International Journal of the Physical Sciences Vol. 6(27), pp. 6234-6240, 2 November, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.543
ISSN 1992 - 1950 ©2011 Academic Journals

*Full Length Research Paper*

# Steady and unsteady exact inverse solutions for the flow of a viscous fluid

**S. Islam[1], T. Haroon[1], Manzoor Elahi[1], Murad Ullah[2]* and A. M. Siddiqui[3]**

[1]COMSATS Institute of Information Technology Park Road, Chakshazad, Islamabad, Pakistan.
[2]Islamia College Peshawar (Charted University), Peshawar, NWFP, Pakistan.
[3]Pennsylvania State University, York Campus, Edgecombe 1703, USA.

Accepted 23 May, 2011

**In this paper, inverse solutions are derived for the equations of two dimensional flows of a viscous incompressible fluid by assuming certain conditions on the stream function. The equation is coupled in terms of the stream function by eliminating the pressure between the component forms of the flow equation. The solutions for steady and unsteady cases are found by prescribing the vorticity proportional to the stream function perturbed by the uniform stream in *x* and *y* directions. The solutions for the stream functions and velocity components are derived in each case and comparison of results have been made with the known results.**

**Key words:** Inverse solutions, viscous fluid, unsteady flow, stream function.

## INTRODUCTION

In spite of mathematical difficulty of the momentum equations for viscous incompressible fluid flows, there exist very few exact solutions. As reported by Wang (1991), exact solutions are important for the following two reasons. First, they provide a solution to be a flow that has technical relevance. Second, such solutions can be used as checks against intricate numerical codes that have been developed for much more complex flows. The usual technique is to assume a particular form of the stream function and apply the so-called inverse method to obtain the exact solutions.

Most of these exact solutions have been derived by variety of methods (Labropulu, 1987). From the historical point of view, the first investigations have been made to these equations by Taylor (1923) by taking the vorticity to be proportional to the stream function, where he obtained the solution of the problem of a double infinite array of vortices decaying exponentially with time. The works of Lin and Tobak (1986), Hamdan (1998), Hui (1987), Dorrepaal (1986), Chandna and Oku-Ukpong (1994), Labropulu (1987) and Siddiqui et al. (2006) are among those who have used inverse methods. The inverse solutions for non-Newtonian fluids have become attractive because of the non-linearities which did not occur only in the inertial part, but also in the viscous part of these equations. Rajagopal (1980), Kaloni and Huschilt (1984), Siddiqui (1986), Labropulu (2002), Islam and Zhou (2007a, 2007b), Islam et al. (2007, 2008), Kamran et al. (2011) and Chandna and Oku-Ukpong (1994) are among those that have received considerable attention in the theory of exact solutions for the non-Newtonian fluid flows.

In this paper, we are interested in studying the flows of a two dimensional incompressible viscous fluid. We obtain a class of exact solutions in an unbounded domain which do not require additional boundary conditions. The velocity field having two components yields the flow equation in two components. The equations then are coupled in the vorticity form known as the compatibility equation. Knowing the incompressibility from the continuity equation, we introduce the stream function in terms of the velocity components, which gives the Newtonian flow equation. The equation is non-dimensionalized by introducing the dimensionless factors with respect to the reference velocity. The solutions are found by equating the vorticity distribution proportional to

*Corresponding author. E-mail: muradullah90@yahoo.com.

the stream function perturbed into the directions of axis.

## BASIC EQUATIONS

The basic equations governing the flow of an incompressible fluid are:

$$\nabla\cdot\mathbf{V}=0, \tag{1}$$

$$\rho\left[\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}.\nabla)\mathbf{V}\right]=\nabla.\tau, \tag{2}$$

where $\rho$ is the constant density, $\mathbf{V}$ is the velocity vector, $\mu$ is the dynamic viscosity, $\tau=-\mathrm{p}\mathbf{I}+\mu\mathbf{A}_1$, where $p$ is the static pressure, $\mathbf{I}$ is the identity matrix and $\mathbf{A}_1=\nabla\mathbf{V}+(\nabla\mathbf{V})^{\mathrm{T}}$ is the Rivlin Erickson tensor.

