academicJournals

Vol. 9(9), pp. 214-223, 16 May, 2014
DOI: 10.5897/IJPS2013.4092
Article Number:638CA8144589
ISSN 1992 - 1950

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Angular symmetry of space-time and the spinor representation of Poincaré group

R. A. Sventkovsky

Technical Sciences, CITiS, Presnenskii val str., 19, Moscow 123557, Russia.

Received 11 December, 2013; Accepted 24 February, 2014

In the paper, a relativistic theory is suggested; we add three independent angles to the four coordinates of the Minkowski space that define the (x, t) - position of a moving local observer. These angular coordinates define the orientation of an observer under free rotations and they allow us to introduce the generators of the Poincaré group in the angular representation. Instead of a multi-component wave function of any spin-component wave function, one component wave function Ψ (and its Lorentz transform) is introduced, depending on the four coordinates of Minkowski space and three angular coordinates. Poincaré invariant first-order linear differential equations are derived. The matrix representation of the above operator equations based on appropriate angular is equivalent to Dirac and Maxwell equations. It is predicted and proven that the number of generations of leptons is three.

Key words: Spinor representation, generators of the Poincaré group, dimensions of Minkowski space.

INTRODUCTION

It is assumed that states with spin j and the corresponding equations describing these states possess the angular symmetry of fields and particles. Diverse formulations are used for spin and Dirac and Maxwell equations (Fushchich and Nikitin, 1994; Varadarajan, 1989); however, the dependence of angular was represented implicitly in previous formulations. Beginning with Kaluza-Klein, numerous compactified and unobservable dimensions were introduced to explain the nature of the four types of fundamental forces (electromagnetic, gravitational, strong, and weak forces). For instance, Rumer and Fet (1977) introduced frames of reference of free rotation at any point of space-time with variable metric. The position of these frames of reference is defined by angles. The equivalence between the Schrödinger operator equation and Heisenberg's matrix mechanics was proved in 1927 for operators depending on the time-space coordinates (Teschl, 2009). A similar construction (substitution matrices by operators) dealing only with the operators acting on the angular variables (which are introduced to explain the nature of spinors) is considered in the present paper.

Complete knowledge of free particle states and their behaviour can be obtained once all the unitary irreducible representations of the Poincaré group are found (Ohnuki, 1988). The relationship between the Lorentz group and Poincaré group in the angular representation and the equations for relativistic particles is focused on in this article, as well as the obtaining of generalized Lorentz group and Poincaré group.

*E-mail: ras@inevm.ru

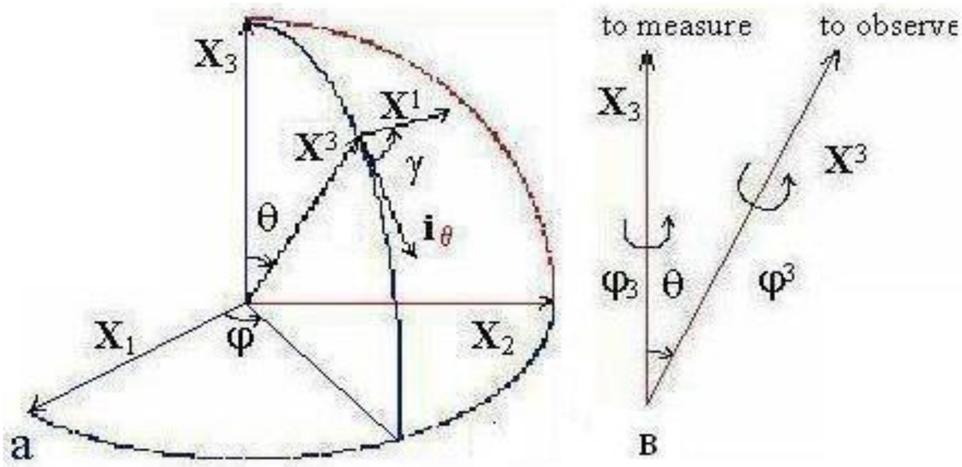

Figure 1. Orientation of the local observer in relation to the ordinary observer.

WAVE FUNCTION

The transformation properties of a multicomponent wave function that describes the transformation of fields and states with spin under rotations of ordinary observer or the original Cartesian of the coordinate system x_1, x_2, x_3 with unit vector $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are related to the existence of 'angular structure' for the spin states. At the center of the point x_1, x_2, x_3 of ordinary observer, we introduce a complementary local observer $x^{(1)}, x^{(2)}, x^{(3)}$. Rotation of local observer does not change x_1, x_2, x_3, t. As a rule, a rotating local observer corresponds to a local (right) Cartesian system of coordinates $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}$ (the isovector space) centered at point x_1, x_2, x_3, with the orientation determined by the three independent Euler angles φ, θ, ν. The position and the orientation of the local observer taken with respect to ordinary observer are shown in Figure 1 and set as $\varphi_3 = \varphi, \varphi^{(3)} = \nu$.

Since the theory of relativity states it is equivalent to rotate the ordinary observer or the local observer (the state, object), then the transformation properties will be described from the point of view of the local observer.

