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The partial eigenvalue assignment for non-symmetric quadratic pencil in multi-input case

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In this paper, we propose a parametric solution to the partial eigenvalue assignment problem by state feedback control for non-symmetric quadratic pencil in multi-input case using orthogonality relations between eigenvectors. Our solution can be implemented with only a partial knowledge of the spectrum and the corresponding eigenvectors of non-symmetric quadratic pencil. We show that the number of eigenvalues and eigenvectors that need to remain unchanged will not be affected by feedback matrices. A numerical example is given to illustrate the proposed method.

Key-words: Non-symmetric quadratic pencil, eigenvectors, eigenvalues.

INTRODUCTION

The matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form:

$$M\frac{d^2}{dt^2}v+(D+G)\frac{d}{dt}v+Kv=0 \tag{1}$$

where $v(t)\in R^{n\times 1}$ M, D, G and K are respectively mass, damping, gyroscopic and stiffness matrices (Datta et al., 2000; Datta and Sarkissian, 1999). The system represented by Equation 1 is called damped gyroscopic system. The gyroscopic matrix G is always skew-symmetric $(G=-G^T)$; the mass matrix M can be assumed to be symmetric and positive definite ($M=M^T>0$). In special cases where D and K are also symmetric, then the system of Equation 1 is called symmetric definite system. If the gyroscopic force is not present, then the system is called non-gyroscopic (Datta and Sarkissian, 1999).

The system of Equation 1 leads, with the separation of variables $v(t)=xe^{\lambda t}$, x a constant vector to the problem of finding the eigenvalues and eigenvectors of the non-symmetric quadratic pencil:

$$P(\lambda)=\lambda^2 M+\lambda(D+G)+K \tag{2}$$

The scalar λ_i is called an eigenvalue, and the corresponding vector $x_i\neq 0$ is called an eigenvector if they satisfy:

$$P(\lambda_i)x_i=0 \qquad i=1,2,3,\ldots,2n$$

For notational convenience, we write $C=D+G$, throughout the rest of the paper.

The system modeled by Equation 1 can be controlled with the application of a forcing function $Bu(t)$, in which case Equation 1 is replaced by:

$$M\frac{d^2}{dt^2}v+C\frac{d}{dt}v+Kv=Bu(t) \tag{3}$$

The system of Equation 3 is called the time-invariant second-order control system in multi-input case that arises naturally in a wide variety of applications (Nichols and Kautsky, 2001), where a matrix $B\in R^{n\times m}$ is multi-input matrix (if m=1 single input) and $u(t)$ is a time dependent $m\times 1$ vector that needs to be applied to

Equation 3. Let $u(t)$ be chosen as:

$$u(t) = F_1^T \frac{d}{dt} v + F_2^T v, \qquad (4)$$

$F_1, F_2 \in R^{n\times m}$ are constants that lead to the closed-loop system:

$$M \frac{d^2}{dt^2} v + \left(C - BF_1^T\right)\frac{d}{dt} v + \left(K - BF_2^T\right)v = 0, \qquad (5)$$

The closed-loop matrix quadratic pencil corresponding to equation 5 is:

$$P_c(\lambda) = \lambda^2 M + \lambda\left(C - BF_1^T\right) + \left(K - BF_2^T\right) \qquad (6)$$

Given m complex numbers $\{\mu_1, \mu_2, \cdots, \mu_m\}$ closed under complex conjugation, $m \le n$ and the matrix $B \in R^{n\times m}$ were required to find $F_1, F_2 \in R^{n\times m}$ such that the closed loop pencil of Equation 6 has spectrum:

$$\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_{2n}\}. \qquad (7)$$

This is the partial eigenvalue assignment problem in which we use the matrices $F_1, F_2 \in R^{n\times m}$ to replace the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ of the open loop pencil:

$$P(\lambda) = M\lambda^2 + C\lambda + K,$$

by $\{\mu_1, \mu_2, \cdots, \mu_m\}$, while leaving the other eigenvalues unchanged.

Recently, Datta et al. (2000) and Datta and Sarkissian (1999) introduced the parametric solution to the partial eigenvalue assignment problem for symmetric quadratic pencil in multi-input case. Datta (1999) and Datta et al. (1997) introduced an explicit solution to the partial eigenvalue assignment problem for symmetric quadratic pencil in single input case. This solution can be implemented with only a partial knowledge of the spectrum and the corresponding eigenvectors of the damped non-gyroscopic $P(\lambda) = \lambda^2 M + \lambda D + K$, where $\left(M = M^T > 0,\ D = D^T, G = 0, K = K^T\right)$. Ramadan and El-Sayed (2010a, b) introduced an explicit solution to the partial eigenvalue assignment problem for non symmetric quadratic pencil in single input case.