For unsteady plane flows in Cartesian coordinates, $\mathbf{V}$ can be expressed as:

$$\mathbf{V}=[u(x,y,t),v(x,y,t),0], \tag{3}$$

where $u$ and $v$ are the velocity components in the $x$ and $y-$ directions, respectively.

Substituting Equation 3 into Equations 1 and 2, furnish the following equations in its component forms:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4}$$

$$\frac{\partial\tilde{p}}{\partial x}+\rho\left[\frac{\partial u}{\partial t}-v\omega\right]=\mu\nabla^2\omega, \tag{5}$$

$$\frac{\partial\tilde{p}}{\partial y}+\rho\left[\frac{\partial v}{\partial t}+u\omega\right]=\mu\nabla^2\omega, \tag{6}$$

where the vorticity $\omega$ and generalized pressure $\tilde{p}$ functions are defined as:

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega, \tag{7}$$

$$\tilde{p}=\frac{1}{2}\rho\left(u^2+v^2\right)+p. \tag{8}$$

**Remark 1**

Equations 4 to 6 are three partial differential equations for three unknown functions $u$ , $v$ and $\tilde{p}$ of the variables $(x,y,t)$. Once $u$ , $v$ and $\tilde{p}$ are obtained, the pressure $p$ can be calculated from Equation 8.

Eliminating the generalized pressure from the Equations 5 and 6 by cross differentiation, we get:

$$\rho\left[\frac{\partial\omega}{\partial t}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\omega\right]=\mu\nabla^2\omega. \tag{9}$$

Let us introduce a stream function $\psi(x,y,t)$ such that:

$$u=\frac{\partial\psi}{\partial y},\ \ v=-\frac{\partial\psi}{\partial x}. \tag{10}$$

Then the continuity equation is identically satisfied and Equation 9 becomes a single partial differential equation as follows:

$$\frac{\partial}{\partial t}\nabla^2\psi-\left\{\psi,\nabla^2\psi\right\}=\nu\nabla^4\psi, \tag{11}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid and

$$\left\{\psi,\nabla^2\psi\right\}=\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi-\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\nabla^2\psi, \quad 12)$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{13}$$

Equation 11 is non-linear and represents the planar motion of a viscous incompressible couple stress fluid in the absence of body force.

For universal use, we non-dimensionalized the Equation 11 by introducing

$$\psi^*=\frac{\psi}{\upsilon},\ u^*=\frac{u}{V},v^*=\frac{v}{V},x^*=\frac{xV}{\nu},\ y^*=\frac{yV}{\nu},\ t^*=\frac{tV^2}{\nu}, \tag{14}$$

where $V$ is some characteristic or reference velocity. With the non-dimensional factors, the Equation 11 after dropping the asteriks, become non-dimensional in the following form:

$$\frac{\partial}{\partial t}\nabla^2\psi-\left(\psi,\nabla^2\psi\right)=\nu\nabla^4\psi. \tag{15}$$

In the preceding discussion, we obtain the solutions of Equation 15 by applying the inverse method. To the study of inverse method, the boundary conditions in the flow field are not given and the solution of the partial differential equation is found by assuming geometrical and physical properties of the flow field. The proposed solution is applied both for the stream function and the vorticity distribution. However, in this paper we are interested to impose a particular solution to the vorticity, which will be discussed subsequently.

## SOLUTIONS METHODOLOGY

We assume the vorticity proportional to the stream function perturbed by the uniform stream in *x* and *y* that is:

$$\nabla^2\psi=\upsilon\left(\psi+ax+by\right),\qquad \upsilon\neq 0, \tag{16}$$

where $a$ , $b$ and $\upsilon$ are real constants. If $\upsilon=0$, the flow is irrotational $\left(\nabla^2\psi=0\right)$ and the Equation 15 is satisfied. For $\upsilon\neq 0$, we proceed.

