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*Review*

# Pseudo-prime and pseudo-irreducible submodules

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**In this paper, we will extend the pseudo-prime and pseudo-irreducible ideals to pseudo-prime and pseudo-irreducible submodules in multiplication modules. After this, we give the correspondence between these submodules and prime submodules almost prime submodules.**

**Key words:** Almost prime submodules, multiplication modules, pseudo prime submodules, pseudo irreducible submodules.

## INTRODUCTİON

Throughout this paper, all rings are considered commutative ring with non-zero identity and all modules are unitary. Let $R$ be a ring and $M$ be an $R$-module. An $R$-module $M$ is called multiplication if for any submodule $N$ of $M$ we get $N = IM$, where $I$ is an ideal of $R$. One can easily show that, if $M$ is a multiplication module, then $N = (N : M)M$ for every submodule $N$ of $M$. Here $(N : M)$ is the ideal $\{r \in R \mid rM \subseteq N\}$ of $R$. A proper submodule $N$ of $M$ is called prime if $rm \in N$, for some $r \in R$ and $m \in M$, implies that either $m \in N$ or $r \in (N : M)$. Abd El-Bast and Smith (1988) have shown that $N$ is a prime submodule, if and only if $(N : M)$ is a prime ideal of $R$, where $M$ is a multiplication $R$-module. Recall that, multiplication of submodules in a multiplication module is defined by Ameri (2003). Let $M$ be a multiplication $R$-module and $N$, $K$ are submodules of $M$. Then there exist ideals $I$ and $J$ of $R$,such that $N = IM$ and $K = JM$. Hence, the multiplication of $N$ and $K$ is defined as, $NK = (IM)(JM) = (IJ)M$. Similarly, multiplication of two elements of $M$ is defined as; let $x, y \in M$ then for the cyclic submodules $Rx$ and $Ry$ we have some ideals $A$ and $B$ of $R$ such that $Rx = AM$ and $Ry = BM$ and so $xy = (Rx)(Ry) = (AM)(BM) = (AB)M$. With these definitions, Ameri (2003) has characterized prime submodules. Ameri has shown that a submodule $N$, is prime if and only if for some submodules $U$ and $V$ such that $UV \subseteq N$ implies that $U \subseteq N$ or $V \subseteq N$, if and only if for all $x, y \in M$ such that $xy \subseteq N$ implies that $x \in N$ or $y \in N$. We call a proper submodule $N$ of $M$ an almost prime submodule, if for all $r \in R$ and $m \in M$ such that $rm \in N \setminus N^2$ implies $m \in N$ or $r \in (N : M)$. It is clear that all prime submodules are almost prime submodules.

In this paper, we will extend pseudo-irreducible and pseudo-prime ideals to pseudo-irreducible and pseudo-prime submodules respectively, in multiplication modules. Pseudo-irreducible and pseudo-prime ideals are defined by McAdam and Swan (2004), "Unique co-maximal factorization". After this work, Bhatwadekar and Sharma (2005) gives some properties of these ideals. Recall that, an ideal $I$ of a ring $R$ is called pseudo-irreducible if one cannot write $I = JK$ with $J + K = R$ where $I \neq R, J \neq R$ be ideals of $R$. And an ideal $I$ of a ring $R$ is called pseudo-prime if whenever $J + K = R$ with $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$ for some ideals $I, J$ of $R$.

### Pseudo-prime and pseudo-irreducible submodules

**Definition 1**

A proper submodule $N$ of the multiplication $R$-module

$M$ is called strictly non-prime if there exist $m \notin N$, $r \notin (N:M)$, such that $rm \in N$ and $rM + Rm = M$.

**Theorem 1**

Let $N$ be a strictly non-prime submodule of the multiplication $R$-module $M$ where $R$ is a PID. Then $(N:M)$ is a strictly non-prime ideal of $R$.