In this paper, we introduce the parametric solution to the partial eigenvalue assignment problem for damped case, such that C and K are non-symmetric matrices gyroscopic second-order control system in multi-input and $M = M^T > 0$. This solution can be implemented with only a partial knowledge of the spectrum and the corresponding eigenvectors of $P(\lambda) = \lambda^2 M + \lambda C + K$.

It is well known (Datta et al., 1997; Laub and Arnold, 1984) that the system of equation 3 is completely controllable if and only if *rank* $\{\lambda^2 M + \lambda C + K, B\} = n,$ for every eigenvalue λ of the pencil (Equation 2). Complete controllability is necessary and sufficient for the existence of F_1 and F_2, such that the closed-loop pencil has a spectrum that can be assigned arbitrarily. However, if the system is only partially controllable, that is, if *rank* $\{\lambda^2 M + \lambda C + K, B\} = n,$ only for m of the eigenvalues $\lambda = \lambda_j, \quad j = 1, 2, \cdots, m. \quad m < n,$ of the pencil, then only those eigenvalues can be arbitrarily assigned by an appropriate choice of F_1 and F_2.

ORTHOGONALITY RELATIONS BETWEEN THE EIGENVECTORS OF QUADRATIC PENCIL

Here, we introduce three orthogonality relations (Sarkissian, 2001; Ramadan and El-Sayed, 2010a, b) between the eigenvectors of non-symmetric definite quadratic pencil. One of these results plays a key role in our later developments (Datta et al., 1997; Datta and Sarkissian, 2001) as well as the well known results on the orthogonality relations between eigenvectors of symmetric definite quadratic pencil (Datta et al., 1997; Datta and Sarkissian, 2001) that follows as special cases.

Definition 1

A scalar $\lambda \in C$ such that $\det(P(\lambda)) = 0$ is called an eigenvalue of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$. The set of eigenvalues is called the spectrum of $P(\lambda)$ (Sarkissian, 2001).

Definition 2

The non-zero matrices x and y are, respectively, called the right and left eigenvectors, corresponding to the eigenvalue λ of the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ (Sarkissian, 2001) if

$$\left(\lambda^2 M + \lambda C + K\right)x = 0 \qquad (8)$$

and

$$y^H\left(\lambda^2 M+\lambda C+K\right)=0 \qquad (9)$$

where y^H is the conjugate transpose of the vector y.

Theorem 1: Orthogonality of the eigenvectors of non-symmetrical quadratic pencil

Let $\lambda_1,\lambda_2,\cdots,\lambda_{2n}$ be the eigenvalues of the $n\times n$ quadratic pencil $P(\lambda)=\lambda^2 M+\lambda C+K$ and let X and Y be, respectively the right and left eigenvector matrices. Assume that $\{\lambda_1,\cdots,\lambda_m\}\bigcap\{\lambda_{m+1},\cdots,\lambda_{2n}\}=\Phi$, partition $X=\left(X_1,X_2\right)$ and $Y=\left(Y_1,Y_2\right)$, where $X_1=\left(x_1,\cdots,x_m\right)$; $X_2=\left(x_{m+1},\cdots,x_{2n}\right)$, $Y_1=\left(y_1,\cdots,y_m\right)$ and $Y_2=\left(y_{m+1},\cdots,y_{2n}\right)$ (Sarkissian, 2001). Then,

$$\Lambda_1 Y_1^H M X_2\Lambda_2-Y_1^H K X_2=0 \qquad (10)$$

and

$$\Lambda_1 Y_1^H M X_2+Y_1^H M X_2\Lambda_2+Y_1^H C X_2=0 \qquad (11)$$

where $\Lambda=diag\left(\Lambda_1,\Lambda_2\right)$ with $\Lambda_1=diag\left(\lambda_1,\cdots,\lambda_m\right)$ and $\Lambda_2=diag\left(\lambda_{m+1},\cdots,\lambda_{2n}\right)$.