Using expression of Equation 16 in Equation 15, we get:

$$\frac{\partial\psi}{\partial t}-b\frac{\partial\psi}{\partial x}+a\frac{\partial\psi}{\partial y}-\upsilon\left(\psi+ax+by\right)=0. \tag{17}$$

By setting

$$\Psi=\psi+ax+by \tag{18}$$

we obtain from Equation 16:

$$\nabla^2\Psi=\upsilon\Psi. \tag{19}$$

Therefore Equation 17 becomes:

$$\frac{\partial\Psi}{\partial t}-b\frac{\partial\Psi}{\partial x}+a\frac{\partial\Psi}{\partial y}-\upsilon\Psi=0. \tag{20}$$

### Remark 2

On setting $a=0$ , $b=-U$ $(U\geq 0)$ in the aforementioned equation we get the differential equation (Hui, 1987). Therefore, we study the non-trivial general case by introducing:

$$\alpha=ax+by,\quad \eta=y \tag{21}$$

we find that the Jacobean of the transformation is not zero,

$$\frac{\partial(\alpha,\eta)}{\partial(x,y)}=a\neq 0.$$

In view of Equation 21, Equation 20 simplifies to

$$\frac{\partial\Psi}{\partial t}+a\frac{\partial\Psi}{\partial\eta}-\upsilon\Psi=0. \tag{22}$$

which is a first order linear partial differential equation in $\Psi$. The solution of $\Psi=\psi+ax+by,$ which satisfies both the Equations 19 and 22 would be the solution of the unsteady two-dimensional Newtonian fluid equations. To find the solution of Equation 22, we discuss the steady and non-steady flows, subsequently.

## STEADY FLOW SOLUTIONS

For steady flow, $\partial\Psi/\partial t=0$ and Equation 22 becomes

$$\frac{\partial\Psi}{\partial\eta}-\frac{\upsilon}{a}\Psi=0.$$

Integration with respect to $\eta$ yields the solution:

$$\Psi=g(\alpha)e^{\left\{\frac{\upsilon}{a}\eta\right\}}, \tag{23}$$

where $g(\alpha)$ is an arbitrary function to be determined. To find this we substitute Equation 23 in Equation 19 to obtain:

$$\left(a^2+b^2\right)g''(\alpha)+\frac{2\upsilon b}{a}g'(\alpha)+\left[\frac{\upsilon^2-\upsilon a^2}{a^2}\right]g(\alpha)=0 \tag{24}$$

whose auxiliary equation is $a^2\left(a^2+b^2\right)m^2+2\upsilon abm+\upsilon^2-\upsilon a^2=0.$

Solving for $m$, the roots of the equation are

$$m_{1,2}=\frac{-\upsilon b\pm a\sqrt{\upsilon\left(a^2+b^2\right)-\upsilon^2}}{a\left(a^2+b^2\right)}. \tag{25}$$

Here arise three cases depending upon the sign of $\left(a^2+b^2\right)-\upsilon$.

**Case 1**

$\left(a^2+b^2\right)>\upsilon>0$. In this case the solution of Equation 24 is given by $g(\alpha)=A_1e^{(m_1\alpha)}+A_2e^{(m_2\alpha)}$, and stream function and velocity components are, respectively, given by:

$$\psi=-ax-by+\exp\left(\frac{\upsilon}{a}y\right)\left[A_1e^{\{m_1(ax+by)}\}+A_2e^{\{m_2(ax+by)}\}\right], \tag{26}$$

$$u=-b+\exp\left(\frac{\upsilon}{a}y\right)\left[\left(\frac{\upsilon}{a}+m_1b\right)A_1e^{\{m_1(ax+by)\}}+\left(\frac{\upsilon}{a}+m_2b\right)A_2e^{\{m_2(ax+by)\}}\right], \tag{27}$$

$$v=a-\exp\left(\frac{\upsilon}{a}y\right)\left[A_1am_1e^{\{m_1(ax+by)\}}+A_2am_2e^{\{m_2(ax+by)\}}\right]. \tag{28}$$

For $a<0$ ($u=-b$; $v=a$) solution of Equation 26 represent a uniform stream with a perturbation part which decays exponentially as $y$ increase.

