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Full Length Research Paper

Sâlâgean-type harmonic univalent functions

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The purpose of the present paper is to introduce new classes of harmonic univalent functions defined by using the Sâlâgean differential operator and to investigate various properties of these classes.

Key words: Harmonic functions, Sâlâgean derivative, coefficient bounds, distortion bounds, extreme points, neighborhood.

INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (|b_1| < 1). \tag{1}$$

In 1984, Clunie and Sheil-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficients bounds. Since then, there have been several related papers on S_H and its subclasses. The differential operator D^m was introduced by Sâlâgean (1983). For $f = h + \overline{g}$ given by Equation (1), Jahangiri et al. (2002) defined the modified Sâlâgean operator of f as:

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}, \tag{2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \text{ and } D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

For $0 \le \alpha < 1$, $\beta \ge 0$, $m \in N$, $n \in N_0$, $m > n$ and $z \in U$, let $S_H(m,n,\alpha,\beta)$ denote the family of harmonic functions f of the form (1) such that

$$Re\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \beta \left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \alpha, \tag{3}$$

where D^m is defined by (2).

If the co-analytic part of $f = h + \overline{g}$ is identically zero, then the family $S_H(m,n,\alpha,\beta)$ turns out to be the class $N_{m,n}(\alpha,\beta)$ introduced by Eker and Owa (2009) for the analytic case. Let we denote the subclass $\overline{S_H}(m,n,\alpha,\beta)$ consist of harmonic functions $f_m = h + \overline{g_m}$ in $\overline{S_H}(m,n,\alpha,\beta)$ so that h and g_m are of the $f_m = h + \overline{g_m}$ for

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k \quad (a_k, b_k \ge 0). \tag{4}$$

The class $\overline{S_H}(m,n,\alpha,\beta)$ includes a variety of well known subclasses of S_H. For example $\overline{S_H}(1,0,\alpha,0) \equiv HS(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U, $\overline{S_H}(2,1,\alpha,0)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U and $\overline{S_H}(n+1,n,\alpha,0) \equiv \overline{H}(n,\alpha)$ is the class of Sâlâgean-type harmonic univalent functions.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avci and Zlotkiewicz (1991) showed that if $\sum_{k=2}^{\infty} k(|a_k|+|b_k|) \le 1$ then $f \in HS(0)$ and if $\sum_{k=2}^{\infty} k^2(|a_k|+|b_k|) \le 1$ then $f \in HK(0)$. Silverman (1998) proved that the above two coefficient conditions are also necessary if $f = h + \overline{g}$ has negative coefficients. Later, Silverman and Silvia (1999) improved the results of Avci and Zlotkiewicz (1991) and Silverman (1998) to the case b_1 not necessarily zero. For the harmonic functions f of the form (4) with $m = 1, \beta = 0$, Jahangiri (1999) showed that $f \in HS(\alpha)$ iff

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \le 1-\alpha$$

and $f \in HK(\alpha)$ iff

$$\sum_{k=2}^{\infty}k(k-\alpha)|a_k| + \sum_{k=1}^{\infty}k(k+\alpha)|b_k| \le 1-\alpha.$$

In this paper, we will give a sufficient condition for $f = h + \overline{g}$ given by (1) to be in $S_H(m,n,\alpha,\beta)$ and it is shown that this condition is also necessary for functions in $\overline{S_H}(m,n,\alpha,\beta)$. Distortion theorems, extreme points, convolution conditions, convex combinations and neighborhoods of such functions are considered. The following results will be required in our investigation Ahujai et al.(2002).

Lemma 1

If α is a real number and ω is a complex number , then

$$Re(\omega) \ge \alpha \Leftrightarrow |\omega + (1-\alpha)| - |\omega - (1+\alpha)| \ge 0.$$

Lemma 2

If ω is a complex number and α, β are real numbers, then

$$Re(\omega) \ge \beta|\omega - 1| + \alpha \Leftrightarrow Re\{\omega(1+\beta e^{i\theta}) - \beta e^{i\theta}\} \ge \alpha \quad (-\pi \le \theta \le \pi).$$

MAIN RESULTS

In our first theorem, we introduced a sufficient coefficient bound for harmonic functions in $S_H(m,n,\alpha,\beta)$.

