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Full Length Research Paper

Quartic spline method for solving second-order boundary value problems

Eisa A. Al-Said[1], Muhammad A. Noor[1,2]*, Anwar H. Almualim[4], B. Kokkinis[3] and John Coletsos[3]

[1]Mathematics Department, College of Science, King Saud University, Riyadh 11451, Saudi Arabia.
[2]Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.
[3]Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece.
[4]Women Students Medical Studies (Sciences Sections), Mathematics Department, King Saud University, Riyadh 11495, Saudi Arabia.

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In this paper, uniform quartic spline polynomial functions are used to develop some consistency relations which are then used to derive a numerical method for approximating the solution and its first, second, third and fourth derivatives of second order boundary value problems. The present method is capable of producing fourth order accurate approximations for the solution, as well as its first and second derivatives, and the second order accurate approximations for its third and fourth derivatives. The main consistency relation for the study's quartic spline method is the same relation derived by Usmani et al. (1987) using quartic spline function. Their spline method has a stability problem which affects the accuracy of the computed approximations, whereas for the study's new method, such stability problem does not appear. The present method produces more accurate approximations for the first and third derivatives of the solution than those produced by the other quartic spline method. A numerical example is included to demonstrate the efficiency and implementation of the proposed method.

Key words: Boundary value problems, quartic spline function, finite difference approximations, convergence analysis.

INTRODUCTION

We consider using quartic spline functions to develop a numerical method for obtaining smooth approximations for the solution and derivatives of a second order boundary value problem of types:

$$y''(x) = f(x)\,y + g(x), \tag{1}$$

$$y(a) = \alpha,\; y(b) = \beta, \tag{2}$$

Where $f(x)$ and $g(x)$ are continuous functions on the interval $[a,b]$. Here, α, β, a and b are real constants. The use of spline functions to construct numerical methods for solving boundary value problems of form (1) started in the late 1960's, when Ahlberg et al. (1967) briefly discussed the possibility of using spline to solve boundary value problems in ordinary differential equations and followed by Albasiny et al. (1969), where they developed a cubic spline method for solving second order boundary value problems. Following this, several authors investigated and developed spline methods for solving problem (1) (Almulaim, 2009; Al-Said, 1996, 1998;

*Corresponding author. E-mail: mnoor.c@ksu.edu.sa, noormaslam@hotmail.com.

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Al-Said et al., 2002; Usmani et al., 1980, 1987). The main advantage of the spline methods is its capability of producing approximations for the solution and its derivatives over the whole interval $[a,b]$. Usmani et al. (1987) used quartic spline polynomial functions to construct a numerical method for solving problem (1). The spline method discussed in Usmani et al. (1987) is capable of producing approximations for the solution, as well as its first, second and third derivatives.

In the present paper, we use quartic spline polynomial functions to develop a numerical technique for computing approximations for the solution of (1) and its derivatives. The new method is of order four, and the fourth order accuracy holds for approximating the solution and its first and second derivatives. Also, our new method is capable of producing second order accurate approximations for the third and fourth derivatives. As expected, the main consistency relation for this study's method is the same one given in Usmani et al. (1987), which is Numerov's approximation. However, the method produces better approximations for the first and third derivatives than the other quartic spline methods considered in Usmani et al. (1987). This is due to the fact that the quartic spline method has stability problem when computing the approximation of the first derivatives. This fact shows that the method affects the computations of the approximations for the third derivatives. Also, another advantage of the present method is capable of computing approximations for the fourth derivative. The outline of the paper is as follows. Subsequently, the new consistency relations were derived and the quartic spline method was developed for solving problem (1), after which the convergence analysis of the study's method was shown. The numerical experiments and the comparison with other methods were then given afterwards.

QUARTIC SPLINE METHOD

In order to develop the quartic spline method for solving the boundary value of problem (1), the interval $[a,b]$ was divided into $(N+1)$ equal subintervals using the grid points $x_i = a + i\,h$, for $i = 0,1,2,\ldots,N+1$, $\quad x_0 = a,\ x_{n+1} = b,$ where

$$h = \frac{b-a}{N+1}, \tag{3}$$

and N is a positive integer. Let s_i be the approximation to $y_i = y(x_i)$ obtained by the quartic spline $Q_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}). As such, $Q_i(x)$ is written in the form that follows:

$$Q_i(x) = a_i(x-x_i)^4 + b_i(x-x_i)^3 + c_i(x-x_i)^2 + d_i(x-x_i) + e_i,\quad i = 0,1,2,\ldots,N. \tag{4}$$