**Case 2**

If $\upsilon=\left(a^2+b^2\right)$ and $\upsilon>0$. In this case the Equation 24 has the solution of the form:

$$g(\alpha)=\left(R_1+R_2\alpha\right)e^{\left(\frac{-b}{a}\alpha\right)}, \tag{29}$$

and as before, the stream function and velocity components are, respectively, given by:

$$\psi=-ax-by+\left(R_1+R_2(ax+by)\right)e^{(ay-bx)}, \tag{30}$$

$$u=-b+\left[bR_2+a\left(R_1+R_2(ax+by)\right)e^{(ay-bx)}\right], \tag{31}$$

$$v=a-\left[R_2a-b\left(R_1+R_2(ax+by)\right)e^{(ay-bx)}\right]. \tag{32}$$

**Case 3**

$\upsilon\left(a^2+b^2\right)-\upsilon^2<0$

Here two possibilities arise:

1. $\upsilon>0$ , $\left(a^2+b^2\right)<\upsilon$
2. $\upsilon<0$ , $\left(a^2+b^2\right)>\upsilon$

***Subcase 1***

$\upsilon>0$, $\left(a^2+b^2\right)<\upsilon$ .

Then the Equation 24 has the solution of the form:

$$g(\alpha)=e^{\left\{\frac{-\upsilon b}{a(a^2+b^2)}\alpha\right\}}\left[C_1\cos m\alpha+C_2\sin m\alpha\right], \tag{33}$$

where

$$m=\frac{\sqrt{\upsilon^2-\upsilon\left(a^2+b^2\right)}}{(a^2+b^2)}.$$

Thus, the stream function and velocity components are, respectively, given by:

$$\psi=-ax-by+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\left[C_1\cos m(ax+by)+C_2\sin m(ax+by)\right], \tag{34}$$

$$u=-b+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\begin{bmatrix}\left(\frac{C_1\upsilon a}{a^2+b^2}+mbC_2\right)\ \cos m(ax+by)\\+\left(\frac{C_2\upsilon a}{a^2+b^2}-C_1mb\right)\ \sin m(ax+by)\end{bmatrix}, \tag{35}$$

$$v=a+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\begin{bmatrix}\left(\frac{C_1\upsilon a}{a^2+b^2}-C_2ma\right)\ \cos m(ax+by)\\+\left(\frac{C_2\upsilon a}{a^2+b^2}+C_1ma\right)\ \sin m(ax+by)\end{bmatrix}. \tag{36}$$

Here in all these expressions of $g(\alpha)$, $A_1$, $A_2$, $R_1$, $R_2$, $C_1$ and $C_2$ are arbitrary constants.

***Sub case 2***

$\upsilon<0$ , $\left(a^2+b^2\right)>\upsilon$

In this case the Equation 24 has the solution of the form:

$$g(\alpha)=e^{\left\{\frac{\upsilon_1}{a(a^2+b^2)}\alpha b\right\}}\left[D_1\cos n\alpha+D_2\sin n\alpha\right], \qquad (37)$$

where

$$n=\frac{\sqrt{\upsilon_1^2+\upsilon_1\left(a^2+b^2\right)}}{(a^2+b^2)},\ \ \upsilon=-\upsilon_1,\upsilon_1>0.$$

Thus, the stream function and velocity components are, respectively, given by:

$$\psi=-ax-by+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\left[\alpha_1\cos n(ax+by)+\alpha_2\sin n(ax+by)\right], \qquad (38)$$

$$u=-b+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\begin{bmatrix}\left(\frac{\alpha_1\upsilon a}{a^2+b^2}+\alpha_2 nb\right)\cos n(ax+by)\\+\left(\frac{\alpha_2\upsilon a}{a^2+b^2}-\alpha_1 nb\right)\sin n(ax+by)\end{bmatrix}, \qquad (39)$$

$$v=a+e^{\left(\frac{\upsilon(ay-bx)}{a^2+b^2}\right)}\begin{bmatrix}\left(\frac{b\alpha_1\upsilon}{a^2+b^2}-\alpha_2 ma\right)\cos n(ax+by)\\+\left(\frac{b\alpha_2\upsilon}{a^2+b^2}+\alpha_1 ma\right)\sin n(ax+by)\end{bmatrix}. \qquad (40)$$