In the Minkowski space, the states of spin particles or fields with spin can be described by using a one component wave function, depending on the position x_1, x_2, x_3, t and the orientation φ, θ, ν of a moving and free rotating local observer. All variables $x_1, x_2, x_3, t, \varphi, \theta, \nu$ are independent. φ, θ, ν are internal variables and fully describe the degree of freedom of the spin. Let $X^{(i)}{}_k = (\mathbf{X}^{(i)}\mathbf{X}_k)$ be the entries of the corresponding rotation matrix (or the projections of the corresponding unit vectors to one another) that has the following form (Biedenharn and Louck, 1984):

$$X^{(3)}{}_3 = \cos(\theta), X^{(1)}{}_1 = -\sin(\varphi)\sin(\nu) + \cos(\theta)\cos(\varphi)\cos(\nu),$$
$$X^{(1)}{}_3 = -\sin(\theta)\cos(\nu), X^{(1)}{}_2 = \cos(\varphi)\sin(\nu) + \cos(\theta)\sin(\varphi)\cos(\nu),$$
$$X^{(2)}{}_3 = \sin(\theta)\sin(\nu), X^{(2)}{}_1 = -\sin(\varphi)\cos(\nu) - \cos(\theta)\cos(\varphi)\sin(\nu),$$
$$X^{(3)}{}_1 = \sin(\theta)\cos(\varphi), X^{(2)}{}_2 = \cos(\varphi)\cos(\nu) - \cos(\theta)\sin(\varphi)\sin(\nu),$$
$$X^{(3)}{}_2 = \sin(\theta)\sin(\varphi)$$

To the multicomponent wave function $C = (C_1, C_2, .. C_k)$ with the amplitudes $C_i(x_1, x_2, x_3, t)$, we assign the one-component wave function $\Psi = \psi C$. The set of the basis $\psi = (\psi_1, \psi_2, .. \psi_k)$ depends only on φ, θ, ν and $<\psi_n|\psi_i> = \delta_{ni}$. The expansion of Ψ with respect to the basis corresponds Ψ to the multicomponent wave function C, $C_i = <\Psi|\psi_i>$, and by the Sommerfeld condition, the wave function $\Psi = \psi C$ must be invariant under the Lorentz transformations. Obviously, the transformation properties of the basis Ψ and of the amplitudes C are dual. The basis of the states with the spin j, which consists of 2j+1 functions $\psi = (\psi_1, \psi_2, .. \psi_{2j+1})$ must have the following transformation properties (Biedenharn and Louck, 1984):

$$\psi' = \psi D^j(\alpha, \beta, c) \quad (1)$$

Where α, β, c stand for the angles defining the new orientation of the ordinary observer, the angles φ', θ', ν' define the new variables in the coordinate system of the ordinary observer, and the matrix of the Wigner functions (or the rotation matrix) D^j has the following properties (Biedenharn and Louck, 1984):

$$\overline{D}^j(\varphi', \theta', \nu') = D^j(\alpha, \beta, c)^T \overline{D}^j(\varphi, \theta, \nu) \quad (2)$$

Here, T stands for the transposition, the over line means complex conjugation, and S_k denotes the matrix operator of the spin angular momentum j in the z-representation. For example, the transformation (Equation 1) for j = 1/2 is of the form:

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \cos(\omega/2)\sigma_0 - i\sin(\omega/2)\sigma_k \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (3)$$

Where ω stands for the rotation angle about the axis X_k and σ k are the Pauli matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

denoted by $\xi^{(k)}$, which is the column of the Wigner matrix $\overline{D}^j$ with an index $k = j, j-1,..,-j$. Let $\xi^{(k)}{}_m = \overline{D}^j{}_{m,k}$, the Wigner functions satisfy the normalization conditions

$$< \xi^{(m)}{}_i \left| \xi^{(k)}{}_n \right. >= \iiint \xi^{(m)}{}_i \bar{\xi}^{(k)}{}_n d\Omega d\nu = \delta_{mk}\delta_{in} / b^2$$

Where $d\Omega = \sin(\theta) d\theta\, d\varphi$ stands for the solid angle, δ_{ik} _is the Kronecker delta, $b = \sqrt{(2j+1)}/(4\pi)$ is the known normalizing coefficient, and the domains of the angles in the arguments of the Wigner D-function are equal to $0 \le \varphi \le 4\pi, 0 \le \theta \le \pi, 0 \le \nu \le 2\pi$, respectively. Moreover, $X^{(i)}{}_k$ stands for the basis with j = 1, and we also have $\mathbf{X}^{(\pm)} = \mathbf{X}^{(1)} \mp \mathbf{X}^{(2)}, < X^{(m)}{}_i \left| X^{(k)}{}_n \right. >= \delta_{mk}\delta_{in} / b^2$

Proposition 1

The basis of spinors with transformation properties of Equation 1 consists either of the columns $\xi^{(m)}$ or of their linear combinations.

Proof

Equation 2 for a column of the matrix D^j (Biedenharn and Louck, 1984) is equivalent to the relation $\xi^{(m)'} = D^j(\alpha,\beta,c)^T \xi^{(m)}$ or Equation 1.

