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Full Length Research Paper

A new iterative method for solving absolute value equations

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In this paper, we suggest and analyze a new iterative method for solving the absolute value equations $Ax-|x|=b$, where $A\in R^{n\times n}$ is symmetric matrix, $b\in R^n$ and $x\in R^n$ is unknown. This method can be viewed as a modification of Gauss-Seidel method for solving the absolute value equations. We also discuss the convergence of the proposed method under suitable conditions. Several examples are given to illustrate the implementation and efficiency of the method. Some open problems are also suggested.

Key words: Absolute value equations, minimization technique, Gauss-Seidel method, Iterative method.

INTRODUCTION

We consider the absolute value equations of the type

$$Ax-|x|=b, \tag{1}$$

where $A\in R^{n\times n}$ is symmetric matrix $b\in R^n$ and $|x|$ will denote the vector in R^n with absolute values of components of x, where $x\in R^n$ is unknown. The absolute value Equation (1) was investigated in detail theoretically in Mangasarian et al (2006) and a bilinear program was prescribed there for the special case when the singular values of *A* are not less than one. The Equation (1) is a special case of the generalized absolute value system of equations of the type

$$Ax+B|x|=b, \tag{2}$$

which was introduced in Rohn (2004) where B is a square Matrix and further investigated in a more general form in Mangasarian (2007, 2007a). The significance of the absolute value Equation (1) arises from the fact that linear programs, quadratic programs, bimatrix games and other problems can all be reduced to an linear complementarity problem (Cottle et al., 1992; Mangasarian, 1995). Mangasarian (2007, 2009) has shown that the absolute value equations are equivalent to the linear complementarity problems. This equivalence formulation has been used by Mangasarian (2007, 2009) to solve the absolute equation and the linear complementarity problem. If B is the zero matrix, then (2) reduces to system of linear equations $Ax=b$, which have several applications in pure and applied sciences.

In this paper, we suggest and analyze an iterative method for solving the absolute value Equations (1) using minimization technique. This new iterative method can be viewed as the modified Gauss-Seidel method for solving the absolute value equations (1). The modified method is faster than the iterative method in Noor et al. (2011). In the modified method, we form a sequence which updates two component of approximate solution at the same time. This method is also called the two-step method for solving the absolute value equations. We also give some examples to illustrate the implementation and efficiency of the new proposed iterative method. It is an open problem to extend this method for solving the generalized absolute value equations of the type (2). This is another

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direction for future research. It is well known that the complemetarity problems are equivalent to the variational inequalities. This equivalence may be used to extend the new iterative method for solving the variational inequalities and related optimization problems. The interested readers are advised to explore the applications of these new methods in different areas of pure and applied sciences.

Let R^n be a finite dimension Euclidean space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. By $x^*, x_k, x_{k+1} \in R^n$ for any nonnegative integer k, we denote the exact, current approximate and later approximate solution to (1) respectively. For $x \in R^n$, *sign(x),* will denote a vector with components equal to $1, 0, -1$ depending on whether the corresponding component of *x* is positive, zero or negative. The diagonal matrix $D(x)$ is defined as $D(x) = \partial |x| = diag(sign(x))$ (Diagonal matrix corresponding to $sign(x)$), where $\partial |x|$ represent the generalized Jacobiean of $|x|$ based on a subgradient (Polyak, 1987; Rockafellar, 1971). We consider *A* such that $C = A - D(x_k)$ is positive definite. If A, $D(x_k)$ are symmetric matrices, then *C* is symmetric. For simplicity, we denote the following by

$$a = \langle Cv_1, v_1 \rangle, \tag{3}$$

$$c = \langle C v_1, v_2 \rangle = \langle C v_2, v_1 \rangle, \tag{4}$$

$$d = \langle C v_2, v_2 \rangle, \tag{5}$$

$$p_1 = \langle Ax_k - |x_k| - b, v_1 \rangle \tag{6}$$

$$p_2 = \langle Ax_k - |x_k| - b, v_2 \rangle, \tag{7}$$

where

$$v_1 \neq v_2 \in R^n, \ D(x_k) = diag(sign(x_k)) \text{ and note that } D(x_k)x_k = |x_k|, \ k = 0, 1, 2, \ldots$$

We need the following Lemma of Jing et al. (2008).