Here, in all these expressions of $g(\alpha)$, $A_1$, $A_2$, $R_1$, $R_2$, $C_1$, $C_2$, $\alpha_1$ and $\alpha_2$ are arbitrary constants.

## UNSTEADY FLOW SOLUTIONS

For the unsteady case, we write Equation 22 as:

$$\frac{\partial\Psi}{\partial t}+a\frac{\partial\Psi}{\partial\eta}-\upsilon\Psi=0, \qquad (41)$$

and set $Y=\eta-at$, to find that

$$\frac{\partial(Y,\eta)}{\partial(y,t)}=-a\neq 0.$$

In view of this, Equation 41 becomes:

$$\frac{\partial\Psi}{\partial\eta}-\frac{\upsilon}{a}\Psi=0 \qquad (42)$$

whose solution is given by:

$$\Psi=g(Y,\alpha)e^{\left\{\frac{\upsilon}{a}\eta\right\}}, \qquad (43)$$

where $g(Y,\alpha)$ is a function to be determined. Differentiating Equation 43 partially with respect to $x$ and $y$ and substituting into Equation 19, we have:

$$\left(a^2+b^2\right)\frac{\partial^2 g}{\partial\alpha^2}+\frac{\partial^2 g}{\partial Y^2}+2b\frac{\partial^2 g}{\partial Y\partial\alpha}+\frac{2\upsilon}{a}\left(\frac{\partial g}{\partial\alpha}b+\frac{\partial g}{\partial Y}\right)+\left(\frac{\upsilon}{a^2}-\upsilon\right)g=0. \qquad (44)$$

$$g(Y,\alpha)=G(X_1),\qquad X_1=Y\cos\theta+\alpha\sin\theta, \qquad (45)$$

where $\theta$ is a constant. Substituting Equation 45 in Equation 44, we get:

$$\begin{aligned}&a^2\left[\left(a^2+b^2\right)\sin^2\theta+\cos^2\theta+b\sin 2\theta\right]G''\\&+2\upsilon a\left[\cos\theta+b\sin\theta\right]G'+\left[\upsilon^2-\upsilon a^2\right]G=0,\end{aligned} \qquad (46)$$

where prime denotes differentiation with respect to $\mathfrak{m}$. Characteristic equation of Equation 46 is:

$$\begin{aligned}&a^2\left[\left(a^2+b^2\right)\sin^2\theta+\cos^2\theta+b\sin 2\theta\right]m^2\\&+2\upsilon a\left[\cos\theta+b\sin\theta\right]m+\left[\upsilon^2-\upsilon a^2\right]=0,\end{aligned} \qquad (47)$$

whose roots are

$$m_{1,2}=\frac{-\upsilon(\cos\theta+b\sin\theta)\pm\sqrt{\upsilon a^2\left[\left(a^2+b^2\right)\cos^2\theta+\sin^2\theta+b\sin 2\theta\right]-\upsilon^2a^2\sin^2\theta}}{a\left[\left(a^2+b^2\right)\sin^2\theta+\cos^2\theta+b\sin 2\theta\right]}. \qquad (48)$$

Depending upon the sign of $\upsilon a^2\left[\left(a^2+b^2\right)\cos^2\theta+\sin^2\theta+b\sin 2\theta\right]-\upsilon^2a^2\sin^2\theta$, the following cases arise.

