

Full Length Research Paper

Solution of free non-linear vibration of beams

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In this paper, we have considered the nonlinear governing equation of tapered beams, attempt has been made to analyze the nonlinear behavior of tapered beams analytically. The nonlinear governing equation is solved by employing the variational approach method (VAM) and Improved Amplitude-Formulation (IAFF). Despite the increasing expenses of building structures to maintain their linear behavior, nonlinearity has been inevitable and therefore, nonlinear analysis has been of great importance to the scientists in the field. The major concern is to assess excellent approximations to the exact solutions for the whole range of the oscillation amplitude, reducing the respective error of angular frequency in comparison with the VAM and IAFF. The effect of vibration amplitude on the non-linear frequency is discussed. It is predicted that there can be wide application of VAM and IAFF in engineering problems, as indicated in this paper.

Key words: Non-linear vibration, analytical solution, beam vibration.

INTRODUCTION

Tapered members are widely used in high-rise buildings, long-span bridges, aerospace vehicles, steel braced frames equipped with ADAS devices and other energy dissipating devices under the far field and near field records (Bayat and Abdollahzadeh, 2011), etc. The study on the nonlinear vibration of tapered beams has been widely mentioned in the past few decades (Goorman, 1975; Evensen, 1968; Pillai and Rao, 1992). As the amplitude of oscillation increases, these structures are subjected to non-linear vibrations which often lead to material fatigue and structural damage.

Obtaining the natural frequencies of the systems become more significant because of the result of these effects. Therefore, it is very important to provide an accurate analysis towards the understanding of the non-linear vibration characteristics of these structures. Generally, it is extremely difficult to find an exact or closed-form solution for nonlinear equations. Many researchers have been concentrated on approximate analytical methods (Bayat et al., 2010, 2011a, b, c, d; Pakar et al., 2011; Kimiaefar et al., 2009; Kimiaefar, 2010; Ibsen et al., 2010; Momeni et al., 2011).

The main objective of this study is to obtain analytical expressions for geometrically non-linear tapered beams.

In dimensionless form, Goorman is given the governing differential equation corresponding to fundamental vibration mode of a tapered beam (Goorman, 1975):

$$\left(\frac{d^2u}{dt^2}\right) + \varepsilon_1 \left(u^2 \left(\frac{d^2u}{dt^2}\right) + u \left(\frac{du}{dt}\right)^2\right) + u + \varepsilon_2 u^3 = 0 \quad (1)$$

Where u is displacement and ε_1 and ε_2 are arbitrary constants. Subject to the following initial conditions:

$$u(0) = A, \quad \frac{du(0)}{dt} = 0 \quad (2)$$

Basic idea of He's variational approach method

He suggested a variational approach which is different from the known variational methods in open literature (He, 2007). Hereby we give a brief introduction of the method:

$$u'' + f(u) = 0 \quad (3)$$

Its variational principle can be easily established utilizing the semi-inverse method (He, 2007):

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$$J(u) = \int_0^{T/4} \left(-\frac{1}{2} u'^2 + F(u) \right) dt \tag{4}$$

Where T is period of the nonlinear oscillator, $\frac{\partial F}{\partial u} = f$. Assume that its solution can be expressed as:

$$u(t) = A \cos(\omega t) \tag{5}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Equation (5) into Equation (4) results in:

$$\begin{aligned} J(A, \omega) &= \int_0^{T/4} \left(-\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A \cos \omega t) \right) dt \\ &= \frac{1}{\omega} \int_0^{\pi/2} \left(-\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A \cos \omega t) \right) dt \\ &= -\frac{1}{2} A^2 \omega \int_0^{\pi/2} \sin^2 t dt + \frac{1}{\omega} \int_0^{\pi/2} F(A \cos \omega t) dt \end{aligned} \tag{6}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A} = 0 \tag{7}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

But with a careful inspection, for most cases we find that:

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2} A^2 \int_0^{\pi/2} \sin^2 t dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A \cos \omega t) dt < 0 \tag{9}$$

Thus, we modify conditions Equations (7) and (8) into a simpler form:

$$\frac{\partial J}{\partial \omega} = 0 \tag{10}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