PARTIAL EIGENVALUE ASSIGNMENT PROBLEM

Suppose $C,K\in R^{n\times n}$ are non-symmetric matrices and $M=M^T>0$ is non-singular matrix. Let the non-symmetric quadratic pencil $P(\lambda_i)x_i=\left(\lambda_i^2 M+\lambda_i C+K\right)x_i=0 \quad i=1,2,\cdots,2n$, be written in the matrix form as follows:

$$MX\Lambda^2+CX\Lambda+KX=0 \qquad (12)$$

where $X=\left(x_1,x_2,\cdots,x_{2n}\right)\in C^{n\times 2n}$, $\Lambda=diag\left(\lambda_1,\lambda_2,\cdots,\lambda_{2n}\right)\in C^{2n\times 2n}$ and λ_i are distinct. Let us partition the $n\times 2n$ right eigenvector matrix X, the $2n\times n$ left eigenvector matrix Y^H and $2n\times 2n$ eigenvalues matrix Λ as follows:

$$X=\left(X_1\quad X_2\right),\quad Y^H=\begin{pmatrix}Y_1^H\\ Y_2^H\end{pmatrix},\quad \Lambda=diag\left(\Lambda_1,\Lambda_2\right)$$

where $X_1=\left(x_1,\cdots,x_m\right)$, $X_2=\left(x_{m+1},\cdots,x_{2n}\right)$, $Y_1=\left(y_1,\cdots,y_m\right)$ and $Y_2=\left(y_{m+1},\cdots,y_{2n}\right)$ with $\Lambda_1=diag\left(\lambda_1,\cdots,\lambda_m\right)$ and $\Lambda_2=diag\left(\lambda_{m+1},\cdots,\lambda_{2n}\right)$.

Theorem 2

If $\{\lambda_1,\cdots,\lambda_m\}\cap\{\lambda_{m+1},\cdots,\lambda_{2n}\}=\Phi$ and the feedback matrices F_1 and F_2 is defined by,

$$F_1^T=\beta\Lambda_1 Y_1^H M \quad F_2^T=-\beta Y_1^H K,\quad \beta\in C^{m\times m}. \qquad (13)$$

Then, for any choice of β, the last $2n-m$ eigenvalues $\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_{2n}$ of the closed loop pencil $P_c(\lambda)=M\lambda^2+\left(C-BF_1^T\right)\lambda+\left(K-BF_2^T\right)$, are same as those of the open loop pencil $P(\lambda)=M\lambda^2+C\lambda+K$.

Proof

Let (X,Λ) be the eigenvector-eigenvalue matrix pair of the open loop pencil,

$$P(\lambda)=M\lambda^2+C\lambda+K. \qquad (14)$$

Then,

$$MX\Lambda^2+CX\Lambda+KX=0. \qquad (15)$$

Let X and Y be, respectively the right and the left eigenvector matrices of the open loop pencil. Our goal is to prove that:

$$MX_2\Lambda_2^2+\left(C-BF_1^T\right)X_2\Lambda_2+\left(K-BF_2^T\right)X_2=0 \qquad (16).$$

By substituting $F_1^T=\beta\Lambda_1 Y_1^H M$ and $F_2^T=-\beta Y_1^H K$ in the left hand side of Equation 16, then we obtain:

$$MX_2\Lambda_2^2 + \left(C - BF_1^T\right)X_2\Lambda_2 + \left(K - BF_2^T\right)X_2 = MX_2\Lambda_2^2 + CX_2\Lambda_2 + KX_2 - B\beta\left(\Lambda_1 Y_1^H MX_2\Lambda_2 - Y_1^H KX_2\right)$$

Since, $MX_2\Lambda_2^2 + CX_2\Lambda_2 + KX_2 = 0$ and $\left(\Lambda_1 Y_1^H MX_2\Lambda_2 - Y_1^H KX_2 = 0\right)$ from the Theorem 1, thus, $MX_2\Lambda_2^2 + \left(C - BF_1^T\right)X_2\Lambda_2 + \left(K - BF_2^T\right)X_2 = 0$. The theorem is proved.

Choosing β

In order to use Theorem 2 to solve the partial eigenvalue assignment problem, we need to choose β which will move $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ of the open loop pencil $P(\lambda) = M\lambda^2 + C\lambda + K$ to $\{\mu_1, \mu_2, \cdots, \mu_m\}$ in the closed loop pencil $P_c(\lambda) = M\lambda^2 + \left(C - BF_1^T\right)\lambda + \left(K - BF_2^T\right)$, if that is possible. If there is such β, then there exist an eigenvector matrix $Z \in C^{n\times m}$;

$$Z = (z_1, z_2, \cdots z_m), \qquad z_j \neq 0, \qquad j = 1,2,\cdots,m.$$

Matrix $D = diag(\mu_1, \mu_2, \cdots, \mu_m)$ such that:

$$MZD^2 + \left(C - BF_1^T\right)ZD + \left(K - BF_2^T\right)Z = 0 \tag{17}$$

Substituting $F_1^T = \beta\Lambda_1 Y_1^H M$ and $F_2^T = -\beta Y_1^H K$ in equation 17, we have:

$$MZD^2 + CZD + KZ = B\beta\left(\Lambda_1 Y_1^H MZD - Y_1^H KZ\right)$$

$$MZD^2 + CZD + KZ = B\beta W^H = B\Gamma \tag{18}$$

where $W^H = \left(\Lambda_1 Y_1^H MZD - Y_1^H KZ\right)$ and $W\beta^H = \Gamma^H$