### Case 1

If $\upsilon>0$ and $\left(a^2+b^2\right)\sin^2\theta+\cos^2\theta+b\sin 2\theta>\upsilon\sin^2\theta$. We find the following solution:

$$G(\upsilon_1)=A_1e^{(m_1\upsilon_1)}+A_2e^{(m_2\upsilon_1)}, \qquad (49)$$

where $m_1$ and $m_2$ are given by Equation 48.

The corresponding stream function and velocity components are given by:

$$\psi = -ax - by + e^{\left(\frac{\upsilon}{a}y\right)}\left[A_1 e^{\{m_1\upsilon_1\}} + A_2 e^{\{m_2\upsilon_1\}}\right], \tag{50}$$

$$u = -b + e^{\left(\frac{\upsilon}{a}y\right)}\begin{bmatrix} \left(\frac{\upsilon}{a}A_1 + m_1 A_1 (b\sin\theta + \cos\theta)\right)e^{\{m_1\upsilon_1\}} \\ +\left(\frac{\upsilon}{a}A_2 + m_2 A_2 (b\sin\theta + \cos\theta)\right)e^{\{m_2\upsilon_1\}} \end{bmatrix}, \tag{51}$$

$$v = a - e^{\left(\frac{\upsilon}{a}y\right)}\left[A_1 a m_1 \sin\theta e^{\{m_1\upsilon_1\}} + A_2 a m_2 \sin\theta e^{\{m_2\upsilon_1\}}\right], \tag{52}$$

$$\upsilon_1 = (y - at)\cos\theta + (ax + by)\sin\theta.$$

***Remark 3***

For $\theta = \frac{\pi}{2}$, the Equations 51 and 52 reduces to the steady flow solutions as given in Equations 27 and 28.

**Case 2**

If $\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta = \upsilon \sin^2\theta$. The solutions with their stream function and velocity components are

$$G(\upsilon_1) = (w_1 + w_2\upsilon_1)\exp\left(\frac{-\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right) \tag{53}$$

$$\psi = -ax - by + (w_1 + w_2\upsilon_1)e^{\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\}}, \tag{54}$$

$$u = -b + \left[w_2(b\sin\theta + \cos\theta) + (w_1 + w_2\upsilon_1)a\right]e^{\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\}}, \tag{55}$$

$$v = a - \left[w_2 a\sin\theta - (w_1 + w_2\upsilon_1)\left(\frac{b\sin\theta + \cos\theta}{\sin\theta}\right)\right]e^{\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\}} \tag{56}$$

$$\upsilon_1 = (y - at)\cos\theta + (ax + by)\sin\theta.$$

If $\theta = \frac{\pi}{2}$, solutions of Equations 55 and 56 reduce to the steady solutions of Equation 31 and 32.

**Case 3**

If ♍☉0 and $\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta < \upsilon \sin^2\theta$. The solutions with their stream function and velocity components are

$$G(\upsilon_1) = (C_1\cos m\upsilon_1 + C_2\sin m\upsilon_1)e^{\left(\frac{-\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right)}, \tag{57}$$

where

$$m = \sqrt{\upsilon^2 a^2\sin^2\theta - \upsilon a^2\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}.$$

$$\psi = -ax - by + (C_1\cos m\upsilon_1 + C_2\sin m\upsilon_1)e^{\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\}}, \tag{58}$$

$$u = -b + \mathrm{exponent}\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\} \times \begin{bmatrix} \left(\frac{\upsilon a^2\sin^2\theta C_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]} + C_2 m(b\sin\theta + \cos\theta)\right)\cos m\upsilon_1 \\ +\left(\frac{\upsilon a^2\sin^2\theta C_2}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]} + C_1 m(b\sin\theta + \cos\theta)\right)\sin m\upsilon_1 \end{bmatrix}, \tag{59}$$

$$v = a + \mathrm{exponent}\left\{\frac{\upsilon}{a}y - \frac{\upsilon(b\sin\theta + \cos\theta)\upsilon_1}{a\left[\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta\right]}\right\} \times \begin{bmatrix} \left(\frac{\upsilon(b\sin\theta + \cos\theta)\sin\theta C_1}{\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta} - C_2 m a\sin\theta\right)\cos m\upsilon_1 \\ +\left(\frac{\upsilon(b\sin\theta + \cos\theta)\sin\theta C_2}{\left(a^2 + b^2\right)\sin^2\theta + \cos^2\theta + b\sin 2\theta} - C_1 m a\sin\theta\right)\sin m\upsilon_1 \end{bmatrix}, \tag{60}$$