Then the quartic spline function is defined by

$$s(x) = Q_i(x),\ i = 0,1,\ldots,N \quad s(x) \in C^3[a,b].$$

We first develop explicit expressions for the five coefficients in (4), in terms of $s_i,\ s_{i+1},\ D_i,\ D_{i+1},\ F_i,\ F_{i+1},$ where

$$Q_i(x_i) = s_i,\quad Q_i(x_{i+1}) = s_{i+1},\quad Q_i''(x_i) = D_i,$$
$$Q_i''(x_{i+1}) = D_{i+1},\quad Q_i^{(4)}(x_i) = \frac{1}{2}[F_i + F_{i+1}],\quad i = 0,1,2,\ldots,N. \tag{5}$$

Where $D_i = f_i\, s_i + g_i$ with $f_i = f(x_i)$ and $g_i = g(x_i)$.

From Equations (4) and (5), we get;

$$a_i = \frac{1}{48}[F_i + F_{i+1}], \tag{6}$$

$$b_i = \frac{1}{6h}[D_{i+1} - D_i] - \frac{h}{24}[F_i + F_{i+1}], \tag{7}$$

$$c_i = \frac{1}{2}D_i, \tag{8}$$

$$d_i = \frac{1}{h}[s_{i+1} - s_i] - \frac{h}{6}[D_{i+1} + 2D_i] + \frac{h^3}{48}[F_i + F_{i+1}], \tag{9}$$

$$e_i = s_i, \tag{10}$$

Now, from the continuity of the first derivatives of the quartic spline $s(x)$ at point (x_i, s_i), where the two quartics $Q_{i-1}(x)$ and $Q_i(x)$ are joined together, we have $\quad Q'_{i-1}(x_i) = Q'_i(x_i)$.

This equation gives

$$h^2[D_{i+1} + 4D_i + D_{i-1}] = 6[s_{i+1} - 2s_i + s_{i-1}] + \frac{h^2}{8}[F_{i+1} + 2F_i + F_{i-1}]. \tag{11}$$

Also, from equation (4) and from the continuity of the third derivatives of the quartic spline $s(x)$ at point (x_i, s_i), we get

$$h^2[D_{i+1} - 2D_i + D_{i-1}] = \frac{h^2}{4}[F_{i+1} + 2F_i + F_{i-1}], \tag{12}$$

From equations (11) and (12), we have

$$6h^2 D_i = [s_{i+1} - 2s_i + s_{i-1}] - \frac{h^2}{48}[F_{i+1} + 2F_i + F_{i-1}], \tag{13}$$

The elimination of D_i from (12) and (13) yields:

$$[s_{i+1}-2s_i+s_{i-1}]=\frac{h^2}{12}[D_{i+1}+10D_i+D_{i-1}],\quad i=0,1,2,\ldots,N. \tag{14}$$

The recurrence relation in (14) forms a system of N linear equations in the N unknowns $s_i,\ i=0,1,2,\ldots,N.$ The quartic solution of the boundary value problem (1) is based on linear equation (14). Having the values of $s_i,\ i=0,1,2,\ldots,N$, by solving system (14), we can compute $D_i,\ i=0,1,2,\ldots,N+1$, using the differential equation (1). Now, we can compute $F_i,\ i=0,1,2,\ldots,N$, using (12). Note that for computing F_0 and F_{N+1}, the following relations:

$$F_0=\frac{2}{h}f_0' s_1+\left[f_0+2f_0''-\frac{2}{h}f_0'-hf_0f_0'\right]s_0+\left[f_0-hf_0'\right]g_0+g_0'', \tag{15}$$

$$F_{N+1}=-\frac{2}{h}f_{N+1}' s_N+\left[f_{N+1}+2f_{N+1}''+\frac{2}{h}f_{N+1}''+hf_{N+1}f_{N+1}'\right]s_{N+1}+\left[f_{N+1}+hf_{N+1}'\right]g_{N+1}+g_{N+1}'', \tag{16}$$

were used respectively. We remark that the knowledge of $s_i,\ D_i,$ and $F_i,\quad i=0,1,2,\ldots,N+1$, enabled us to write down $Q_i(x),\ i=0,1,2,\ldots,N$ as given by (4). Thus, the quartic spline $s(x)$, approximating the solution of problem (1) was determined.

