+\frac{|\bar{a}_2|\,\beta\eta^{q+1}}{(q+1)}+|\delta_1\bar{a}_2|+q\,|\delta_2\bar{a}_2|\right\},$$

Where $\sup\{\zeta(t):t\in[0,1]\}=\|\zeta\|$ ($\zeta(t)$ is defined in $(\mathbf{A}_3)$), we consider $B_{\bar{r}} = \{x \in E : \| x \| \le \bar{r}\}$. We define the operators T_1 and T_2 on $B_{\bar{r}}$

$$
\begin{aligned}
(T_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))ds\,,\\
(T_2x)(t) &= -\alpha\, a_1(t)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m,\, x(m),\, (\phi_1 x)(m),\, (\phi_2 x)(m))dm\right)ds\\
&\quad + a_2(t)\left\{\beta\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m,\, x(m),\, (\phi_1 x)(m),\, (\phi_2 x)(m))dm\right)ds\right.\\
&\quad -\delta_1\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))ds\\
&\quad \left. -\delta_2\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))ds\right\}.
\end{aligned}
$$

For $x, y \in B_{\bar{r}}$, we find that

$$
\| T_1 x + T_2 y \| \le \frac{\|\zeta\|}{(1-\kappa)\Gamma(q+1)}\left\{1 + \frac{|\alpha\bar{a}_1|\xi^{q+1}}{(q+1)} + \frac{|\bar{a}_2|\beta\eta^{q+1}}{(q+1)} + |\delta_1\bar{a}_2| + q|\delta_2\bar{a}_2|\right\} \le \bar{r}.
$$

Thus, $T_1x + T_2y \in B_{\bar{r}}$. It follows from the assumption $\mathbf{(A_1)}$ together with Equation 10 that T_2 is a contraction mapping. Continuity of f implies that the operator T_1 is continuous. Also, T_1 is uniformly bounded on $B_{\bar{r}}$ as:

$$
\| T_1 x \| \le \frac{\|\zeta\|}{\Gamma(q+1)}.
$$

Now, we prove the compactness of the operator T_1. In view of $\mathbf{(A_1)}$, we define

$$
\sup\{|f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))| : (t, x, \phi_1 x, \phi_2 x) \in [0,1]\times B_{\bar{r}}\times B_{\bar{r}}\times B_{\bar{r}}\} = M_1 < \infty.
$$

Consequently, we have

$$
\begin{aligned}
\|(T_1x)(t_1) - (T_1x)(t_2)\| &= \left\| \frac{1}{\Gamma(q)}\int_0^{t_1}\left((t_2-s)^{q-1} - (t_1-s)^{q-1}\right) f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))ds \right.\\
&\quad \left. + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1} f(s,\, x(s),\, (\phi_1 x)(s),\, (\phi_2 x)(s))ds \right\|\\
&\le \frac{M_1}{\Gamma(q+1)}\left|\left(2(t_2-t_1)\right)^q + t_1^q - t_2^q\right|,
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, T_1 is relatively compact on $B_{\bar{r}}$. Hence, by the Arzel´a-Ascoli theorem, T_1 is compact on $B_{\bar{r}}$. Thus, all the assumptions of Theorem 2 are satisfied. So, by the conclusion of Theorem 2, the boundary value Problem 1 has at least one solution on $[0, 1]$. This completes the proof.

Theorem 4

Let $f : [0, 1]\times R\times R\times R \to R$. Assume that there exist constants $0 \le \kappa_1 < 1/\rho$ such that $|f(s, x, \phi_1 x, \phi_2 x)| \le \kappa_1 |x| + M,\ M > 0,\ \forall t \in [0, 1], x \in C[0,1]$,

Where

$$
\rho = \frac{1}{\Gamma(q+1)}\left\{1 + \frac{|\alpha\bar{a}_1|\xi^{q+1}}{(q+1)} + \frac{|\bar{a}_2|\beta\eta^{q+1}}{(q+1)} + |\delta_1\bar{a}_2| + q|\delta_2\bar{a}_2|\right\},\ \kappa_1 = N(\mu_0 + \upsilon_0),\ N > 0. \tag{11}
$$

Then the boundary value Problem 1 has at least one solution.

Proof

Define a fixed point problem

$$
x = Tx, \tag{12}
$$

Where T is given by Equation 9. In view of Problem 12, we just need to prove the existence of at least one solution $x \in C[0, 1]$ satisfying Equation 12. Define a suitable ball $B_R \subset C[0, 1]$ with radius $R > 0$ as

$$
B_R = \{x \in C[0, 1],\ \max_{t\in[0,1]} |x(t)| < R\},
$$

Where R will be fixed later. Then, it is sufficient to show that $T : \bar{B}_R \to C[0, 1]$ satisfies

$$
x \ne \lambda Tx,\ \ \forall x \in \partial B_R,\ \text{ and }\ \forall\lambda \in [0, 1]. \tag{13}
$$