Theorem 1

Let $f = h + \overline{g}$ be given by (1). Furthermore, let

$$\sum_{k=1}^{\infty}\left(\frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha}|a_k| + \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}|b_k|\right) \le 2 \quad (5)$$

($a_1 = 1$, $\alpha(0 \le \alpha < 1)$, $\beta \ge 0$, $m \in N$, $n \in N_0$ and $m > n$). Then f is sense-preserving, harmonic univalent in U and $f \in S_H(m,n,\alpha,\beta)$.

Proof

If $z_1 \ne z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \ge 1 - \frac{\sum_{k=1}^{\infty}\frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}|b_k|}{1 - \sum_{k=2}^{\infty}\frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha}|a_k|} \ge 0$$

which proves univalence. Note that f is sense-preserving in U. This is because

$$|h'(z)| \ge 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty}\frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha}|a_k|$$

$$\ge \sum_{k=1}^{\infty}\frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}|b_k|$$

$$> \sum_{k=1}^{\infty}\frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}|b_k||z|^{k-1}$$

$$\ge \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \ge |g'(z)|.$$

Let f of the form given by (1) satisfy the condition (5). We will show that the inequality (3) is satisfied and so $f \in S_H(m,n,\alpha,\beta)$. Using Lemma 2, it is enough to show that

$$Re\left\{\frac{D^m f(z)}{D^n f(z)}(1+\beta e^{i\theta})-\beta e^{i\theta}\right\}>\alpha\ (-\pi\le\theta\le\pi). \tag{6}$$

That is, $Re\left\{\frac{A(z)}{B(z)}\right\}\ge\alpha$, where

$$A(z)=(1+\beta e^{i\theta})D^m f(z)-\beta e^{i\theta}D^n f(z)$$
$$=z+\sum_{k=2}^{\infty}\left[k^m+\beta e^{i\theta}(k^m-k^n)\right]a_k z^k$$
$$+(-1)^n\sum_{k=1}^{\infty}\left[(-1)^{m-n}k^m+\beta e^{i\theta}((-1)^{m-n}k^m-k^n)\right]\overline{b_k z^k},$$
$$B(z)=D^n f(z)=z+\sum_{k=2}^{\infty}k^n a_k z^k+(-1)^n\sum_{k=1}^{\infty}k^n\overline{b_k z^k}.$$

In view of Lemma 1, we only need to prove that

$$\left|A(z)+(1-\alpha)B(z)\right|-\left|A(z)-(1+\alpha)B(z)\right|\ge 0.$$

It is easy to show that;

$$\left|A(z)+(1-\alpha)B(z)\right|-\left|A(z)-(1+\alpha)B(z)\right|\ge 2(1-\alpha)|z|$$
$$-2\sum_{k=2}^{\infty}[(1+\beta)k^m-(\alpha+\beta)k^n]|a_k||z|^k$$
$$-\sum_{k=1}^{\infty}\left[|(-1)^{m-n}k^m+(1-\alpha)k^n|+|(-1)^{m-n}k^m-(1+\alpha)k^n|\right]|b_k||z|^k$$
$$-\sum_{k=1}^{\infty}\left[2\beta|(-1)^{m-n}k^m-k^n|\right]|b_k||z|^k$$

$$=\begin{cases}2(1-\alpha)|z|-2\sum_{k=2}^{\infty}[(1+\beta)k^m-(\alpha+\beta)k^n]|a_k||z|^k \\ \qquad -2\sum_{k=1}^{\infty}[(1+\beta)k^m+(\alpha+\beta)k^n]|b_k||z|^k;\ m-n \text{ is odd} \\ 2(1-\alpha)|z|-2\sum_{k=2}^{\infty}[(1+\beta)k^m-(\alpha+\beta)k^n]|a_k||z|^k \\ \qquad -2\sum_{k=1}^{\infty}[(1+\beta)k^m-(\alpha+\beta)k^n]|b_k||z|^k;\ m-n \text{ is even}\end{cases}$$

$$=2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{[(1+\beta)k^m-(\alpha+\beta)k^n]}{1-\alpha}|a_k||z|^{k-1}\right.$$
$$\left.-\sum_{k=1}^{\infty}\frac{[(1+\beta)k^m-(\alpha+\beta)k^n]}{1-\alpha}|b_k||z|^{k-1}\right\}$$
$$>2(1-\alpha)\left\{1-\sum_{k=2}^{\infty}\frac{[(1+\beta)k^m-(\alpha+\beta)k^n]}{1-\alpha}|a_k|\right.$$
$$\left.-\sum_{k=1}^{\infty}\frac{[(1+\beta)k^m-(\alpha+\beta)k^n]}{1-\alpha}|b_k|\right\}$$