Basic idea of improved amplitude-frequency formulation

We consider a generalized nonlinear oscillator in the form (He, 2008):

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{11}$$

We use the following trial functions

$$u_1(t) = A \cos(\omega_1 t), \tag{12}$$

And

$$u_2(t) = A \cos(\omega_2 t), \tag{13}$$

The residuals are

$$R_1(\omega t) = -A \omega_1^2 \cos(\omega_1 t) + f(A \cos(\omega_1 t)), \tag{14}$$

And

$$R_2(\omega t) = -A \omega_2^2 \cos(\omega_2 t) + f(A \cos(\omega_2 t)), \tag{15}$$

The original frequency-amplitude formulation reads (He, 2004, 2006):

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1}, \tag{16}$$

He used the following formulation (He, 2004, 2006) and Geng and Cai (2007) improved the formulation by choosing another location point.

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = 0) - \omega_2^2 R_1(\omega_1 t = 0)}{R_2 - R_1}, \tag{17}$$

This is the improved form by Geng and Cai (2007).

$$\omega^2 = \frac{\omega_1^2 R_2(\omega_2 t = \pi/3) - \omega_2^2 R_1(\omega_1 t = \pi/3)}{R_2 - R_1}, \tag{18}$$

The point is: $\cos(\omega_1 t) = \cos(\omega_2 t) = k$

Substituting the obtained ω into $u(t) = A \cos(\omega t)$, we can obtain the constant k in ω^2 equation in order to have the frequency without irrelevant parameter.

To improve its accuracy, we can use the following trial function when they are required.

$$u_1(t) = \sum_{i=1}^m A_i \cos(\omega_i t),$$

and

$$u_2(t) = \sum_{i=1}^m A_i \cos(\Omega_i t), \tag{19}$$

or

$$\text{and } u_2(t) = \frac{\sum_{i=1}^m A_i \cos(\Omega_i t)}{\sum_{j=1}^m B_j \cos(\Omega_j t)}, \tag{20}$$

But in most cases because of the sufficient accuracy, trial functions are as follow and just the first term:

$$u_1(t) = A \cos t, \text{ and}$$

$$u_2(t) = a \cos(\omega t) + (A - a) \cos(2\omega t), \tag{21}$$

And

$$u_1(t) = A \cos t, \text{ and } u_2(t) = \frac{A(1+c) \cos(\omega t)}{1+c \cos(2\omega t)}, \tag{22}$$

Where a and c are unknown constants. In addition we can set: $\cos t = k$ in u_1 , and $\cos(\omega t) = k$ in u_2

APPLICATIONS

Solution using VAM

Its variational formulation can be readily obtained as follows:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2} \left(\frac{du}{dt} \right)^2 - \frac{1}{2} \varepsilon_1 \left(\frac{du}{dt} \right)^2 u^2 + \frac{1}{2} u^2 + \frac{1}{4} \varepsilon_2 u^4 \right) dt \tag{23}$$

Choosing the trial function $u(t) = A \cos(\omega t)$ into Equation (23) we obtain

$$J(A, \omega) = \int_0^{T/4} \left(-\frac{1}{2} (A \omega \sin(\alpha))^2 - \frac{1}{2} \varepsilon_1 (A \omega \sin(\alpha))^2 (A \cos(\alpha))^2 + \frac{1}{2} (A \cos(\alpha))^2 + \frac{1}{4} \varepsilon_2 (A \cos(\alpha))^4 \right) dt \tag{24}$$

The stationary condition with respect to A reads:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(-A \omega^2 \sin^2(\omega t) - 2 \omega^2 A^3 \sin^2(\omega t) \cos^2(\omega t) + A \cos^2(\omega t) + \varepsilon_1 A^3 \cos^4(\omega t) \right) dt = 0 \tag{25}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(-A \omega^2 \sin^2 t - 2 \varepsilon_1 \omega^2 A^3 \sin^2 t \cos^2 t + A \cos^2 t + \varepsilon_2 A^3 \cos^4 t \right) dt = 0 \tag{26}$$