Spherical harmonics $Y(\theta,\varphi)^j{}_m = \overline{D}^j(\varphi,\theta,\nu)_{m,0}\sqrt{(2j+1)}/\sqrt{(4\pi)}$, where j – integer is the eigen functions of the operators angular moment of L that depends only on two angles φ,θ. It is known that 2j+1 of spherical harmonics of $Y = (Y^j{}_j, Y^j{}_{j-1},..Y^j{}_{-j})$ will be transformed at rotation in space on a representation of a group of SO(3), coinciding with Equation 1, $Y' = YD^j(\alpha,\beta,c)$.

Spinor representation of a group of rotation is given by matrix $D^{(j)}$ dimension 2j + 1, the Lie algebra which is isomorphic to the Lie algebra of three-dimensional rotations SO(3), (Biedenharn and Louck, 1984). Quantities-(2j +1)-dimensional vector transformations according to the spinor representation of Equation 1 are called spinors. For matrices of even dimension, this corresponds to an irreducible representation unitary group SU(2). Irreducible unitary representations of group SO(3) always have odd dimension.

Theorem 1

Two spherical angles (φ stands for the azimuthal angle and θ for the polar angle) are insufficient to describe the transformation properties of state with spin by using the wave function $\Psi(\varphi,\theta)$. For non-integer j, there are no 2j + 1 functions $\psi = (\psi_1, \psi_2,..\psi_{2j+1})$ of the variables φ,θ that are transformed according to some representation of the group SU(2) by (1).

Proof

Substituting $\xi^{(m)}(\varphi,\theta,\nu) = \exp(im\nu)\xi^{(m)}(\varphi,\theta,0)$ into Equation 1, we obtain

$$\xi^{(m)}(\varphi',\theta',0) = \exp(imf)D^j(\alpha,\beta,c)\xi^{(m)}(\varphi,\theta,0), f = \nu' - \nu.$$

This means that Relation 1 holds for $\xi^{(m)}(\varphi,\theta,0)$ up to the phase factor $\exp(imf)$. Further, let us define the dependence of the basis $\Psi_\pm$ of eigen functions S_3 on φ,θ for j = 1/2. To this end, consider the rotation by the angle α around X_3. On the section of the sphere θ = const we have

$$\varphi' = \varphi - \alpha, \psi(\varphi - \alpha)_\pm = \exp(\mp\alpha/2)\psi(\varphi)_\pm.$$

Therefore, $\psi(\varphi)_\pm = \exp(\pm\varphi/2)\psi(0)_\pm$.

Let us now define the dependence of the basis $\eta_\pm$ of the eigen functions S_2 for j = 1/2. The bases of the eigen functions S_2 and S_3 are connected in accordance with Equation 3, $\eta_\pm = \psi_+ \mp \psi_-$. Making a rotation in the section of the sphere $\varphi = 0$ by the angle β around $\mathbf{X}_2$, we have

$$\theta' = \theta + \beta, \eta(\theta + \beta)_\pm = \exp(\mp\beta/2)\eta(\theta)_\pm.$$

Therefore, $\eta(\theta)_\pm = \exp(\mp i\theta/2)\eta(0)_\pm$. Moreover, we use the rotation around X_2 by the angle π, which is equivalent to the transformation $\hat{G}: \varphi' = \pi - \varphi, \theta' = \pi - \theta$. Here, we have $\hat{G}\psi_\pm = \pm\psi_\mp$. We finally obtain the form of a basis whose existence is assumed, namely, $(\psi_+, \psi_-)^T = C_1\xi^{(1/2)}(\varphi,\theta,0) + C_2\xi^{(-1/2)}(\varphi,\theta,0)$.. This basis is transformed according to Equation 1 up to the phase factor exp(i f/2).

The generators of the rotation groups of the ordinary observer and local observer or of the vector and isovector rotations for the functions φ,θ,γ are the operators of angular momentum J_k, $J^{(n)}$ (Biedenharn and Louck, 1984).

$$J_1 = i\cos(\varphi)ctg(\theta)\partial/\partial\varphi + i\sin(\varphi)\partial/\partial\theta - i\cos(\varphi)/\sin(\theta)\partial/\partial\nu,$$
$$J_2 = i\sin(\varphi)ctg(\theta)\partial/\partial\varphi - i\cos(\varphi)\partial/\partial\theta - i\sin(\varphi)/\sin(\theta)\partial/\partial\nu,$$
$$J_3 = -i\partial/\partial\varphi, J^{(3)} = -i\partial/\partial\nu,$$
$$J^{(1)} = -i\cos(\nu)ctg(\theta)\partial/\partial\nu - i\sin(\nu)\partial/\partial\theta + i\cos(\nu/\sin(\theta)\partial/\partial\varphi,$$
$$J^{(2)} = i\sin(\nu)ctg(\theta)\partial/\partial\nu - i\cos(\nu)\partial/\partial\theta - i\sin(\nu)/\sin(\theta)\partial/\partial\varphi.$$

The rotation groups of the ordinary observer and the local observer commute with each other and the spatial and isovector rotations are realized. To be more precise,

$$[J_i, J_n] = i\varepsilon_{ink}J_k, [J_i, J^{(n)}] = 0, [J^{(i)}, J^{(n)}] = -i\varepsilon_{ink}J^{(k)}, \quad (4)$$