The last expression is non-negative by (5), and so the proof is complete. The harmonic univalent function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\beta)k^m-(\alpha+\beta)k^n}x_k z^k$$
$$+\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}\overline{y_k z^k}, \tag{7}$$

($m\in N$, $n\in N_0$, $m>n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$) shows that the coefficient bound given by (5) is sharp. The functions of the form (7) are in $S_H(m,n,\alpha,\beta)$ because

$$\sum_{k=1}^{\infty}\left(\frac{(1+\beta)k^m-(\alpha+\beta)k^n}{1-\alpha}|a_k|+\frac{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}|b_k|\right)$$
$$=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

In the following theorem it is shown that the condition (5) is also necessary for functions $f_m=h+\overline{g_m}$ where h and g_m are of the form (4).

Theorem 2

Let $f_m=h+\overline{g_m}$ be given by (4). Then $f_m\in\overline{S}_H(m,n,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty}\left(\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]a_k\right.$$
$$\left.+\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]b_k\right)\le 2(1-\alpha) \tag{8}$$

($a_1=1$, $\alpha(0\le\alpha<1)$, $\beta\ge 0$, $m\in N$, $n\in N_0$ and $m>n$).

Proof

Since $\overline{S}_H(m,n,\alpha,\beta)\subset S_H(m,n,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. Note that a necessary and sufficient condition for $f_m=h+\overline{g_m}$ to be in $\overline{S_H}(m,n,\alpha,\beta)$ is that

$$Re\left\{(1+\beta e^{i\theta})\frac{D^m f_m(z)}{D^n f_m(z)}-\beta e^{i\theta}\right\}\ge\alpha.$$

This is equivalent to

$$Re\left\{\frac{(1+\beta e^{i\theta})D^m f_m(z)-\beta e^{i\theta}D^n f_m(z)-\alpha D^n f_m(z)}{D^n f_m(z)}\right\}$$

$$= Re\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[(1+\beta e^{i\theta})k^m-\beta e^{i\theta}k^n-\alpha k^n\right]a_k z^{k-1}}{1-\sum_{k=2}^{\infty}k^n a_k z^{k-1}+\frac{\bar{z}}{z}(-1)^{m+n-1}\sum_{k=1}^{\infty}k^n \overline{b_k z^{k-1}}} \quad (9)$$

$$-\frac{\frac{\bar{z}}{z}\sum_{k=1}^{\infty}\left[(1+\beta e^{i\theta})k^m-\beta e^{i\theta}(-1)^{m-n}k^n-\alpha(-1)^{m-n}k^n\right]\overline{b_k z^{k-1}}}{1-\sum_{k=2}^{\infty}k^n a_k z^{k-1}+\frac{\bar{z}}{z}(-1)^{m+n-1}\sum_{k=1}^{\infty}k^n\overline{b_k z^{k-1}}}\}\geq 0.$$

The above required condition (9) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$Re\frac{(1-\alpha)-\left[\sum_{k=2}^{\infty}\left(k^m-\alpha k^n\right)a_k r^{k-1}+\sum_{k=1}^{\infty}(k^m-(-1)^{m-n}\alpha k^n)b_k r^{k-1}\right]}{1-\sum_{k=2}^{\infty}k^n a_k r^{k-1}+(-1)^{m+n-1}\sum_{k=1}^{\infty}k^n b_k r^{k-1}}$$

$$-\frac{\beta e^{i\theta}\left[\sum_{k=2}^{\infty}(k^m-k^n)a_k r^{k-1}+\sum_{k=1}^{\infty}(k^m-(-1)^{m-n}k^n)b_k r^{k-1}\right]}{1-\sum_{k=2}^{\infty}k^n a_k r^{k-1}+(-1)^{m+n-1}\sum_{k=1}^{\infty}k^n b_k r^{k-1}}\geq 0.$$