Lemma

Let a, c, d defined by Equations (1), (4) and (5) respectively satisfy the following conditions

$$a = \langle C v_1, v_1 \rangle > 0, \quad d = \langle C v_2, v_2 \rangle > 0,$$

then

$$ad - c^2 > 0.$$

NEW ITERATIVE METHOD

Here, we use the technique of updating the solution in conjunction with minimization to suggest and analyze a new iterative method for solving the absolute value Equation (1), which is the main motivation of this paper. Using the idea and technique of Ujevic (2006) as extended by Noor et al. (2011), we now construct the iteration method. For this purpose, we consider the function

$$f(x) = \langle Ax, x \rangle - \langle |x|, x \rangle - 2\langle b, x \rangle. \tag{8}$$

and

$$x_{k+1} = x_k + \alpha v_1 + \beta v_2. \quad k = 0, 1, 2, ... \quad \text{for}$$

$$v_1 \neq 0, v_2 \neq 0 \in R^n, \quad \alpha, \beta \in R \tag{9}$$

We use Equation (9) to minimize the function (8). That is, we have to show that

$$f(x_{k+1}) \leq f(x_k).$$

Now, using the Taylor series, we have

$$f(x_{k+1}) = f(x_k + \alpha v_1 + \beta v_2)$$
$$= f(x_k) + \langle f'(x_k), \alpha v_1 + \beta v_2 \rangle + \frac{1}{2}\langle f''(x_k)(\alpha v_1 + \beta v_2), \alpha v_1 + \beta v_2 \rangle. \tag{10}$$

Also, using Equation (8), we have

$$f'(x_k) = 2(Ax_k - |x_k| - b) \tag{11}$$

$$f''(x_k) = 2(A - D(x_k)) = 2C, \tag{12}$$

And

$$\langle \partial |x|, x \rangle = |x|$$

From Equations (10) to (12), we have

We have used the fact that $A - D(x_k)$,

$$f(x_k + \alpha v_1 + \beta v_2) = f(x_k) + 2\langle Ax_k - |x_k| - b, \alpha v_1 + \beta v_2 \rangle + \langle \alpha C v_1 + \beta C v_2, \alpha v_1 + \beta v_2 \rangle$$
$$= f(x_k) + 2\alpha \langle Ax_k - |x_k| - b, v_1 \rangle + 2\beta \langle Ax_k - |x_k| - b, v_2 \rangle + \alpha^2 \langle C v_1, v_1 \rangle$$
$$+ \alpha\beta \langle C v_2, v_1 \rangle + \alpha\beta \langle C v_1, v_2 \rangle + \beta^2 \langle C v_2, v_2 \rangle.$$
$$= f(x_k) + 2\alpha \langle Ax_k - |x_k| - b, v_1 \rangle + 2\beta \langle Ax_k - |x_k| - b, v_2 \rangle$$
$$+ \alpha^2 \langle C v_1, v_2 \rangle + 2\alpha\beta \langle C v_1, v_2 \rangle + \beta^2 \langle C v_2, v_2 \rangle. \tag{13}$$

is symmetric for each Equations k . Now from (1), (2), (3), (4), (5) and (13), we have

$$f(x_k+\alpha v_1+\beta v_2)=f(x_k)+2\alpha p_1+2\beta p_2+a\alpha^2+2c\alpha\beta+d\beta^2. \quad (14)$$