Where ε_{ijk} stands for the antisymmetric tensor with $\varepsilon_{123} = 1$ and the eigen functions of the operators J^2, $J^{(3)}$ and J_3 are Wigner D-functions, $\mathbf{J}^2 = \sum J_k J_k$ and

$$J_3\overline{D}^j{}_{m,m'} = m\overline{D}^j{}_{m,m'}, J^{(3)}\overline{D}^j{}_{m,m'} = m'\overline{D}^j{}_{m,m'}, J^2\overline{D}^j{}_{m,m'} = j(j+1)\overline{D}^j{}_{m,m'}.$$

The operators of raising and lowering indices m,m' act by the usual rules,

$$J_+ = J_1 + iJ_2, J^{(-)} = J^{(1)} + iJ^{(2)}, \bar{J}^{(-)} = -J^{(+)}, \bar{J}_- = -J_+,$$
$$J_+\overline{D}^j{}_{m,m'} = \sqrt{(j-m)(j+m+1)}\overline{D}^j{}_{m+1,m'}, J^{(+)}\overline{D}^j{}_{m,m'} = \sqrt{(j-m')(j+m'+1)}\overline{D}^j{}_{m,m'+1}$$

and the columns $\overline{D}^j$, j = 1/2 are the spinors $\xi^{(1/2)} = (\psi_1, \psi_2), \xi^{(-1/2)} = (\psi_3, \psi_4)$.

$$\psi_1 = \cos(\theta/2)\exp(i\nu/2 + i\varphi/2), \psi_3 = -\sin(\theta/2)\exp(-i\nu/2 + i\varphi/2),$$
$$\psi_2 = \sin(\theta/2)\exp(i\nu/2 - i\varphi/2), \psi_4 = \cos(\theta/2)\exp(-i\nu/2 - i\varphi/2), \quad (5)$$

In z-representation, the spin operator S_p and its eigenvectors are identical to the matrix representation J_p on the basis of the spinor $\psi = \xi^{(m)}$ for any index $m = j, j-1,..,-j$.

We refer to the operator J_p as the spin operator in the operator representation $J_p\psi = \psi S_p$. In terms of J_p, the transformation of Equation 1 acquires the form $\xi^{(m)'} = \exp(-i\omega J_i)\xi^{(m)}$, because the action of the rotation operator $\exp(-i\omega S_i)$ is identical to the operator $\exp(-i\omega J_i)$.

Proposition 2

The rotation of a spin state (spinor) by the angle π around the axis x_2 corresponds to the transformation $\widehat{G}_2 = \exp(-i\,\pi J_2)$, which is equivalent to the angles $\widehat{G}_2 : \varphi' = \pi - \varphi, \theta' = \pi - \theta, \nu' = \nu - \pi$.

For $j = 1/2, \widehat{G}_2\,\psi_1 = \psi_2, \widehat{G}_2\,\psi_2 = \psi_1$, it suffices to verify the relation $\widehat{G}_2\,\xi^{(j)}{}_m = (-1)^{j-m}\xi^{(j)}{}_{-m}$. For example, $j = 1/2, \widehat{G}_2\,\psi_1 = \psi_2, \widehat{G}_2\psi_2 = -\psi_1$.

Theorem 2

The spatial inversion $\widehat{P} : t' = t, x_i' = -x_i$ leads to an internal inversion $\widehat{I} : \varphi' = \pi + \varphi, \theta' = \pi - \theta, \nu' = \pi - \nu$ – equivalent to the rotation in the isovector space by the angle −π around the axis $X^{(2)}, \widehat{I} = \exp(i\,\pi J^{(2)})$.

Proof

Let us prove the properties. We have:

$$\widehat{I}X^{(k)} = (-1)^k X^{(k)},\ \widehat{I}\xi^{(\pm)} = (-1)^{j\mp j}\,\xi^{(\mp)}, \widehat{I}J_i = J_i, \widehat{I}^2\xi^{(\pm)} = (-1)^{2j}\,\xi^{(\pm)}.$$

The conservation law for parity is immediately related to the conservation for symmetry between left and right. The bases $\overline{D}^j{}_{m,m'}$ and $\overline{D}^j{}_{m,-m'}$ are related by the inversion transformation, $\widehat{I}\overline{D}^j{}_{m,m'} = (-1)^{j-m'}\overline{D}^j{}_{m,-m'}$. This enables one to decompose the entire basis into two equal groups of bases in all cases except for $m' = 0$, the integer j. The group of bases $m' > 0$ is said to be left, and its mirror part $m' < 0$ to be right. The difference between left and right with weight m' is evaluated as the mean value of the operator $J^{(3)}$. For example, $\Psi = C^{(m')}\overline{D}^j{}_{m,m'} + C^{(-m')}\overline{D}^j{}_{m,-m'}, <\Psi|J^{(3)}|\ \Psi> b^2 = m'(|C^{(m')^2}|\ -|C^{(-m')^2}|\ )$.

Any translation of coordinates x_i, t does not change the angular variables φ, θ, ν.

THE GENERALIZED SPIN LORENTZ GROUP

The Lie algebra of the Lorentz group and its generators in the coordinate representation $\mathbf{L} = (M_{23}, M_{31}, M_{12}), \mathbf{K} = (M_{01}, M_{02}, M_{03})$ have this form (Ohnuki, 1988):

$$[L_i, L_n] = i\varepsilon_{ink}L_k, [K_i, L_n] = i\varepsilon_{ink}K_k, [K_i, K_n] = -i\varepsilon_{ink}L_k \quad (6)$$

Where $M_{ij} = x_iP_j - x_jP_i, x_0 = t, P_i = -i\partial/\partial x_i, P_0 = i\partial/\partial t$ stand for the operators of angular momentum, momentum and energy.