Since $Re(-e^{i\theta})\geq -\left|e^{i\theta}\right|=-1$, the above inequality reduces to

$$\frac{1-\alpha-\sum_{k=2}^{\infty}[(1+\beta)k^m-(\alpha+\beta)k^n]a_k r^{k-1}}{1-\sum_{k=2}^{\infty}k^n a_k r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}k^n b_k r^{k-1}}$$

$$-\frac{\sum_{k=1}^{\infty}[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n]b_k r^{k-1}}{1-\sum_{k=2}^{\infty}k^n a_k r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}k^n b_k r^{k-1}}\geq 0. \quad (10)$$

If the condition (8) does not hold, then the expression in (10) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in (0,1) for which the quotient in (10) is negative. This contradicts the required condition for $f_m\in \overline{S}_H(m,n,\alpha,\beta)$. And so the proof is complete. Next we determine the extreme points of the closed convex hull of $\overline{S}_H(m,n,\alpha,\beta)$, denoted by $clco\overline{S}_H(m,n,\alpha,\beta)$.

Theorem 3

Let f_m be given by (4). Then $f_m\in\overline{S}_H(m,n,\alpha,\beta)$ if and only if

$$f_m(z)=\sum_{k=1}^{\infty}(x_k h_k(z)+y_k g_{m_k}(z))$$

where $h_1(z)=z,$

$$h_k(z)=z-\frac{(1-\alpha)}{(1+\beta)k^m-(\alpha+\beta)k^n}z^k \quad (k=2,3,\ldots) \text{ and}$$

$$g_{m_k}(z)=z+(-1)^{m-1}\frac{(1-\alpha)}{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}\overline{z^k} \ (k=1,2,\ldots),$$

$x_k\geq 0,\ y_k\geq 0$ and $x_1=1-\sum_{k=2}^{\infty}(x_k+y_k)\geq 0$. In particular, the extreme points of $\overline{S}_H(m,n,\alpha,\beta)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof

Suppose

$$f_m(z)=\sum_{k=1}^{\infty}\left[x_k h_k(z)+y_k g_{m_k}(z)\right]$$

$$=\sum_{k=1}^{\infty}(x_k+y_k)z-\sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\beta)k^m-(\alpha+\beta)k^n}x_k z^k$$

$$+(-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}y_k\overline{z^k}.$$

Then

$$\sum_{k=2}^{\infty}\frac{(1+\beta)k^m-(\alpha+\beta)k^n}{1-\alpha}\left(\frac{1-\alpha}{(1+\beta)k^m-(\alpha+\beta)k^n}x_k\right)$$

$$+\sum_{k=1}^{\infty}\frac{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha}\left(\frac{1-\alpha}{(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n}y_k\right)$$

$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1-x_1\leq 1$$

and so $f_m(z)\in clco\overline{S}_H(m,n,\alpha,\beta)$. Conversely, if $f_m(z)\in clco\overline{S}_H(m,n,\alpha,\beta)$, then

$$a_k \le \frac{1-\alpha}{(1+\beta)k^m - (\alpha+\beta)k^n}$$

and

$$b_k \le \frac{1-\alpha}{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}.$$

Set

$$x_k = \frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} a_k \ (k = 2,3,\ldots)$$

and

$$y_k = \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} b_k \ (k = 1,2,3,\ldots).$$

Then note that by Theorem 2, $0 \le x_k \le 1$, $(k = 2,3,\ldots)$ and $0 \le y_k \le 1$ $(k = 1,2,\ldots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and note that by Theorem 2, $x_1 \ge 0$.

Consequently, we obtain

$x_1 = 1 - \sum_{k=2}^{\infty}\left(x_k h_k(z) + y_k g_{m_k}(z)\right)$ as required.

The following theorem gives the distortion bounds for functions in $\overline{S}_H(m,n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 4

Let $f_m \in \overline{S}_H(m,n,\alpha,\beta)$. Then for $|z| = r < 1$ we have

$$|f_m(z)| \le (1+b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{(1+\beta)2^{m-n} - (\alpha+\beta)} - \frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)}{(1+\beta)2^{m-n} - (\alpha+\beta)} b_1\right) r^2$$

and

$$|f_m(z)| \ge (1-b_1)r - \frac{1}{2^n}\left(\frac{1-\alpha}{(1+\beta)2^{m-n} - (\alpha+\beta)} - \frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)}{(1+\beta)2^{m-n} - (\alpha+\beta)} b_1\right) r^2$$

Proof

We only prove the right hand inequality. The proof for the left hand inequality is similar.