$\upsilon_1 = (y - at)\cos\theta + (ax + by)\sin\theta.$ With $\theta = \frac{\pi}{2}$, solutions of Equations 59 and 60 reduce to steady solutions of Equations 35 and 36.

## CONCLUIDNG REMARKS

We study two-dimensional incompressible steady and unsteady flows in terms of the vorticity distribution. The vorticity distribution is proportional to the stream function with the linear combination of the axis of geometry. For steady flow three different cases are discussed. The physical interpretations of the flows are given in each part. For unsteady flow, also three different solutions are given with their physical interpretation. It is interesting to note that the results of the steady cases are followed from the unsteady solutions. Also, the results of Hui (1987) become a particular case of our solutions.

### REFERENCES

Chandna OP, Oku-Ukpong (1994). Unsteady second grade aligned MHD fluid. Acta Mechanica., 107: 77-91.

Chandna OP, Oku-Ukpong EO (1994). Flows for Chosen Vorticity Functions-Exact Solutions of the Navier-Stokes Equations. Int. J. Math. Math. Sci., 17: 155-164.

Dorrepaal JM (1986). An exact solution of the Navier-Stokes equations which describes non orthogonal stagnation-point flow in two dimensions. J. Fluid Mech., 163: 141-147.

Hamdan HM (1998). An alternative approach to exact solutions of a class of Navier-Stokes flows. Appl. *Math. Computation*, 93: 83-90.

Hui WH (1987). Exact solutions of the 2-dimension Navier Stokes equations, *ZAMP*, 38: 689-702.

Islam S, Mohyuddin MR, Zhou CY (2008). Few exact solutions of non-Newtonian fluid in Porous Medium with Hall effects. J. Porous Media, 11(7): 669-680.

Islam S, Zhou CY (2007a). Exact solutions for two dimensional flows of couple stress fluids. Z. Angew Math. Phys., 60: 1178-1193.

Islam S, Zhou CY (2007b). Certain inverse solutions of a second grade MHD aligned fluid flows in a porous medium. J. Porous Media, 10(4): 401-408.

Islam S, Zhou CY, Ran XJ (2008). Exact Solutions for Different Vorticity Functions of Couple Stress Fluids, J Zhejiang Univ. Sci., 9(5): 672-680.

Kaloni PN, Hushchilt K (1984). Semi-Inverse solutions of non-Newtonian fluid. Int. J. Non-Linear Mech., 19: 373-381.

Labropulu F (1987), Riabouchinsky flows in magneto hydrodynamics, M.Sc. Thesis, University of Windsor, Canada.

Labropulu F (2002). Generalized Beltrami Flows and Other Closed-form Solutions of an Unsteady Viscoelastic Fluid, Int. J. Math. Math. Sci., 30: 271-282.

Lin SP, Tobak M (1986). Reversed flow above a plate with suction. AIAAJ, 24: 334-335.

Rajagopal KR (1980). On the decay of vortices of a second grade fluid. Mechanics, 9: 185-188.

Siddiqui AM (1986). Some inverse solutions of a non-Newtonian fluid. Mech. Res. Comm., 17: 157-163.

Siddiqui AM, Islam S, Ghori QK (2006). Two-Dimensional Viscous Incompressible flows in a Porous Medium, J. Porous Media, 9(6): 591-596.

Taylor GI (1923). On the decay of vortices in a viscous fluid. *Phil. Mag.*, 46: 671-674.

Wang CY (1991). Exact solutions of the steady state Navier-Stokes equations, *Ann. Rev. Fluid. Mech.*, 23: 159-177.