Which leads to the result

$$\omega^2 = \frac{\int_0^{\pi/2} (A \cos^2 t + \varepsilon_2 A^3 \cos^4 t) dt}{\int_0^{\pi/2} (A \sin^2 t + 2 \varepsilon_1 A^3 \sin^2 t \cos^2 t) dt} \tag{27}$$

Solving Equation (27), according to ω , we have:

$$\omega = \frac{\sqrt{2} \sqrt{(\varepsilon_1 A^2 + 2)(3 \varepsilon_2 A^2 + 4)}}{2(\varepsilon_1 A^2 + 2)} \tag{28}$$

Hence, the approximate solution can be readily obtained:

$$u(t) = A \cos \left(\frac{\sqrt{2} \sqrt{(2 + \varepsilon_1 A^2)(4 + 3 \varepsilon_2 A^2)}}{2(\varepsilon_1 A^2 + 2)} t \right) \tag{29}$$

Solution using IAFF

We use trial functions, as follows:

$$u_1(t) = A \cos t, \tag{30}$$

And

$$u_2(t) = A \cos(2t), \tag{31}$$

Respectively, the residual equations are:

$$R_1(t) = -A^3 \cos(t) (2 \varepsilon_1 \cos^2(t) - \varepsilon_2 \cos^2(t) - \varepsilon_1), \tag{32}$$

And

$$R_2(t) = -A \cos(2t) (8 \varepsilon_1 A^2 \cos^2(2t) - \varepsilon_2 A^2 \cos^2(2t) - 4 \varepsilon_1 A^2 + 3) \tag{33}$$

Considering $\cos t_1 = \cos 2t_2 = k$, we have:

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = \frac{\varepsilon_2 A^2 k^2 + 1}{2 A^2 k^2 \varepsilon_1 - A^2 \varepsilon_1 + 1}, \tag{34}$$

We can rewrite $u(t) = A \cos(\omega t)$ in the form:

$$u(t) = A \cos \left(\sqrt{\frac{\varepsilon_2 A^2 k^2 + 1}{2 A^2 k^2 \varepsilon_1 - A^2 \varepsilon_1 + 1}} t \right), \tag{35}$$

In view of the approximate solution, we can rewrite the main equation in the form:

$$\left(\frac{d^2 u}{dt^2} \right) + \left(\frac{\varepsilon_2 A^2 k^2 + 1}{2 A^2 k^2 \varepsilon_1 - A^2 \varepsilon_1 + 1} \right) u = -\varepsilon_1 \left(u^2 \left(\frac{d^2 u}{dt^2} \right) + u \left(\frac{du}{dt} \right)^2 \right) - \varepsilon_1 u^3 + \left(\frac{\varepsilon_2 A^2 k^2}{2 A^2 k^2 \varepsilon_1 - A^2 \varepsilon_1 + 1} \right) u = 0 \tag{36}$$

If by any chance Equation (35) is the exact solution, then the right side of Equation (36) vanishes completely. Considering our approach which is just an approximation one, we set:

$$\int_0^{T/4} \left(-\varepsilon_1 \left(u^2 \left(\frac{d^2 u}{dt^2} \right) + u \left(\frac{du}{dt} \right)^2 \right) - \varepsilon_1 u^3 + \left(\frac{\varepsilon_2 A^2 k^2}{2 A^2 k^2 \varepsilon_1 - A^2 \varepsilon_1 + 1} \right) u \right) \cos \omega t dt = 0 \tag{37}$$

Where $T = 2\pi/\omega$. Substituting the Equation (35) in (37) and solving the integral term, we have:

$$k = \frac{\sqrt{2} \sqrt{(2 \varepsilon_1^2 \omega^2 A^2 - 2 \varepsilon_1 \omega^2 - 4 \varepsilon_1 + 3 \varepsilon_2 - 3 \varepsilon_2 A^2 \varepsilon_1)}}{2 \sqrt{(2 \varepsilon_1^2 \omega^2 A^2 - 4 \varepsilon_1 + 2 \varepsilon_2 - 3 \varepsilon_2 A^2 \varepsilon_1)}}, \tag{38}$$

So, substituting Equation (39) into (34), and simplifying, we have:

$$\omega = \frac{1}{2} \sqrt{\frac{3 \varepsilon_2 A^2 + 4}{\varepsilon_1 A^2 + 2}} \tag{39}$$

Table 1. Comparison of frequency corresponding to various parameters of system.