Γ is a matrix $m\times m$ that will depend on the scaling chosen for the eigenvectors in Z. To obtain Z, we choose the matrix Γ as $\Gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m)$. Then Equation 18 becomes:

$$MZD^2 + CZD + KZ = B(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

We can solve for each of the eigenvectors z_j using these equations:

$$\left(M\mu_j^2 + C\mu_j + K\right)z_j = B\gamma_j, \quad \gamma_j \in C^{n\times 1} \quad j = 1,2,\cdots,m \tag{19}$$

We choose arbitrary vectors $\gamma_1, \gamma_2, \ldots, \gamma_m$ in such a way that $\mu_{\boldsymbol{j}} = \bar{\mu}_{\boldsymbol{k}}$ implies $\gamma_{\boldsymbol{j}} = \bar{\gamma}_{\boldsymbol{k}}$ for $k = 1,2,...m$.

So, we obtain the eigenvectors of $Z = (z_1, \ldots, z_m)$, and hence we compute the matrix W from $W^H = \left(\Lambda_1 Y_1^H MZD - Y_1^H KZ\right)$ if W is ill-conditioned, then we select different vectors $\gamma_1, \gamma_2, \ldots, \gamma_m$. We solve the $m\times m$ square linear system,

$$W\beta^H = (\gamma_1, \gamma_2, \cdots, \gamma_m)^H \tag{20}$$

for β^H, and hence determine the matrices F_1 and F_2. We summarize the solution in the following algorithm.

Algorithm 1

This is an algorithm for the multi-input partial eigenvalue assignment algorithm for non-symmetric quadratic pencil.

Inputs:

1) The $n\times n$ real non-symmetric constant matrices $\boldsymbol{C}$ and K, $M = M^T > 0$,
2) The $n\times m$ control (input) matrix $B \in R^{n\times m}$, and
3) $D = diag(\mu_1, \mu_2, \cdots, \mu_m)$ is closed under complex conjugation.

Outputs:

The feedback matrices $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ such that the spectrum of non-symmetric quadratic pencil $P_c(\lambda) = M\lambda^2 + \left(C - BF_1^T\right)\lambda + \left(K - BF_2^T\right)$, are

$\{\mu_1, \cdots, \mu_m; \lambda_{m+1}, \cdots, \lambda_{2n}\}$, where $\lambda_{m+1},, \cdots, \lambda_{2n}$ are the last $2n - m$ eigenvalues of matrix pencil

$P(\lambda) = M\lambda^2 + C\lambda + K$.

Assumptions:

Let M be non-singular matrix and the numbers $\mu_1, \cdots, \mu_m$ and $\lambda_1, \cdots, \lambda_m$ are all distinct and closed under complex conjugation, where $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are the eigenvalues of matrix pencil $P(\lambda) = M\lambda^2 + C\lambda + K$.

Step1: Obtain the first m eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ of matrix pencil $P(\lambda) = M\lambda^2 + C\lambda + K$ that need to be reassigned and the corresponding left eigenvectors $y_1, y_2, \cdots, y_m$.

Step 2: Choose arbitrary vectors $\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_m$ in such a way that $\mu_j = \bar{\mu}_k$ implies $\gamma_j = \bar{\gamma}_k$ for $k = 1,2,...m$ and solve for $z_1, \ldots, z_m$.

$$\left(M\mu_j^2 + C\mu_j + K\right)z_j = B\gamma_j \qquad j = 1,2,\cdots,m$$

Step 3: Form $W^H = \left(\Lambda_1 Y_1^H MZD - Y_1^H KZ\right)$ if W is ill-conditioned, then return to Step 2 and select different vectors $\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_m$.