We define the function $h(\alpha,\beta)=2\alpha p_1+2\beta p_2+a\alpha^2+2c\alpha\beta+d\beta^2.$ To minimize $h(\alpha,\beta)$ in term of, α,β, we proceed as follows

$$\frac{\partial h}{\partial\alpha}=2p_1+2c\beta+2a\alpha=0, \quad (15)$$

$$\frac{\partial h}{\partial\beta}=2p_2+2c\alpha+2d\beta=0. \quad (16)$$

Using Lemma, it is clear that $h(\alpha,\beta)$ assumes a minimal value because

$$\left(\frac{\partial^2 h}{\partial\alpha^2}\right)\left(\frac{\partial^2 h}{\partial\beta^2}\right)-\left(\frac{\partial^2 h}{\partial\alpha\partial\beta}\right)=4(ad-c^2)>0.$$

From Equations (15) and (16), we have

$$\alpha=\frac{cp_2-dp_1}{ad-c^2}, \quad (18)$$

$$\beta=\frac{cp_1-ap_2}{ad-c^2}. \quad (19)$$

From Equations (18), (19) and (14), we have

$$f(x_k)-f(x_{k+1})=\frac{dp_1^2+ap_2^2-2cp_1p_2}{ad-c^2}. \quad (20)$$

We have to show that $f(x_k)\geq f(x_{k+1})$. If this is not true, then, for $a>0$, we have

$$0>a(dp_1^2+ap_2^2-2cp_1p_2)$$
$$=adp_1^2-c^2p_1^2+(cp_1-ap_2)^2$$
$$\geq p_1^2(ad-c^2).$$

This show that $ad-c^2<0$, which is impossible thus $f(x_k)\geq f(x_{k+1})$. This complete the proof.

We now suggest and analyze the iterative method for solving the absolute value equation.

Algorithm

Choose an initial guess $x_0\in R^n$ to Equation (1)

For $i=1,2,\ldots,n$ do

$j=i-1$

if $i=1$ then $j=n.$ stop.

For $k=0,1,2,\ldots$ do

$p_i=\langle Ax_k-|x_k|-b,e_i\rangle$

$p_j=\langle Ax_k-|x_k|-b,e_j\rangle$

$\alpha_i=\dfrac{cp_j-dp_i}{ad-c^2}$

$\beta_i=\dfrac{cp_i-ap_j}{ad-c^2}$

$x_{k+1}=x_k+\alpha_i e_i+\beta_i e_j$

do for i

Stopping criteria

do for k .

In the algorithm, we consider $v_1=e_i,\ v_2=e_j$ where j depends on $i, i\neq j,\ i=1,2,\ldots,n.$ $j=i-1$ for $i>1,$ and $j=n$ when $i=1.$ Here e_i,e_j denote the ith and jth column of identity matrix respectively. We now consider the convergence of the algorithm under the condition that $D(x_{k+1})=D(x_k)$, where $D(x_{k+1})=diag(sign(x_{k+1})),\ k=0,1,2,\ldots.$

Theorem 1

If $D(x_{k+1})=D(x_k)$ for some k, and f is defined by Equation (8), then (9) converges linearly to a solution x^* of (1) in *C*-norm.

Proof

Consider

$$\|x_{k+1}-x^*\|_C^2-\|x_k-x^*\|_C^2=\langle Cx_{k+1}-Cx^*,x_{k+1}-x^*\rangle-\langle Cx_k-Cx^*,x_k-x^*\rangle$$
$$=\langle Cx_{k+1},x_{k+1}\rangle-\langle Cx_{k+1},x^*\rangle-\langle Cx^*,x_{k+1}\rangle+\langle Cx^*,x^*\rangle-$$
$$\langle Cx_k,x_k\rangle+\langle Cx_k,x^*\rangle+\langle Cx^*,x_k\rangle-\langle Cx^*,x^*\rangle$$
$$=\langle Cx_{k+1},x_{k+1}\rangle-2\langle b,x_{k+1}\rangle-\langle Cx_k,x_k\rangle+2\langle b,x_k\rangle$$