Constant parameters		Approximate solution		Exact solution	Relative error (%)
A	ε_1	ε_2	$\omega_{VAM=IAFF}$	ω_{Exact}	$\frac{\omega_{VAM=IAFF} - \omega_{Ex}}{\omega_{Ex}}$
0.5	0.1	0.1	1.003082	1.003094	0.00119
0.5	1	0.2	0.960324	0.961203	0.09142
1	0.5	1	1.183216	1.18593	0.22885
1	1	0.5	0.957427	0.96655	0.94387
2	1	1	1.154701	1.205874	4.24368
2	2	5	1.788854	1.915915	6.63184
3	5	1	0.574271	0.644779	10.9353
5	10	5	0.86717	0.994212	12.7781
10	20	50	1.935782	2.232558	13.2931

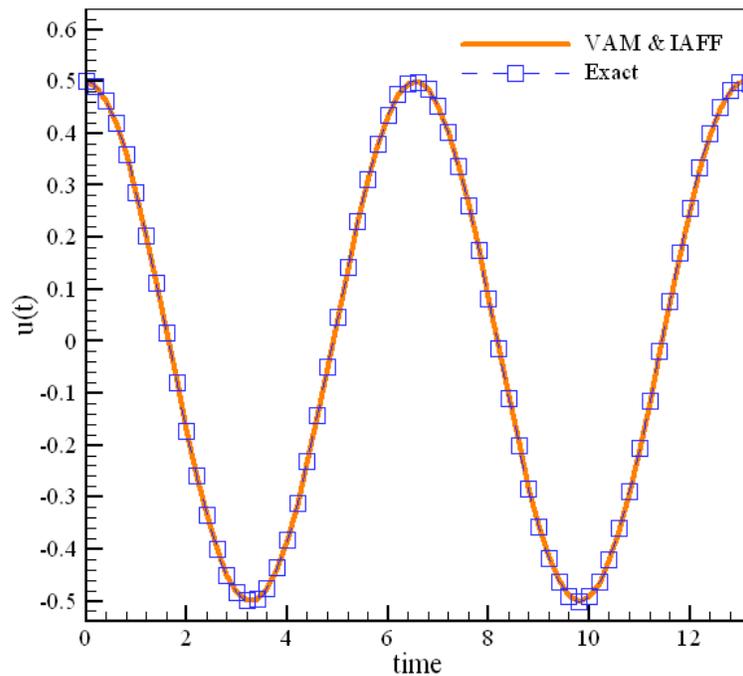


Figure 1. Comparison of analytical solution of $u(t)$ based on time with the exact solution for $\varepsilon_1=1, \varepsilon_2=0.2, A=0.5$.

RESULTS AND DISCUSSION

To illustrate and verify the accuracy of the Variational approach method and Improved Amplitude-Formulation (IAFF), comparison with published data and exact solutions is presented. The exact frequency ω_e for a dynamic system governed by Equation (1) can be derived, as shown in Equation (40), as follows:

$$\omega_{Exact} = 2\pi / 4\sqrt{2}A \int_0^{\pi/2} \frac{\sqrt{1+\varepsilon_1 A^2 \cos^2 t} \sin t}{\sqrt{A^2(1-\cos^2 t)(\varepsilon_2 A^2 \cos^2 t + \varepsilon_2 A^2 + 2)}} dt \quad (40)$$

The results obtained by VAM and IAFF are tabulated in Table 1 for different value of A, ε_1 and ε_2 . Figures 1 and 2 shows the displacement of the system for different

