

Full Length Research Paper

Mannheim partner curves in 3-dimensional Heisenberg group

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In this study, we give some basic theorems for Mannheim partner curves in three dimensional Heisenberg group. We also found the relations between the curvatures and torsions of these associated curves.

Key words: Heisenberg group, Mannheim curve.

INTRODUCTION

In mathematics, the Heisenberg group, named after Werner Heisenberg, is the group of 3x3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

or its generalizations under the operation of matrix multiplication. In 1987, L. Bianchi classified the homogeneous metrics. L. Bianchi, E. Cartan and G. Vranceanu found the following 2-parameter family of homogeneous Riemannian metrics on the cartesian 3-space $\mathbb{R}^3(x, y, z)$:

$$g_{\lambda, \mu} = \frac{dx^2 + dy^2}{\{1 + \mu(x^2 + y^2)\}} + \left\{ dz + \frac{\lambda}{2} \frac{ydx - xdy}{1 + \mu(x^2 + y^2)} \right\}^2, \quad \forall \lambda, \mu \in \mathbb{R}.$$

In this family, if $\lambda = \mu = 0$, the Euclidean metric is obtained, and if $\lambda \neq 0, \mu = 0$, the Heisenberg metric is obtained. Inoguchi et al. (1999) studied on the differential geometry of Heisenberg metric.

As it is well known, many studies relating to the differential geometry of curves have been carried out. Especially, by establishing relations between the Frenet frames in mutual points of two curves. Several theories

have been obtained. The best known of these is Bertrand curves discovered by Bertrand in 1850 and are defined as a special curve which shares its principal normals with another special curve (called Bertrand mate) and other involute -evolute curves discovered by Huygens in 1658 are defined as a special curve which shares its principal normals with another special curve. These curves are investigated by mathematicians and they have gotten many characterizations related to these curves (Burke, 1960; Görgülü and Özdamar, 1986; Matsuda and Yorosu, 2003).

Liu and Wang (2008) and Wang and Lui (2007) gave a new definition of the curves known as Mannheim curves. Let (α) and (α^*) be two curves in E^3 . According to the definition given by Liu and Wang (2008), the principal normal vector field of (α) is linearly dependent with the binormal vector field of (α^*) . Then (α) is called a Mannheim curve and (α^*) a Mannheim partner curve of (α) . The pair $\{\alpha, \alpha^*\}$ is said to be a Mannheim pair. Furthermore, they showed that the curve is a Mannheim curve (α) if and only if its curvature and torsion satisfy the formula $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant. Orbay and Kasap (2009) gave many characterizations related to these curves in IE^3 .

PRELIMINARIES

Definition 1

The Heisenberg group n^3 can be seen as the Euclidean

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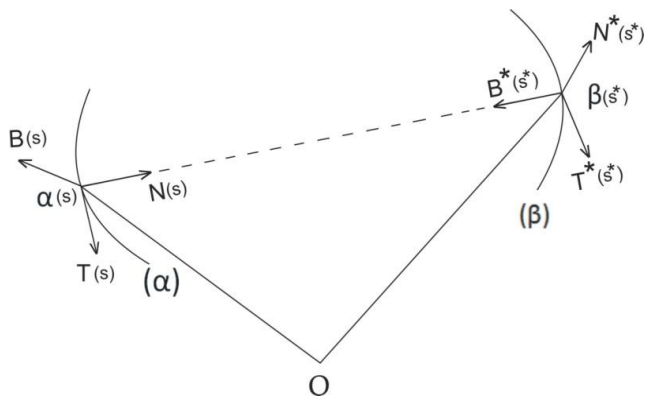


Figure 1. The Mannheim partner curves.

space with the multiplication

$$(x', y', z')(x, y, z) = \left(x' + x, y' + y, z' + z + \frac{1}{2}x'y - \frac{1}{2}y'x \right)$$

And with the Riemann metric g given by

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy \right)^2$$

n^3 is a three dimensional, connected, simply connected and 2-step nilpotent Lie group. The Lie algebra of n^3 has a basis

$$e_1 = \frac{\partial}{\partial x} - \frac{1}{2}ydz, \quad e_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

which is dual to

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

For this basis, Lie brackets are:

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = [e_2, e_3] = 0.$$

Definition 2

Let α and β be two curves in N^3 . If there exists a corresponding relationship between α and β such that, at the corresponding points of the curves, the principal normal of α coincides with the binormal lines of β , then α is called a Mannheim curve, β is called Mannheim partner curve of α and (α, β) is called Mannheim pair.

Mannheim partner curves in N^3

To study the Mannheim curves in N^3 , we shall use their Frenet vector fields and equations. Let $\alpha: I \rightarrow N^3$ be a differentiable curve parametrized by arc length and let $\{T, N, B\}$ be the orthonormal frame field tangent to n^3 along α defined as follows: by T we denote the tangent to α , by N the unit vector field in the direction ∇T normal to α and we choose $B = T \times N$, so that $\{T, N, B\}$ is a positive oriented orthonormal basis. Thus, we have the following Frenet equations:

$$\begin{cases} D_T T = \kappa N \\ D_T N = -\kappa T - \tau B \\ D_T B = \tau N \end{cases} \quad (1)$$

where κ and τ are the curvature and torsion of the curve α , respectively .

Theorem 1

Let (α, β) be a Mannheim pair in N^3 . The distance between corresponding points of the Mannheim partner curves in n^3 is constant where $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are the Frenet frames of the curves α and β , respectively.