CONVERGENCE ANALYSIS

The convergence analysis of the quartic spline method described previously was investigated. For this purpose, the study first allowed $\mathbf{y}=(y_i)$, $\mathbf{s}=(s_i)$, $\mathbf{c}=(c_i)$, $\mathbf{t}=(t_i)$, and $\mathbf{e}=(e_i)$, to be N-dimensional column vectors. Here, $e_i=y_i-s_i$, is the discretization error and t_i is the local truncation error for the consistency relation of (13) and is given by:

$$t_i=-\frac{1}{240}h^6y^{(vi)}(\xi_i)+O(h^8),\ x_{i-1}<\xi_i<x_{i+1},\ i=0,1,2,\ldots,N. \tag{17}$$

Using these notations, the study's quartic spline method can be written as:

$$\mathbf{Ay}=r+\mathbf{t}, \tag{18}$$

$$\mathbf{As}=\mathbf{r}, \tag{19}$$

$$\mathbf{Ae}=\mathbf{t}. \tag{20}$$

$$\mathbf{A}=\mathbf{B}+\frac{1}{12}h^2\mathbf{PF}, \tag{21}$$

Where $\mathbf{B}=(b_{ij})$ is the tridiagonal matrix given by:

$$b_{ij}=\begin{cases}2, & i=j=1,2,\ldots,N\\ -1, & |i-j|=1\\ 0, & \text{otherwise.}\end{cases}$$

The matrix $\mathbf{P}=(p_{ij})$ is the $N\times N$ tridiagonal matrix defined by:

$$p_{ij}=\begin{cases}10, & i=j=1,2,\ldots,N\\ 1, & |i-j|=1\\ 0, & \text{otherwise.}\end{cases}$$

and the diagonal matrix $\mathbf{F}=\operatorname{diag}(f_i), i=1,2,\ldots,N.$, while the vector $\mathbf{r}$ is defined by:

$$r_i=\begin{cases}\alpha-\frac{1}{12}h^2[f_0\alpha+g_0+10g_1+g_2], & i=1\\ -\frac{1}{12}h^2[g_{i-1}+10g_i+g_{i+1}], & 2\le i\le N-1\\ \beta-\frac{1}{12}h^2[g_{N-1}+10g_N+f_{N+1}\beta+g_{N+1}], & i=N\end{cases} \tag{22}$$

The study's main purpose now is to derive a bound on $\|\mathbf{e}\|$. From the aforeseen relations, we have:

$$\mathbf{e}=\mathbf{A}^{-1}t=\left(\mathbf{B}+\frac{1}{12}h^2\mathbf{PF}\right)^{-1}\mathbf{t}=\left(\mathrm{I}+\frac{1}{12}h^2\mathbf{B}^{-1}\mathbf{PF}\right)^{-1}\mathbf{B}^{-1}\mathbf{t}$$

and

$$\|\mathbf{e}\|\le\frac{\|\mathbf{B}^{-1}\|\|\mathbf{t}\|}{1-\frac{1}{12}h^2\|\mathbf{B}^{-1}\|\|\mathbf{P}\|\|\mathbf{F}\|}, \tag{23}$$

provided that $\frac{1}{12}h^2\|\mathbf{B}^{-1}\|\|\mathbf{P}\|\|\mathbf{F}\|<1.$

However, it is well known (Henrici, 1961) that

$$\|\mathbf{B}^{-1}\|=\frac{(b-a)^2}{8h^2}, \tag{24}$$

Thus, using these equations and the fact that:

Table 1. Observed errors using the study's quartic spline method for problem (28).

h	$\max_i \lvert y_i - s_i \rvert$	$\max_i \lvert y_i' - s_i' \rvert$	$\max_i \lvert y_i'' - s_i'' \rvert$	$\max_i \lvert y_i''' - s_i''' \rvert$	$\max_i \lvert y_i^{(iv)} - s_i^{(iv)} \rvert$
1/8	1.74×10^{-7}	1.65×10^{-6}	6.28×10^{-8}	2.20×10^{-3}	2.15×10^{-2}
1/16	1.10×10^{-8}	1.08×10^{-7}	4.00×10^{-9}	5.72×10^{-4}	6.20×10^{-3}
1/32	6.85×10^{-10}	6.84×10^{-9}	2.50×10^{-10}	1.44×10^{-4}	1.70×10^{-3}

t $\|\mathbf{t}\| = \frac{1}{240} M_6,\ M_6 = \max_x \left| y^{(6)}(x) \right|$, and

$\|\mathbf{P}\| = 12 \text{ and } \|\mathbf{F}\| \le \max_x |f(x)|$, we obtain

$$\|\mathbf{e}\| \le \frac{\lambda M_6 h^4}{12\left[1 - \lambda \max_x |f(x)|\right]} = Kh^4 \cong O\left(h^4\right), \tag{25}$$

Where $K = \frac{\lambda M_6}{12\left[1 - \lambda \max_x |f(x)|\right]}$ and $\lambda = \frac{1}{8}(b-a)^2$.