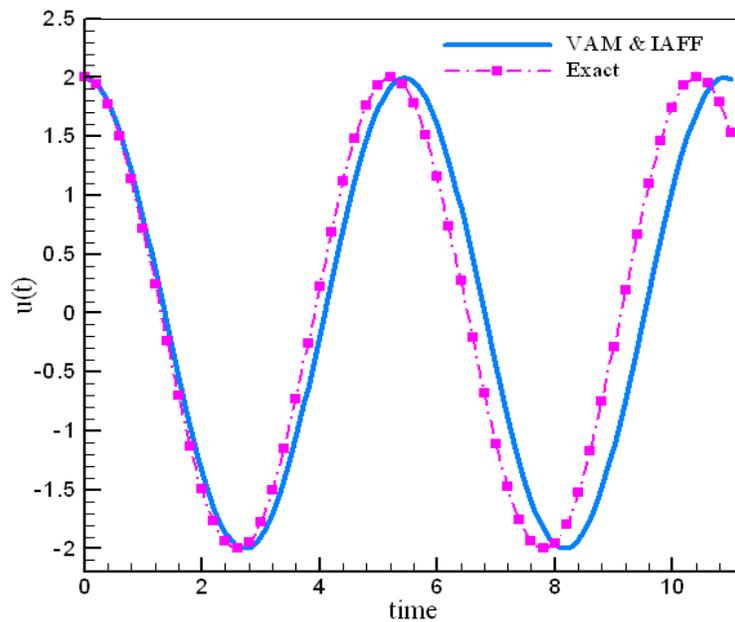


Figure 2. Comparison of analytical solution of $u(t)$ based on time with the exact solution for $\varepsilon_1=1$, $\varepsilon_2=1$, $A=2$.

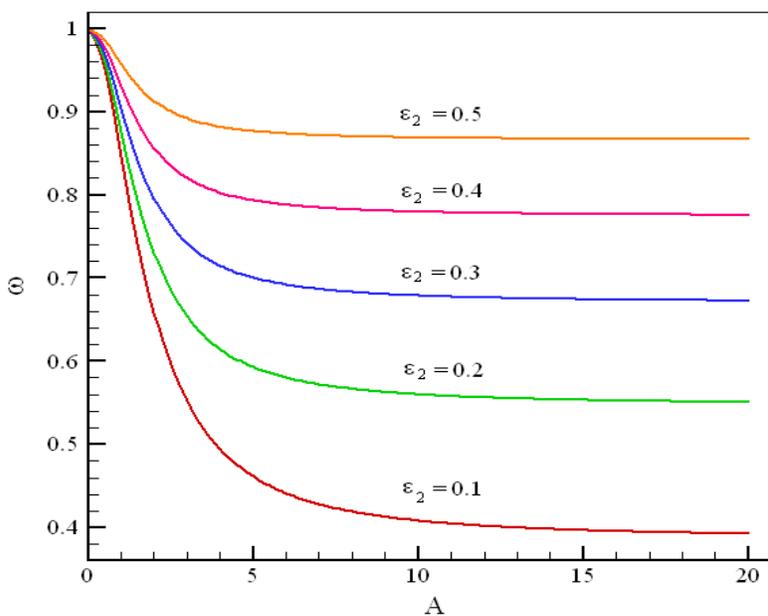


Figure 3. Comparison of frequency corresponding to various parameters of amplitude (A) $\varepsilon_1=1$.

parameter of A , ε_1 and ε_2 .

It can be seen from Figures 1 and 2 VAM and IAFF results have a good agreement with the exact solution. The effect of small parameters ε_1 on the frequency

corresponding to various parameters of amplitude (A) has been studied in Figure 3 and for ε_2 in Figure 4.