Let $f_m \in \overline{S}_H(m,n,\alpha,\beta)$. Taking the absolute value of f_m we have

$$|f_m(z)| \le (1+b_1)r + \sum_{k=2}^{\infty}(a_k + b_k) r^k$$

$$\le (1+b_1)r + r^2 \sum_{k=2}^{\infty}(a_k + b_k)$$

$$= (1+b_1)r + r^2 \frac{1-\alpha}{2^n((1+\beta)2^{m-n} - (\alpha+\beta))} \times \sum_{k=2}^{\infty} \frac{2^n((1+\beta)2^{m-n} - (\alpha+\beta))}{1-\alpha}(a_k + b_k)$$

$$\le (1+b_1)r + r^2 \frac{1-\alpha}{2^n\left((1+\beta)2^{m-n} - (\alpha+\beta)\right)} \times \sum_{k=2}^{\infty}\left(\frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} a_k + \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} b_k\right)$$

$$\le (1+b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{(1+\beta)2^{m-n} - (\alpha+\beta)} - \frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)}{(1+\beta)2^{m-n} - (\alpha+\beta)} b_1\right) r^2.$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 1

Let $f_m \in \overline{S}_H(m,n,\alpha,\beta)$, then for $|z| = r < 1$ we have

$$\left\{ w : |w| < \frac{2^m - 1 - \beta(2^n - 2^m) - \alpha(2^n - 1)}{(1+\beta)2^m - (\alpha+\beta)2^n} - \frac{(1+\beta)(2^m - 1) - (\alpha+\beta)(2^n - (-1)^{m-n})}{(1+\beta)2^m - (\alpha+\beta)2^n} b_1 \right\} \subset f_m(U).$$

Remark 1

If we take $m = n+1$ and $\beta = 0$, then for $n = 0$ and $n = 1$ the above covering result given in Jahangiri (1999) and Jahangiri et al. (2002), respectively. For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \overline{z^k} \quad (11)$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} B_k \overline{z^k}, \quad (12)$$

we define the convolution of two harmonic functions f_m

and F_m as

$$(f_m * F_m)(z) = f_m(z) * F_m(z)$$
$$= z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k B_k \overline{z^k}. \quad (13)$$

Using this definition, we show that the class $\overline{S_H}(m,n,\alpha,\beta)$ is closed under convolution.

Theorem 5

Let $f_m \in \overline{S}_H(m,n,\alpha,\beta)$ and $F_m \in \overline{S}_H(m,n,\gamma,\beta)$. Then

$$\left(f_m(z) * F_m(z)\right) \in \overline{S}_H(m,n,\alpha,\beta) \subset \overline{S}_H(m,n,\gamma,\beta)$$

where $0 \le \gamma \le \alpha < 1$.

Proof

Let $f_m(z)$ given by (11) be in $\overline{S_H}(m,n,\alpha,\beta)$ and $F_m(z)$ given by (12) be in $\overline{S}_H(m,n,\gamma,\beta)$. Then the convolution $f_m * F_m$ is given by (13). We want to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2. For $F_m \in \overline{S}_H(m,n,\gamma,\beta)$, we note that $A_k < 1$ and $B_k < 1$. Now, for the convolution function $f_m * F_m$ we obtain

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^m - (\gamma+\beta)k^n}{1-\gamma} a_k A_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (-1)^{m-n}(\gamma+\beta)k^n}{1-\gamma} b_k B_k$$
$$\le \sum_{k=2}^{\infty} \frac{(1+\beta)k^m - (\gamma+\beta)k^n}{1-\gamma} a_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (-1)^{m-n}(\gamma+\beta)k^n}{1-\gamma} b_k$$
$$\le \sum_{k=2}^{\infty} \frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} b_k$$
$$\le 1$$

since $0 \le \gamma \le \alpha < 1$ and $f_m \in \overline{S}_H(m,n,\alpha,\beta)$. Therefore $\left(f_m(z) * F_m(z)\right) \in \overline{S}_H(m,n,\alpha,\beta) \subset \overline{S}_H(m,n,\gamma,\beta)$.

Now we show that $\overline{S_H}(m,n,\alpha,\beta)$ is closed under convex combinations of its members.

Theorem 6

The class $\overline{S_H}(m,n,\alpha,\beta)$ is closed under convex combination.