Let $Q^{(p)}{}_i$ be the generators of the Lorentz group for arbitrary spin in the angular representation. The Lorentz transformation is of the form $\psi' = \exp(-i\chi Q^{(p)}{}_n)\psi$, where $th(\chi) = v/c, \mathbf{v} = |\ v|\ \mathbf{x}_n$ stands for the velocity, and p = 1, 2, 3, and

$$[J_i, J_n] = i\varepsilon_{ink}J_k, [Q^{(p)}{}_i, J_n] = i\varepsilon_{ink}Q^{(p)}{}_k, [Q^{(p)}{}_i, Q^{(p)}{}_n] = -i\varepsilon_{ink}J_k \quad (7)$$

Let us find the form of generators of Equation 7 satisfying the conditions that these are purely imaginary vector generators (which are Hermitean-first-order linear differential operators) and ensure the transformation properties of the basis Ψ, j =1/2,1,3/2… in accordance with the Lorentz transformations. To this end, we assume that the representation p = 3 has the Lorentz transformation on the basis $\psi = (\xi^{j)}, \xi^{(-j)})$ or on the basis of eigen functions of $J_3, J^{(3)}, \mathbf{J}^2$ in the well-known diagonal form (Weinberg, 2003).

$$B = \exp(-i\chi Q^{(3)}{}_3), B\xi^{(\pm j)}{}_m = \exp(\pm m\chi)\xi^{(\pm j)}{}_m \quad (8)$$

Therefore, the functions $\xi^{(\pm j)}{}_m$ by themselves are eigen functions of the operator $Q^{(3)}{}_3$. We impose the space-time isotropy condition on the generators of the Lorentz group, namely, $Q^{(3)}{}_3$ must be independent of the angles φ and γ. Therefore, $\sin(\theta/2), \cos(\theta/2)$ are also eigen functions of the operator $Q^{(3)}{}_3$. It can readily be shown that there is only one generator of this kind, $Q^{(3)}{}_3 = i\,j\cos(\theta) - i\sin(\theta)\ \partial/\partial\theta,\ \ Q^{(3)}{}_3\xi^{(\pm j)}{}_m = \pm im\xi^{(\pm j)}{}_m$.

Using $Q^{(3)}{}_3$, we find $\mathbf{Q}^{(3)} = i\mathbf{X}^{(3)}j + [\mathbf{X}^{(3)}\mathbf{J}]$. Applying rotation in the isovector space, $\exp(i\,\pi/2\,J^{(2)})\,X^{(3)}{}_k = X^{(1)}{}_k$, $\exp(i\,\pi/2\,J^{(1)})\,X^{(3)}{}_k = -X^{(2)}{}_k$, we obtain two other generators p = 1, 2, the angular representations of the Lorentz group,

$$\mathbf{Q}^{(p)} = i\mathbf{X}^{(p)}j = +[\mathbf{X}^{(p)}\mathbf{J}], Q^{(p)}{}_k = i\,X^{(p)}{}_k\ \ j + \varepsilon_{ijk}\,X^{(p)}{}_i\,J_j \quad (9)$$

The dependence of the operators Q on j is excluded by replacing j with the scalar operator $\widehat{J}$ independent of j and such that $\widehat{J}\Psi = j\Psi$.

Consider action of the operators J, Q for j = 1 on the basis $j = 1, X^0{}_0 = 1$,

$$J^{(i)}X^{(n)}{}_p = -i\varepsilon_{ink}X^{(k)}{}_p, J_iX^{(p)}{}_n = i\varepsilon_{ink}X^{(p)}{}_k, Q^{(n)}{}_kX^0{}_0 = ijX^{(n)}{}_k,$$
$$Q^{(m)}{}_iX^{(p)}{}_n = i\delta_{in}\delta_{mp}X^0{}_0 + i\varepsilon_{ink}\varepsilon_{mpd}X^{(d)}{}_k \quad (10)$$

The operators Q, X, J and N with the superscripts and subscripts +,

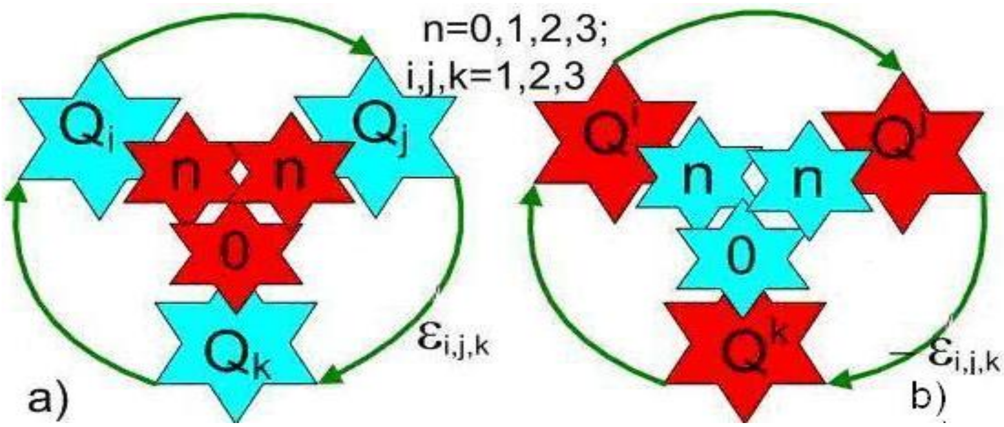

Figure 2. Visual model of Lie algebra and the generalized spin Lorentz group.