Step 4: Form $\Gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m)$ and solve for β when $W\beta^H = \Gamma^H$

Step 5: Form

$$F_1^T = \beta\Lambda_1 Y_1^H M \quad F_2^T = -\beta Y_1^H K, \quad \beta \in C^{m\times m}.$$

NUMERICAL EXAMPLE

We choose randomly generated matrices M, C, K (size 4) as follows:

$$M = \begin{bmatrix} 0.9501 & 0.8913 & 0.8214 & 0.9218 \\ 0.2311 & 0.7621 & 0.4447 & 0.7382 \\ 0.6068 & 0.4565 & 0.6154 & 0.1763 \\ 0.4860 & 0.0185 & 0.7919 & 0.4057 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.4451 & 0.8462 & 0.8381 & 0.8318 \\ 0.9318 & 0.5252 & 0.0196 & 0.5028 \\ 0.4660 & 0.2026 & 0.6813 & 0.7095 \\ 0.4186 & 0.6721 & 0.3795 & 0.4289 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.9355 & 0.0579 & 0.1389 & 0.2722 \\ 0.9169 & 0.3529 & 0.2028 & 0.1988 \\ 0.4103 & 0.8132 & 0.1987 & 0.0153 \\ 0.8936 & 0.0099 & 0.6038 & 0.7468 \end{bmatrix}$$

and a random matrix B as:

$$B = \begin{bmatrix} 0.3046 & 0.1897 \\ 0.1934 & 0.6822 \\ 0.3028 & 0.5417 \\ 0.1509 & 0.6979 \end{bmatrix}.$$

The quadratic pencil $P(\lambda) = M\lambda^2 + C\lambda + K$ has eigenvalues as shown in Table 1.

Now, we assign the last $m = 2$ eigenvalues λ_6, λ_7 to the conjugate pair $\mu_{1,2} = -1 \pm i$. Using Algorithm 1 gives:

$$\beta = \begin{bmatrix} -3.3472 + 2.9940\text{i} & 0.9172 + 0.8287\text{i} \\ -3.3472 - 2.9940\text{i} & 0.9172 - 0.8287\text{i} \end{bmatrix}$$

The random choices of γ_1 and γ_2 produces matrix W with condition number 1.8894 in step 17

From which we compute the feedback matrices F_1 and F_2, in view of equation 15.

$$F_1 = \begin{bmatrix} -0.0578 & 0.2653 \\ -6.4495 & 0.1883 \\ 2.2208 & -0.3297 \\ 3.1225 & -0.3190 \end{bmatrix}$$

and

$$F_2 = \begin{bmatrix} -2.7900 & -0.5501 \\ 2.8545 & 2.2048 \\ -2.1387 & -0.5216 \\ -3.4306 & -1.3347 \end{bmatrix}$$

Table 1. Eigenvalues of pencil $P(\lambda) = M\lambda^2 + C\lambda + K$.

Eigenvalues of pencil:	$P(\lambda) = M\lambda^2 + C\lambda + K$
	0.1191
	0.4284
	-0.5245
	-0.5302 - 0.7218i
	-0.5302 + 0.7218i
	0.6117 - 1.4439i
	0.6117 + 1.4439i
	2.5321

Table 2. Eigenvalues of quadratic open pencil $P(\lambda) = M\lambda^2 + C\lambda + K$ and quadratic closed pencil $P_c(\lambda) = \lambda^2 M + \lambda\left(C - BF_1^T\right) + \left(K - BF_2^T\right)$.

Eigenvalues of $P(\lambda) = M\lambda^2 + C\lambda + K$	**Eigenvalues of** $P_c(\lambda) = \lambda^2 M + \lambda\left(C - BF_1^T\right) + \left(K - BF_2^T\right)$
0.1191	0.1191
0.4284	0.4284
-0.5245	-0.5245
-0.5302 - 0.7218i	-0.5302 - 0.7218i
-0.5302 + 0.7218i	-0.5302 + 0.7218i
0.6117 - 1.4439i	-1.0000 + 1.0000i
0.6117 + 1.4439i	-1.0000 - 1.0000i
2.5321	2.5321

The eigenvalues of quadratic open pencil $P(\lambda) = M\lambda^2 + C\lambda + K$ and quadratic closed pencil $P_c(\lambda) = \lambda^2 M + \lambda\left(C - BF_1^T\right) + \left(K - BF_2^T\right)$ are as shown in Table 2.

Conclusion

In this paper, we derived the parametric solution to the partial eigenvalue problem by using one of the orthogonality relations between eigenvectors for non-symmetric quadratic pencil $P(\lambda) = M\lambda^2 + C\lambda + K$. We need only a partial knowledge of the spectrum (and the associated left eigenvectors) for non-symmetric quadratic pencil $P(\lambda) = M\lambda^2 + C\lambda + K$. These eigenvalues and eigenvectors required to be reassigned.

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