Let us set

$$
H(\lambda,\, x) = \lambda Tx,\quad x \in C(R),\ \ \lambda \in [0, 1].
$$

Then, by the Arzel´a-Ascoli theorem $h_\lambda(x) = x - H(\lambda,\, x) = x - \lambda Tx$ is completely continuous. If Equation 13 is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$
\begin{aligned}
\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda T, B_R, 0) = \deg(h_1, B_R, 0)\\
&= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \ne 0,\ 0 \in B_R,
\end{aligned}
$$

Where I denotes the identity operator. By the non-zero property of Leray-Schauder degree, $h_1(x) = x - \lambda \mathrm{T}x = 0$ for at least one $x \in B_R$. In order to prove Equation 13, we assume that $x = \lambda \mathrm{T}x$ for some $\lambda \in [0, 1]$ and for all $t \in [0, 1]$ so that

$$
\begin{aligned}
\| x \| &= \sup_{t\in[0,1]} |(\mathrm{T}x)(t)| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} | f(s, x(s), (\phi_1 x)(s), (\phi_2 x)(s)) | ds \\
&\quad + |\alpha\, a_1(t)| \int_0^{\xi} \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} | f(m, x(m), (\phi_1 x)(m), (\phi_2 x)(m)) | dm \right) ds \\
&\quad + |a_2(t)| \Bigg\{ |\beta| \int_0^{\eta} \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} | f(m, x(m), (\phi_1 x)(m), (\phi_2 x)(m) | dm \right) ds \\
&\quad + |\delta_1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} | f(s, x(s), (\phi_1 x)(s), (\phi_2 x)(s)) | ds \\
&\quad + |\delta_2| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} | f(s, x(s), (\phi_1 x)(s), (\phi_2 x)(s)) | ds \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\kappa_1 \| x \| + M) \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + |\alpha\, a_1(t)| \int_0^{\xi} \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} dm \right) ds \\
&+ |a_2(t)| \left[|\beta| \int_0^{\eta} \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} dm \right) ds + \delta_1 | \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} ds + |\delta_2| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} ds \right] \Bigg\} \\
&\leq (\kappa_1 \| x \| + M) \rho,
\end{aligned}
$$

which, on solving for $\| x \|$, yields

$$
\| x \| \leq \frac{M\rho}{1 - \kappa_1 \rho},
$$

Where ρ is given by Equation 11. Letting $R = \frac{M\rho}{1-\kappa_1 \rho} + 1,$ Equation 13 holds. This completes the proof.

Example: Consider the following boundary value problem

$$
\begin{cases}
{}^c D^{3/2} x(t) = \dfrac{t|x|}{4(1+|x|)} + \dfrac{1}{4} + \dfrac{1}{5} \displaystyle\int_0^1 (e^{-(s-t)}/5) x(s) ds + + \dfrac{1}{5} \int_0^1 (e^{-(s-t)/2}/5) x(s) ds, \quad t \in [0, 1], \\
x(0) + x'(0) = \dfrac{1}{4} \displaystyle\int_0^{1/4} x(s) ds, \quad x(1) + x'(1) = \dfrac{1}{6} \int_0^{1/3} x(s) ds.
\end{cases}
\tag{14}
$$

Here, $q = 3/2,\ \mu(t, s) = e^{-(s-t)}/5,\ \upsilon(t,s) = e^{-(s-t)/2}/5,\ \xi = 1/4,\ \eta = 1/3,\ \alpha = 1/4,\ \beta = 1/6,\ \delta_1 = \delta_2 = 1.$ With $\mu_0 = (e-1)/5,\ \upsilon_0 = 2(\sqrt{e}-1)/5,$ we find that

$$
\begin{aligned}
&\bar{a}_1 = 2.142302\ ,\ \bar{a}_2 = 1.067726\ , \\
&\mu_0 + \upsilon_0 = (e + 2\sqrt{e} - 3)/5,\ N = 1/5,\ \kappa_1 = N(\mu_0 + \upsilon_0) \approx 0.120629,
\end{aligned}
$$

$$
\rho = \frac{1}{\Gamma(q+1)} \left\{ 1 + \frac{|\alpha \bar{a}_1| \xi^{q+1}}{(q+1)} + \frac{|\bar{a}_2| \beta \eta^{q+1}}{(q+1)} + |\delta_1 \bar{a}_2| + q |\delta_2 \bar{a}_2| \right\} \approx 2.768726.
$$

Clearly $\kappa_1 < 1/\rho$ therefore, by Theorem 4, the boundary value Problem 14 has a solution on $[0,1]$.

Conclusions

We have investigated the existence of solutions for a four-point nonlocal integral boundary value problem of nonlinear fractional integro-differential equations of order $q \in (1, 2]$. The existence of solutions to the given problem is shown by applying Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type. The uniqueness of solutions to the problem is established by using Banach's contraction mapping principle. Our results are new and several special cases can be obtained by fixing the parameters involved in the problem. For instance, our results correspond to the ones for nonlinear fractional integro-differential equations with two-point separated boundary conditions for $\alpha = 0 = \beta.$ We obtain the existence results for a classical second-order nonlocal nonlinear four-point integral boundary value problem by taking $q = 2$ in the results of this paper.

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