Proof: From Figure 1, we can write

$$\alpha(s) = \beta(s^*) + \lambda(s^*)B(s^*) \quad (2)$$

By taking the derivative of Equation 2 and using Equation 1, we obtain

$$T \frac{ds}{ds^*} = T^*(s^*) + \lambda'(s^*)B^*(s^*) + \lambda(s^*)\tau^*(s^*) \quad (3)$$

Since N and B^* are linearly dependent, $g(T, B^*) = 0$, we have $T(s^*) = 0$ and in this case λ is a nonzero constant. On the other hand, from the distance functions between two points, we have

$$\begin{aligned} d(\beta(s^*), \alpha(s)) &= \|\alpha(s) - \beta(s^*)\| \\ &= \|\alpha\beta\| \\ &= \left| \sqrt{g(\lambda B, \lambda B)} \right| \\ &= |\lambda| \left| \sqrt{g(B, B)} \right| \\ &= |\lambda|. \end{aligned}$$

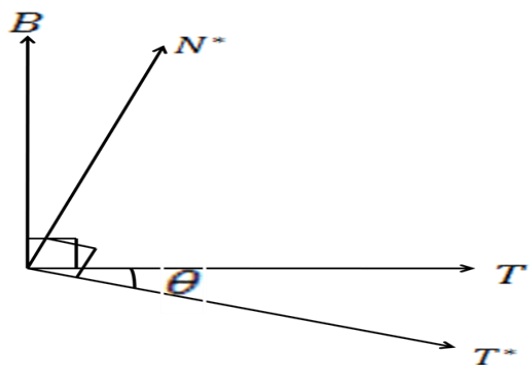


Figure 2. Frenet frame.

Namely, $d(\beta(s^*), \alpha(s)) = \text{constant}$.

Theorem 2

For a curve α in N^3 , there is a curve β so that (α, β) is a Mannheim pair.

Proof: Since N and B^* are linearly dependent, Equation 2 can be written as:

$$B^* = \alpha - \lambda N \tag{4}$$

Now that λ is a nonzero constant, there is a curve β for all values of λ

Theorem 3

Let $\{\alpha, \beta\}$ be a Mannheim pair in N^3 . The torsion of the curve β is

$$T^* = -\frac{\kappa}{\lambda\tau}$$

Proof: We know that $T \frac{ds}{ds^*} = T^* + \lambda T^* N$. By using this equation we have

$$T = \frac{ds^*}{ds} T^* + \lambda \tau^* \frac{ds^*}{ds} N^* \tag{5}$$

Even so, from the Figure 2, we know that:

$$\begin{cases} T = \cos \theta T^* + \sin \theta N^* \\ B = -\sin \theta T^* + \cos \theta N^* \end{cases} \tag{6}$$

where θ is the angle between T and T^* at the corresponding points of α and β .

By taking into consideration Equations 5 and 6, we get

$$\cos \theta = \frac{ds}{ds^*}, \quad \sin \theta = \lambda \tau^* \frac{ds}{ds^*}$$

Besides, by taking the derivative of Equation 4 with respect to s and using Equation 1, we have

$$T^* = (1 + \lambda \kappa) \frac{ds}{ds^*} T + \lambda \tau \frac{ds}{ds^*} B \tag{7}$$

From Equation 5, we obtain,

$$\begin{cases} T^* = \cos \theta T - \sin \theta B \\ N^* = \sin \theta T + \cos \theta B \end{cases} \tag{8}$$

By applying Equations 7 and 8, we get

$$\cos \theta = (1 + \lambda \kappa) \frac{ds}{ds^*}, \quad \sin \theta = \lambda \tau \frac{ds}{ds^*}$$

From both values of $\cos \theta$ and $\sin \theta$, we see that,

$$\cos^2 \theta = (1 + \lambda \kappa), \quad \sin^2 \theta = \lambda \tau \tau^* \tag{9}$$

Here, with the help of the fundamental equation $\cos^2 \theta + \sin^2 \theta = 1$, we reach the equation:

$$\tau^* = -\frac{\kappa}{\lambda \tau}$$

Theorem 4

Let $\{\alpha, \beta\}$ be a Mannheim pair in N^3 . Between the curvature and the torsion of the curve α , there is the relationship:

$$\mu \tau - \lambda \kappa = 1$$

where λ and μ are nonzero real numbers.

Proof: From Equation 9, we have,

$$\frac{\cos \theta}{1 + \lambda \kappa} = \frac{\sin \theta}{\lambda \tau}$$

Arranging this equation, we obtain

$$\lambda \cot \theta - \lambda \kappa = 1 \text{ and } \mu \tau - \lambda \kappa = 1.$$

Theorem 5

Let $\{\alpha, \beta\}$ be a Mannheim pair in N^3 . Here are the following equations for the curvatures and the torsions of the curves α and β :

$$1). \kappa^* = -\frac{d\theta}{ds^*}$$

$$2). \tau^* = (\sin \theta) \kappa \frac{ds}{ds^*} + (\cos \theta) \tau \frac{ds}{ds^*}$$

$$3). \kappa = -(\sin \theta) \tau^* \frac{ds}{ds^*}$$

$$4). \tau = -(\cos \theta) \tau^* \frac{ds}{ds^*}$$

Proof: By taking the derivative of equation $g(T, T^*) = \cos \theta$ with respect to s^* , we have

$$g(T, \kappa^* N^*) + g(\kappa N \frac{ds}{ds^*}, T^*) = -\sin \theta \frac{d\theta}{ds^*}$$

Furthermore, by considering N and B^* that are linearly dependent and using Equation 8, we have

$$\kappa^* = -\frac{d\theta}{ds^*}$$

By considering the equations $g(N, N^*) = 0$, $g(T, B^*) = 0$ and $g(B, B^*) = 0$, we can do the proofs of 2, 3 and 4 of the theorem, respectively as in Proof 1.

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