Proof

Let $f_{m_i} \in \overline{S}_H(m,n,\alpha,\beta)$, where f_{m_i} is given by

$$f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k_i} \overline{z^k} \; (i = 1,2,3,\ldots).$$

Then by (8),

$$\sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} a_{k_i}$$
$$+ \sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} b_{k_i} \le 2. \quad (14)$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1,$ the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^{m-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \overline{z^k}.$$

Then by (14),

$$\sum_{k=1}^{\infty} \frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} \sum_{i=1}^{\infty} t_i a_{k_i}$$
$$+ \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} \sum_{i=1}^{\infty} t_i b_{k_i}$$
$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \left(\frac{(1+\beta)k^m - (\alpha+\beta)k^n}{1-\alpha} a_{k_i} + \frac{(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n}{1-\alpha} b_{k_i} \right) \right\}$$
$$\le 2 \sum_{i=1}^{\infty} t_i = 2,$$

and so $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \overline{S}_H(m,n,\alpha,\beta)$.

Finally, we will give δ-neighborhood of $f \in S_H(m,n,\alpha,\beta)$ which is given by (1). Ruscheweyh (1981) is introduced by the δ-neighborhood of f the set

$$N_\delta(f) = \left\{ F = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} : \sum_{k=2}^{\infty} k\left(|a_k - A_k| + |b_k - B_k|\right) + |b_1 - B_1| \le \delta \right\}.$$

In our case, let us define the generalized δ-neighborhood of f to be the set

$$N_\delta(f) = \{ F : \sum_{k=2}^{\infty} \left(\left[(1+\beta)k^m - (\alpha+\beta)k^n \right] |a_k - A_k| \right.$$
$$\left. + \left[(1+\beta)k^m - (-1)^{m-n}(\alpha+\beta)k^n \right] |b_k - B_k| \right)$$

$$+\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]\left|b_1-B_1\right| \le (1-\alpha)\delta\}.$$

Theorem 7

Let f be given by (1). If f satisfies the conditions

$$\sum_{k=2}^{\infty}k\left(\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]|a_k|+\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]|b_k|\right)$$
$$\le (1-\alpha)-\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|b_1| \quad (15)$$

and

$$\delta \le \frac{1-\alpha}{2-\alpha}\left(1-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)}{1-\alpha}|b_1|\right),$$

then $N_\delta(f)\subset S_H(m,n,\alpha,\beta)$.

Proof

Let f satisfy (15) and $F=z+\overline{B_1 z}+\sum_{k=2}^{\infty}\left(A_k z^k+\overline{B_k z^k}\right)$ belong to $N_\delta(f)$. We have

$$\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|B_1|$$
$$+\sum_{k=2}^{\infty}\left(\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]|A_k|+\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]|B_k|\right)$$

$$\le\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|B_1-b_1|+\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|b_1|$$
$$+\sum_{k=2}^{\infty}\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]|A_k-a_k|$$
$$+\sum_{k=2}^{\infty}\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]|B_k-b_k|$$
$$+\sum_{k=2}^{\infty}\left(\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]|a_k|+\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]|b_k|\right)$$
$$\le(1-\alpha)\delta+\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|b_1|$$
$$+\frac{1}{2-\alpha}\sum_{k=2}^{\infty}k\left(\left[(1+\beta)k^m-(\alpha+\beta)k^n\right]|a_k|+\left[(1+\beta)k^m-(-1)^{m-n}(\alpha+\beta)k^n\right]|b_k|\right)$$
$$\le(1-\alpha)\delta+\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|b_1|$$
$$+\frac{1}{2-\alpha}\left[(1-\alpha)-\left[(1+\beta)-(-1)^{m-n}(\alpha+\beta)\right]|b_1|\right]$$
$$\le(1-\alpha).$$

Hence, for

$$\delta \le \frac{1-\alpha}{2-\alpha}\left(1-\frac{(1+\beta)-(-1)^{m-n}(\alpha+\beta)}{1-\alpha}\left|b_1\right|\right),$$

we have $f\in S_H(m,n,\alpha,\beta)$.

Remark 2

The results of his paper, for $\beta=0$, coincide with the results in Yalçin (2005). Furthermore, if we take $m=n+1$ and $\beta=0$ in our theorems, we obtain the results given in Jahangiri et al. (2002). Therefore our present study is generalization of Yalçin (2005) and Jahangiri et al. (2002).

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