Where we have used the fact that *C* is symmetric and $Cx^*=b.$

$$\|x_{k+1}-x^*\|_C^2-\|x_k-x^*\|_C^2=\langle Ax_{k+1}-|x_{k+1}|,x_{k+1}\rangle-2\langle b,x_{k+1}\rangle-[\langle Ax_k-|x_k|,x_k\rangle-2\langle b,x_k\rangle]$$
$$=f(x_{k+1})-f(x_k).$$

Table 1. Comarsion beween MGSM and IM

Order	No. of iterations (Prob. 1)		No. of iterations (Prob. 2)	
	MGSM	IM	MGSM	IM
10	3	4	4	7
50	3	4	4	9
100	4	5	4	8

MGSM: Modified Gauss-Seidel method; IM: Iterative method.

Using Equation (19), we have

$$\left\|x_{k+1}-x^*\right\|_C^2 \leq \left\|x_k - x^*\right\|_C^2.$$

This shows that $\{x_k\}$ is a Fejer sequence. Thus we conclude the sequence $\{x_k\}$ converges linearly to x^*, in C -norm. ⊔

In the next theorem, we compare our result with the iterative method of Noor et al. (2011)

Theorem 2

The rate of convergence of modified Gauss-Seidel method is better (at least equal) than the iterative method of Noor et al. (2011).

Proof

The iterative method (Algorithm) gives the reduction of (8) as

$$f(x_k) - f(x_{k+1}) = \frac{p_1^2}{a}. \qquad (20)$$

To compare Equations (19) and (20), subtract (20) from (19) we have

$$\frac{dp_1^2 + ap_1^2 - 2cp_1p_2}{ad-c^2} - \frac{p_1^2}{a} = \frac{(cp_1 - ap_2)^2}{a(ad-c^2)} \geq 0.$$

Hence modified Gauss-Seidel method gives better than iterative method. In other words, the rate of convergence of modified Gauss-Seidel method is better than iterative method.

Remark: If $cp_1 = ap_2$, then Algorithm 2.1 reduces to the iterative method of Noor et al. (2011).

NUMERICAL RESULTS

To illustrate the implementation and efficiency of the proposed method, we consider the following examples.

Example 1

Let *A* be a matrix whose diagonal elements are 500 and the non diagonal elements are chosen randomly from the interval [1, 2] such that *A* is symmetric. Let $b=(A-I)e$ where *I* is the identity matrix of order *n* and *e* is $n\times 1$ vector whose elements are all equal to unity such that $x=(1, 1, \ldots, 1)^T$ is the exact solution. The stopping criteria is $\|x_{k+1}-x_k\| < 10^{-6}$ and the initial guess is $x_0 = (0, 0, \ldots, 0)^T$.

Example 3.2

Let the matrix *A* be given by

$$a_{ii} = 4n, \ a_{i,i+1} = a_{i+1,i} = n, \quad a_{ij} = 0.5, \ i = 1, 2, \ldots, n, \ k = 0, 1, 2, \ldots$$

Let $b=(A-I)e$ where *I* is the identity matrix of order *n* and *e* is $n\times 1$ vector whose elements are all equal to unity such that $x=(1, 1, \ldots, 1)^T$ is the exact solution. The stopping criteria is $\|x_{k+1}-x_k\| < 10^{-6}$ and the initial guess is equal to $x_0 = (x_1, x_2, \ldots, x_n)^T$, $x_i = 0.001 * i$. The numerical results are shown in Table 1.

Conclusion

In this paper, we have used the minimization technique to suggest an iterative method for solving the absolute value equations of the form $Ax - |x| = b$. We have shown that the modified method is faster than the iterative method of Noor et al. (2011). We have also considered the convergence criteria of the new method under some suitable conditions. Some nunmerical examples are given to illustrate the efficiency and implementation of the new method for solving the absolute value equations. We remark that the absolute value problem is also equivalent to the linear variational inequalities. It is an open problem to extend this technique for solving the variational inequalities and related optimization problems. Noor (1988, 2004, 2009) and Noor et al. (1993) show the recent advances in variational inequalities.

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