3, and −, correspondingly, raise, preserve, and lower, respectively, the indices m, m' for $\overline{D}^j_{m,m'}$. The result of action of the operators Q on $\overline{D}^j_{m,m'}$ (by using Varshalovich D.A, 1975) is just a linear combination of two summands $\overline{D}^j$ and $\overline{D}^{j-1}$. For example,

$$Q^{(3)}{}_3\overline{D}^j{}_{m,m'} = i\sqrt{(j^2-m^2)(j^2-m'^2)}\,/\,j\overline{D}^{j-1}{}_{m,m'} + imm'/j\overline{D}^j{}_{m,m'},$$
$$Q^{(+)}{}_+\overline{D}^j{}_{m,m'} = i\sqrt{(j-m-1)(j-m)\,(j-m'-1)(j-m')}\,/\,j\overline{D}^{j-1}{}_{m+1,m'+1} +$$
$$i\sqrt{(j-m)(j+m+1)}\sqrt{(j+m'+1)\,(j-m')}\,/\,j\overline{D}^j{}_{m+1,m'+1} \quad (11)$$

The action of the operators $\mathbf{J},\mathbf{Q}^{(3)}$ on the basis $\psi=(\xi^{(j)},\xi^{(-j)}), S_{0ik}=j\delta_{ik}$ is of the form:

$$Q_k{}^{(3)}\psi = i\psi\begin{bmatrix} S_k & 0 \\ 0 & -S_k \end{bmatrix}, J_k\psi=\psi\begin{bmatrix} S_k & 0 \\ 0 & S_k \end{bmatrix}, J^{(3)}\psi=\psi\begin{bmatrix} S_0 & 0 \\ 0 & -S_0 \end{bmatrix}$$

Matrix representation of Equation (7) is identical to the spinor representation of the Lorentz group.

Similarly, we introduce the operators (Ohnuki, 1988) $\widehat{J}^{(1)}{}_i=(J_i+iQ^{(3)}{}_i)/2, \widehat{J}^{(2)}{}_n=(J_n-iQ^{(3)}{}_n)/2$ with the Lie algebra of the Lorentz group which coincides with the algebra of groups SO(3) and SU(2):

$$[\widehat{J}^{(1)}{}_i,\widehat{J}^{(2)}{}_n]=0,[\widehat{J}^{(1)}{}_i,\widehat{J}^{(1)}{}_j]=i\varepsilon_{ijk}\widehat{J}^{(1)}{}_k,[\widehat{J}^{(2)}{}_i,\widehat{J}^{(2)}{}_j]=i\varepsilon_{ijk}\widehat{J}^{(2)}{}_k\,.$$

It is known (Ohnuki, 1988:24) that an irreducible representation of the Lorentz group is uniquely determined by eigen values of some operators $\mathbf{L}^2-\mathbf{K}^2,\mathbf{LK}$, which correspond to two operators $\widehat{\mathbf{J}}^{(1)2},\widehat{\mathbf{J}}^{(2)2}$. So $\widehat{\mathbf{J}}^{(1)2}\Psi=j_1(j_1+1),\widehat{\mathbf{J}}^{(2)2}\Psi=j_2(j_2+1)$. Every irreducible representation of the Lie algebra is characterized by a pair of numbers (j_1,j_2).

The spinors $\xi^{(j)}$ and $\xi^{(-j)}$ are transformed accordingly to the representations of (0, j) and (j, 0), respectively (0,1/2), (1/2,0), which are the Weyl spinors.

Scalar unit operator 1 and the generators of the groups ensuring the vector, isovector and Lorentz rotations of the bases in the new variables generate an generalized spin Lorentz group of 16 generators, Equations (4) and (12).

$$[\mathbf{Q}^{(n)}{}_i,\mathbf{Q}^{(p)}{}_k]=i\delta_{ik}\varepsilon_{npm}\mathbf{J}^{(m)}-i\delta_{np}\varepsilon_{ikr}\mathbf{J}_r,$$
$$[\mathbf{Q}^{(i)}{}_r,\mathbf{J}^{(n)}]=-i\varepsilon_{ink}\mathbf{Q}^{(k)}{}_r,[\mathbf{Q}^{(p)}{}_i,\mathbf{J}_n]=i\varepsilon_{ink}\mathbf{Q}^{(p)}{}_k \quad (12)$$

Visual model of Lie algebra and the generalized spin Lorentz group is shown in Figure 2.