Figure 5 represents the phase plane for this problem obtained from VAM and IAFF for $\varepsilon_2 = 0.1$ to $\varepsilon_2 = 0.5$. It

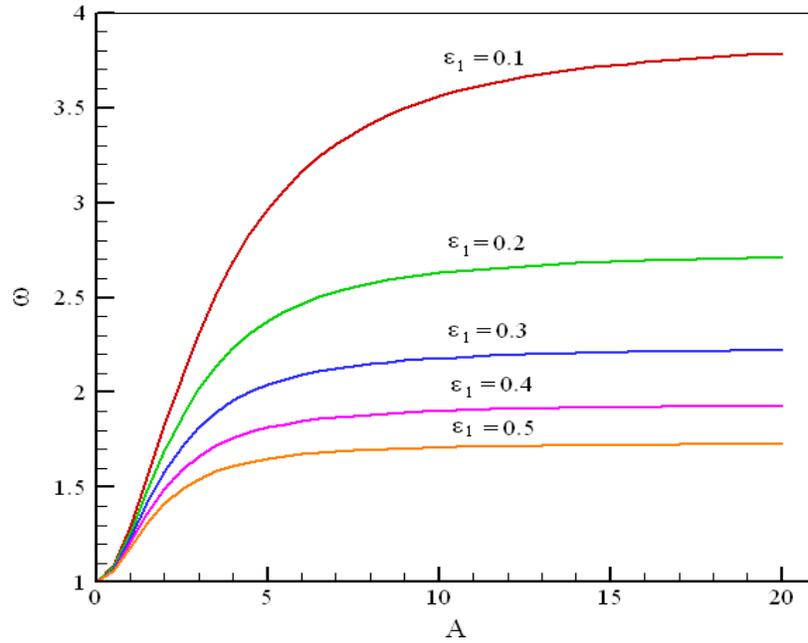


Figure 4. Comparison of frequency corresponding to various parameters of amplitude (A) $\varepsilon_2 = 1$.

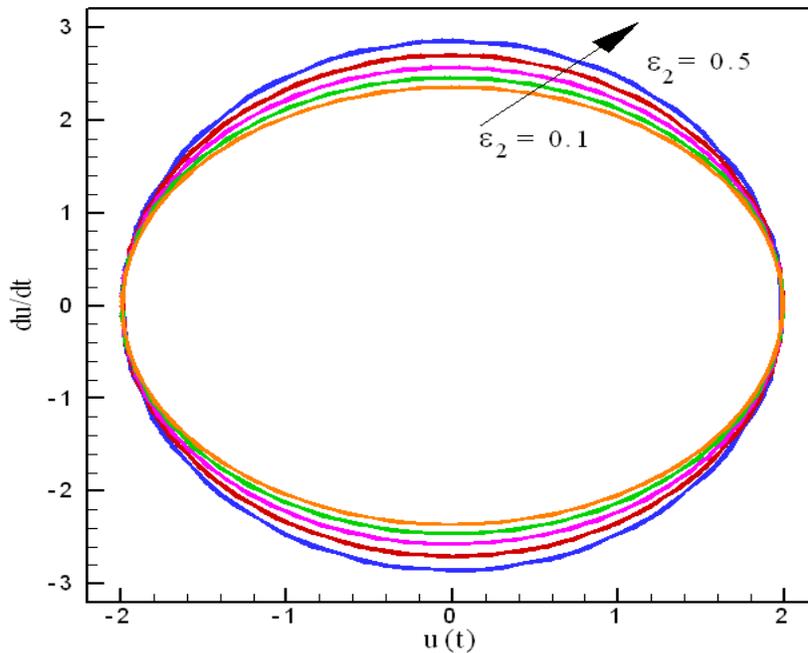


Figure 5. Phase plane, for $A = 2, \varepsilon_1 = 0.5$.

is evident that VAM and IAFF show excellent agreement with the numerical solution using the exact solution and quickly convergent and valid for a wide range of vibration

amplitudes and initial conditions. The accuracy of the results shows that the VAM and IAFF can be potentially used for the analysis of strongly nonlinear oscillation

problems accurately.

Conclusions

In this paper, the Variational approach method and Improved Amplitude-Formulation (IAFF) have been used to obtain analytical solutions for non-linear oscillation of tapered beams. The results of the first iteration led to an excellent solution and both methods provide the same analytical approximations for the nonlinear differential equations. It is obvious from the figures the comparison of these methods with numerical results reveals that the approximations obtained by the VAM and IAFF quickly converge to an exact solution. The VAM and IAFF do not require small parameters in the equations, so the limitations of the traditional perturbation methods can be eliminated. As shown in this study excellent agreement between approximate frequencies and the exact one are demonstrated and discussed. The authors suggest that VAM and IAFF are strong and novel methods to determine approximate periodic solutions for studying of nonlinear oscillators.