Inequality (25) indicates that (18) is a fourth order convergent method.

Now, a brief discussion is given regarding the accuracy of the consistency relations of (18) to (21) and (11), which are used for computing the approximations of y_i', y_i''' and $y_i^{(4)}$, respectively. The local truncation errors are:

$$-\frac{1}{720} h^5 y_i^{(5)}, \quad -\frac{1}{12} h^5 y_i^{(5)} \quad \text{and} \quad -\frac{1}{6} h^6 y_i^{(6)}, \tag{26}$$

respectively

Thus, the order of convergence for the approximation of the first derivative of the solution is four and that of the third and fourth derivatives are two. Also, since $s_i'' = D_i = f_i s_i + g_i$, then

$$\begin{aligned} \max_i \left| y_i'' - s_i'' \right| &= \max_i \left| f_i y_i + g_i - f_i s_i - g_i \right| = \max_i \left| f_i (y_i - s_i) \right| \\ &= \max_i |f_i e_i| \le \max_x |f(x)| \max_i |e_i| \\ &\le K \max_x |f(x)| h^4 \cong O\left(h^4\right) \end{aligned} \tag{27}$$

which indicate that the order of accuracy for the approximation of the second derivative is four. Consequently, these results are summarized in the following theorem.

Theorem

Let $y(x) \in C^6[a,b]$ be the exact solution of the boundary value problem (1), and $s(x)$ the quartic spline solution approximating $y(x)$. Then

$$\max_{1\le i\le n} \left| y_i^{(m)} - s_i^{(m)} \right| = O\left(h^{v(m)}\right), \quad v(m) = 4 - \frac{m(m-1)(m-2)}{2(m-1)+m-4}, \quad m = 0,1,2,3,4.$$

NUMERICAL RESULTS AND DISCUSSION

The use of the quartic spline method described previously to solve a boundary value problem was demonstrated and the numerical results of this study were compared with those produced by the quartic spline method of Usmani et al. (1987).

Example

A consideration is made in solving the boundary value problem

$$\begin{aligned} y'' &= \frac{2}{x^2} y - \frac{1}{x}, \\ y(2) &= 0, \quad y(3) = 0 \end{aligned} \tag{28}$$

The exact solution for this problem is $y(x) = \frac{1}{38}\left[-5x^2 + 19x - \frac{36}{x}\right]$. This problem was solved using the quartic spline method with a variety of h values and the observed maximum errors in absolute values associated with $y_i^{(m)},\ m = 0,1,\ldots 4$ were given in Table 1. As seen in Table 2, it can be observed that if the stepsize h is reduced by a factor $\frac{1}{2}$, then the maximum errors $\max_i \left| y_i^{(m)} - s_i^{(m)} \right|$ are reduced by factor $\frac{1}{16}$, when $m = 0,1,2$ and by $\frac{1}{4}$ when $m = 3,4$. Thus, the numerical results confirm that the study's quartic spline method gives the fourth order accurate approximations

Table 2. Observed errors using quartic spline method for problem (28) (Usmani et al., 1987).

h	$\max_i \lvert y_i - s_i \rvert$	$\max_i \lvert y_i' - s_i' \rvert$	$\max_i \lvert y_i'' - s_i'' \rvert$	$\max_i \lvert y_i''' - s_i''' \rvert$
1/8	1.74×10^{-7}	2.72×10^{-6}	6.28×10^{-8}	2.90×10^{-3}
1/16	1.10×10^{-8}	1.93×10^{-7}	4.00×10^{-9}	8.14×10^{-4}
1/32	6.85×10^{-10}	1.29×10^{-8}	2.50×10^{-10}	2.16×10^{-4}

for the solution and its first and second derivatives, and the second order approximation for the third and fourth derivatives as predicted in convergence analysis.

Conclusion

In this paper, a new method was developed for solving the second-order boundary value problem using the quartc spline technique. Also, the convergence of the new method was considered. Some numerical examples are given to illustrate the efficiency and implementation of the proposed method. The results of this study show a significant improvement of the previous known results. It would be interesting to extend the results of this paper for solving the variational inequalities associated with obstacle and contact problems (Noor, 1988, 2004, 2009; Noor et al. 1993, 2010, 2011, 2011a, 2011b) and the references therein. Nonetheless, this is another direction for future research.

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