In Figure 2, $J^{(k)}$ corresponds to $Q^{(k)}{}_0$; J_n corresponds to $Q^{(0)}{}_n$. The operator $W:\varphi'=\nu,\nu'=\varphi,\theta'=-\theta$ of permutation introduces the ordinary observer and the local observer or the vector and isovector rotations. The operation W preserves the invariance of the Lie algebra of the generalized spin Lorentz group and corresponds to the transposition operation for the group representation and matrix $\overline{D}^j$.

$$J^{(k)}=(-1)^{k+1}WJ_k, W\overline{D}^j=\overline{D}^{jT}, X^{(k)}{}_n=(-1)^{k+n}WX^{(n)}{}_k,\; Q^{(k)}{}_n=(-1)^{k+n}WQ^{(n)}{}_k\,.$$

The ten generators of the Poincaré group $(\mathbf{L},\mathbf{K},\mathbf{P},P_0)$ and the ten operators $(\mathbf{J},\mathbf{Q}^{(3)},\mathbf{Q}^{(\cdot)},iJ^{(\cdot)})$ have the same Lie algebra in Equations 6 and 13; they are preserved under the cyclic permutation of the superscripts 1, 2, 3.

$$[P_i,P_n]=0,[K_k,P_k]=-iP_0,[L_i,P_j]=i\varepsilon_{ijk}P_k,[K_i,P_0]=-iP_i\,, \quad (13)$$

The rotation of the coordinate system x_1,x_2,x_3,t by the angle ζ_k around the x_k axis and the Lorentz transformation corresponding to the velocity v = |v|x_n are of the form

$$\Psi'=\exp(-i\zeta_k(L_k+J_k))\Psi,\Psi'=\exp(-i\chi\;(K_n+Q^{(3)}{}_n))\Psi \quad (14)$$

The extended basis $X^{(0)}{}_0-\mathbf{X}^{(3)}$ is transformed as a four-dimensional vector, the basis $(-\mathbf{X}^{(1)},\mathbf{X}^{(2)})$ is transformed as a Lorentz bivector, and all these objects together form a full system of 10 bases for j = 1. The basis $X^{(0)}{}_0=\psi_1\psi_2-\psi_3\psi_4=1$ differs from the first basis for j = 0 only in the dimension and Lorentz transformation

$$BX^{(3)}{}_3=ch(\chi)X^{(3)}{}_3+sh(\chi)X^{(0)}{}_0, BX^{(0)}{}_0=sh(\chi)X^{(3)}{}_3+ch(\chi)X^{(0)}{}_0, \quad (15)$$
$$X^{(+)'}{}_+=X^{(+)}{}_+\exp(\chi), X^{(+)'}{}_-=X^{(+)}{}_-\exp(-\chi),$$

Transformation operators are performed according to the Baker-

Campbell-Hausdorff (Biedenharn and Louck, 1984): $\exp(A)B\exp(-A) = B + [AB] + 1/2![A[AB]] + 1/3![A[A[AB]]] + ..$

The operators $\mathbf{Q}^{(2)}, \mathbf{J}^{(1)}$ and $\mathbf{Q}^{(1)}, -\mathbf{J}^{(2)}$ or $\mathbf{Q}^{(-)}, i\,\mathbf{J}^{(-)}$ are transformed as a four-dimensional Lorentz vector, $\mathbf{Q}^{(3)}, \mathbf{J}$ as a bivector, and $J^{(3)}$ as a scalar,

$$Q^{(2)}{}_1{}' =' Q^{(2)}{}_1, Q^{(2)}{}_2{}' = Q^{(2)}{}_2, \mathbf{v}/v = \mathbf{x}_3, Q^{(3)}{}_3{}' = Q^{(3)}{}_3, J_3' = J_3,$$

$$\begin{bmatrix} J_1' \\ Q^{(3)}{}_2{}' \end{bmatrix} = \begin{bmatrix} ch(\chi) & sh(\chi) \\ sh(\chi) & ch(\chi) \end{bmatrix} \circ \begin{bmatrix} J_1 \\ Q^{(3)}{}_2 \end{bmatrix}$$

Proposition 3

The Lorentz transformation of the basis $\mathbf{j}, \mathbf{v} = |\mathbf{v}|\mathbf{x_3}$ is equivalent to an angular transformation and a scale transformation of the basis.

$$\varphi' = \varphi, \nu' = \nu, tg(\theta'/2) = \exp(-\chi) tg(\theta/2), \psi'_{eff} = \psi / T(\chi,\theta)^{2j} \quad (16)$$

Where $T(\chi,\theta)^2 = ch(\chi) + sh(\chi)\cos(\theta)$.

The operator B in the above Lorentz transformation does not depend on the angles φ, γ. The Lorentz transformation of the angles φ,θ,γ follows from the Lorentz transformation of Equations 8 and 15 of the bases $\xi^{(\pm j)}{}_m, X^{(k)}{}_i$. The Lorentz transformation of Equation 15 of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}$ is dual to the Lorentz transformation correspondingly, for the vector E of the electric field, the vector H of the magnetic field, and the Umov-Poynting vector [EH], (Pauli W., 1991). Therefore, there exists an analogy between the Lorentz transformation of Equation 16 of the angles φ, θ, ν and the Lorentz transformation of the angles of the polarized light, where the Umov-Poynting vector is directed along the vector $\mathbf{X}^{(3)}$ (Figure 1) and the electric vector along the vector $\mathbf{X}^{(1)}$.