**Proof**

Since $N$ is a strictly non-prime submodule of $M$, there exist $m \notin N$, $r \notin (N:M)$ such that $rm \in N$ and $rM + Rm = M$. Since $m \notin N$ we get $(m) \not\subset N$ where $(m) = IM$ for some ideal $I$ of $R$. Then we have $I \not\subset (N:M)$. So if $I = (a)$ for some $a \in R$, we get $a \in I \setminus (N:M)$. Since $rm \in N$, we have $(rI)M = (rm) \subseteq N$. So $rI \subseteq (N:M)$, thus $ra \in (N:M)$. Now, we will show that $(r) + (a) = R$. Since $rM + Rm = M$, we get $M = rM + Rm = (r)M + (a)M$. Hence $(r) + (a) = R$.

**Definition 2**

Let $N$ be a proper submodule of the multiplication $R$-module $M$. $N$ is called pseudo- irreducible submodule if $N = UV$ with $U + V = M$ for any submodules $U$ and $V$ of $M$, then either $U = M$ or $V = M$.

**Theorem 2**

(El-Bast and Smith, Theorem 3.1) Let $R$ be a commutative ring with identity and $M$ a faithful multiplication $R$-module. Then the following statements are equivalent:

(i) $M$ is finitely generated.
(ii) If $A$ and $B$ are ideals of $R$ such that $AM \subseteq BM$ then $A \subseteq B$.
(iii)For each submodule $N$ of $M$ there exists a unique ideal $I$ of $R$ such that $N = IM$.
(iv) $M \neq AM$ for any proper ideal $A$ of $R$.
(v) $M \neq PM$ for any maximal ideal $P$ of $R$.

**Theorem 3**

Let $M$ be a faithful finitely generated multiplication $R$-module. Then the submodule $N$ of $M$ is pseudo-irreducible, if and only if the ideal $(N:M)$ is pseudo-irreducible.

**Proof**

Suppose that $N$ is a pseudo-irreducible submodule of $M$ and $(N:M) = IJ$ where $I$ and $J$ are co-maximal ideals of $R$. Then $N = (N:M)M = (IJ)M = (IM)(JM)$ and we get $M = RM = (I+J)M = IM + JM$. Since $N$ is a pseudo-irreducible submodule, we have $IM = M$ or $JM = M$. Since $M$ is faithful, we get $I = R$ or $J = R$. For the converse, assume that the ideal $(N:M)$ is pseudo-irreducible in $R$. Suppose that there exist submodules $U$, $V$ of $M$ such that $UV = N$ and $U + V = M$. Since $M$ is a multiplication module, there exist ideals $I$, $J$ of $R$ such that $U = IM$ and $V = JM$. Then we get $N = UV = (IM)(JM) = (IJ)M$ and $M = U + V = IM + JM = (I+J)M$. Therefore, $IJ = (N:M)$ and $I + J = R$. Since $(N:M)$ is a pseudo-irreducible ideal, we obtain $I = R$ or $J = R$ that is $U = IM = RM = M$ or $V = JM = RM = M$.

**Definition 3**

Let $N$ be a proper submodule of the multiplication $R$-module $M$. $N$ is called pseudo-prime submodule if $U + V = M$ and $UV \subseteq N$ for any submodules $U$ and $V$ of $M$, then either $U \subseteq N$ or $V \subseteq N$.

**Theorem 4**

Let $M$ be a multiplication $R$-module. If the submodule $N$ of $M$ is pseudo-prime then the ideal $(N:M)$ of $R$ is pseudo-prime.

**Proof**

Let $N$ be a pseudo-prime submodule of $M$ and suppose that for co-maximal ideals $I$, $J$ of $R$ we have $IJ \subseteq (N:M)$. Then we get $(I+J)M = RM = M$ and $(IJ)M \subseteq (N:M)M = N$. And so $IM + JM = M$ and $(IM)(JM) \subseteq N$. Since $N$ is pseudo-prime submodule we obtain that $IM \subseteq N$ or $JM \subseteq N$. Hence $I \subseteq (N:M)$ or $J \subseteq (N:M)$.

**Theorem 5**

Let $M$ be a faithful finitely generated multiplication $R$-module. Then the submodule $N$ of $M$ is pseudo-prime if and only if the ideal $(N:M)$ of $R$ is pseudo-prime.