REFERENCES

- Bayat M, Abdollahzadeh GR (2011). Analysis of the steel braced frames equipped with ADAS devices under the far field records. *Latin Am. J. Solids Struct.* (In press).
- Bayat M, Shahidi M, Barari A, Domairry G (2010). The Approximate Analysis of Nonlinear Behavior of Structure Under Harmonic Loading. *Int. J. Phys. Sci.*, 5(7): 1074-1080.
- Bayat M, Shahidi M, Barari A, Domairry G (2011a). Analytical Evaluation of the Nonlinear Vibration of Coupled Oscillator Systems. *J. Phys. Sci.*, 1(66a): 1-16.
- Bayat M, Shahidi M, Abdollahzadeh GR (2011b). Analytical Solutions for Free Vibrations of a Mass Grounded by Linear and Nonlinear Springs in Series Using Energy Balance Method and Homotopy Perturbation Method. *J. Appl. Funct. Anal.*, 6(2): 182-194.
- Bayat M, Barari A, Shahidi M (2011c). Dynamic Response of Axially Loaded Euler-Bernoulli Beams. *Sci. J. Mechanika* (In Press).
- Bayat M, Bayat M, Bayat M (2011d). An Analytical Approach on a Mass Grounded by Linear and Nonlinear Springs in Series. *Int. J. Phys. Sci.*, 6(2): 229-236.
- Evensen DA (1968). Nonlinear Vibrations of Beams with Various Boundary Conditions. *AIAA J.*, 6: 370-372.
- Geng L, Cai XC (2007). He's Frequency Formulation for Nonlinear Oscillators. *Eur. J. Phys.*, 28: 923-993.
- Goorman DJ (1975). *Free Vibrations of Beams and Shafts*. Wiley, New York Appl. Mech., ASME, 18: 135-139.
- He JH (2004). Solution of Nonlinear Equations by an Ancient Chinese Algorithm. *Appl. Math. Comput.*, 151(2004): 293-297.
- He JH (2006). Some Asymptotic Methods for Strongly Nonlinear Equations. *Int. J. Mod. Phys. B.*, 20(2006): 1141-1199.
- He JH (2007). Variational Approach for Nonlinear Oscillators. *Chaos. Soliton. Fractals*, 34(5): 1430-1439.
- He JH (2008). An Improved Amplitude-Frequency Formulation for Nonlinear Oscillators. *Int. J. Nonlin. Sci. Numer. Simul.*, 9(2): 211-212.
- Kimiaieifar A, Saidi AR, Bagheri GH, Rahimpour M, Domairry DG (2009). Analytical solution for Van der Pol -Duffing oscillators. *Chaos Solitons Fractals*, 42(5): 2660-2666.
- Kimiaieifar A (2010). An analytical approach to investigate the response and stability of Van der Pol-Mathieu-Duffing oscillators under different excitation functions. *Math. Methods Appl. Sci.*, 33(13): 1571-1577.
- Ibsen LB, Barari A, Kimiaieifar A (2010). Analysis of highly nonlinear oscillation systems using He's max-min method and comparison with homotopy analysis and energy balance methods. *Sadhana. Acad. Proc. Eng. Sci.*, 35(4): 433-448.
- Momeni M, Jamshidi N, Barari A, Ganji DD (2011). Application of He's energy balance method to Duffing-harmonic oscillators. *Int. J. Comput. Math.*, 88(1): 135-144.
- Pakar I, Shahidi M, Ganji DD, Bayat M (2011). Approximate Analytical Solutions for Nonnatural and Nonlinear Vibration Systems Using He's Variational Approach Method. *J. Appl. Funct. Anal.*, 6(2): 225-232.
- Pillai SRR, Rao BN (1992). On Nonlinear Free Vibrations of Simply Supported Uniform Beams. *J. Sound Vib.*, 159: 527-531.