**Proof**

The one part is clear by the above Theorem. Now for the converse, assume that the ideal $(N:M)$ of $R$ is pseudo prime for the submodule $N$ of $M$. Let $U$ and $V$ be submodules of $M$ such that $UV \subseteq N$ and $U+V=M$. Since $M$ is a multiplication module, then there exist ideals $I$ and $J$, such that $U = IM$ and $V = JM$. Then we get $UV = (IJ)M \subseteq N$ and $U+V = (I+J)M = M$. Therefore we obtain $IJ \subseteq (N:M)$ and $I+J=R$. Since $(N:M)$ is a pseudo-prime ideal, we get $I \subseteq (N:M)$ or $J \subseteq (N:M)$. Hence we obtain $U = IM \subseteq N$ or $V = JM \subseteq N$.

**Theorem 6**

Let $M$ be a multiplication $R$-module and $N$ be a pseudo-irreducible submodule of $M$. Then $N$ is pseudo-prime.

**Proof**

Suppose that $N$ is a pseudo-irreducible submodule and $UV \subseteq N$ with $U+V=M$ for any submodules $U$, $V$ of $M$ .Then, $(U+V)(V+N) = UV + VN + UN + N^2 = N$. Since $N$ is a pseudo-irreducible submodule, we get $U+N=M$ or $V+N=M$. Thus $U \subseteq N$ or $V \subseteq N$.

**Proposition 1**

Let $M$ be a multiplication $R$-module and $N$ be a pseudo-prime submodule of $M$. If $x, y \in M$ with $xy \subseteq N$ and $(x)+(y)=M$, then $x \in N$ or $y \in N$.

**Proof**

It is trivial by the definition of pseudo prime submodules.

**Theorem 7**

Let $M$ be a cyclic $R$-module and $N$ be a submodule of $M$ with the property, if $x, y \in M$ with $xy \in N$ and $(x)+(y)=M$ then $x \in N$ or $y \in N$. Then $N$ is a pseudo-irreducible submodule.

**Proof**

Suppose that $N = UV$ with $U+V=M$ for some submodules $U, V$ of $M$. Assume that $M=(m)$ for some $m \in M$ and $U \neq M$, $V \neq M$. Then we have $m \in M \setminus (U \cup V)$ but $m \in U+V$. So $m = u+v$ for some $u \in U$ and $v \in V$. Thus $M = (m) = (u)+(v)$. Since $N = UV$, we get $uv \subseteq N$. Hence $u \in N$ or $v \in N$. And so $v \in N = UV \subseteq U$ or $u \in N = UV \subseteq V$. It follows that $m \in U$ or $m \in V$ which is a contradiction.

**Theorem 8**

Let $M$ be a multiplication $R$-module and $N$ be a submodule of $M$.
(i) $N$ is a pseudo-irreducible submodule
(ii) $N$ is a pseudo-prime submodule
(iii) If $x, y \in M$ with $xy \subseteq N$ and $(x)+(y)=M$ then $x \in N$ or $y \in N$.
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is always true. If $M$ is cyclic then the conditions are equivalent.

**Proposition 2**

(Ameri, Proposition 3.5) Let $M$ be a multiplication $R$-module. Then for submodules $N, K$ of $M$ such that $N+K=M$, we get $NK = N \cap K$.

### Theorem 9

(Oral, 2010; Theorem 5) Let $M$ be a finitely generated multiplication $R$-module and $N, K$ are finitely generated submodules of $M$, then $N$ is not an almost prime submodule.

### Theorem 10

Let $N$ be a finitely generated strictly non-prime submodule in a finitely generated multiplication module $M$. Then $N$ is not an almost prime submodule.

### Proof

Since $N$ is strictly non-prime submodule, there exist $m \notin N$, $r \notin (N:M)$ such that $rm \in N$ and $rM + Rm = M$. Then for the proper submodules $K_1 = N + rM$ and $K_2 = N + Rm$, we get $K_1 + K_2 = M$. So by the above proposition we obtain $K_1 K_2 = K_1 \cap K_2 = N$. Thus by Theorem 10 and since $K_1$ and $K_2$ are finitely generated submodules, we obtain that $N$ is not almost a prime submodule.

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