The independence of the speed of light of the reference system implies that the shape of spin states for a massless particle with spin j that flies along the x_3 is invariant under the Lorentz transformation along x_3. The only eigen functions of the generator of the Lorentz group $Q^{(3)}{}_3, \mathbf{J}^{(2)}, \Psi = \xi^{(\pm)}{}_m, \Psi = \xi^{(\pm j)}{}_m, \Psi = \xi^{(m)}{}_{\pm j}$ can satisfy this invariance condition. The spinors $\xi^{(\pm j)}{}_m$ can be treated as a basis of the massless field. For j =1/2 let us show that the neutrinos are purely left handed particles $<\Psi|J^{(3)}|\ \Psi> b^2 = 1/2$, whereas the antineutrinos are the right-handed ones $<\Psi|J^{(3)}|\ \Psi> b^2 = -1/2$. To this end, we choose the increasing solutions under the Lorentz transformation of Equation 8 along the x_3 axis; these are $\Psi = \xi^{(1/2)}{}_{-1/2}$ for the neutrino and $\Psi = \xi^{(-1/2)}{}_{1/2}$ for the antineutrino.

The vector $\mathbf{P}^{(-)}, P^{(-)}{}_0$ can be considered as the momentum operator in the Lie algebra of the angular variables, as they have the same Lie algebra of the Poincaré group, so that

$$\mathbf{P}^{(-)} = \mathbf{Q}^{(-)}, P^{(-)}{}_0 = i\,J^{(-)}, P^{(1)} = Q^{(1)}, P^{(2)} = Q^{(2)}, P^{(1)}{}_0 = -J^{(2)}, P^{(2)}{}_0 = J^{(1)}, \mathbf{P}^{(-)} = \mathbf{P}^{(1)} + i\mathbf{P}^{(2)}.$$

We introduce the operators in the symmetric form of the Lorentz invariant scalar product of two four-vector momentum operators **P**, P_0/c and $\mathbf{P}^{(-)}$, $P^{(-)}{}_0$ or $P^{(+)}$, $P^{(+)}{}_0$.

In the equation for the wave function with arbitrary spin, we will use this operator $N^{(\pm)} = \mathbf{P}^{(\pm)}\mathbf{P} - P^{(\pm)}{}_0$, where $N^{(-)} = N^{(1)} + iN^{(2)}$ is the complexified operator.

Below we will use the invariant real-valued operators

$$N^{(1)} = \mathbf{Q}^{(1)}\mathbf{P} + J^{(2)}P_0/c, N^{(2)} = \mathbf{Q}^{(2)}\mathbf{P} - J^{(1)}P_0/c, \quad (17)$$

instead of the Lorentz invariant operators $N^{(\pm)}$. These operators have different parity $\widehat{P}\widehat{I}N^{(1)} = N^{(1)}, \widehat{P}\widehat{I}N^{(2)} = -N^{(2)}, \widehat{I}\mathbf{Q}^{(2)} = \mathbf{Q}^{(2)}$ and are connected to each other by rotations around a movable axis $X^{(3)}$ in angles $\pi/2, \nu' = \nu - \pi/2, N^{(1)} = \exp(i\,\pi/2\,J^{(3)})N^{(2)}$

THE GENERALIZED DIRAC EQUATION

Consider the generalized Dirac equation for a particle with arbitrary spin for для $j = 1/2, 1, 3/2...,..$ written either in terms of $N^{(1)}$ or in terms of $N^{(2)}$, k = 1 or k = 2:

$$N^{(k)}\Psi = m_e c/(2h)\Psi \quad (18)$$

This is the first-order linear differential equation for wave functions $\Psi = b\psi C$. Consider the set of the eigenvectors of the operators $J_3, \mathbf{J}^2, J^{(3)}$, consisting of (2j +1)(2j +1) of functions and let me be the mass of the particle and h be the Planck's constant. Equation 11 readily shows that the generalized Dirac equation reduces the staircase equations for the $\Psi^{(j)}, \Psi^{(j-1)}$.... This explains the fact that if j > 1/2 then particles or fields are composite. For example, for j = 3/2, presence of the physical modes is obligatory for the Rarita - Schwinger fields with spin 1/2.

The matrix representation of equations (18) on the basis (5) with j = 1/2, k = 2 coincides with the following Dirac equations in spinor representation (Fushchich and Nikitin, 1994)

$$(gP - g_0 P_0/c + m_e c/h) = 0 \quad (19)$$

Where

$$g_0 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}, g_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \widehat{g}_0 = i\begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix}, \widehat{g}_k = -i\begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix},$$

σ are the Pauli matrices, g_0, g and $\widehat{g}_0$, $\widehat{g}$ are the Dirac matrices in two different spinor representations. To prove Equation 19, it suffices to calculate the action of the operators $N^{(1)}, N^{(2)}$ on the basis vectors of Equation 5, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ for j = 1/2.

$$2N^{(2)}\psi = \psi(\mathbf{gP} - g_0 P_0/c), 2N^{(1)}\psi = \psi(\widehat{\mathbf{g}}\mathbf{P} - \widehat{g}_0 P_0/c).$$

In order to simplify the above equations we have used the following identities:

$$2J^{(1)}\psi = \psi g_0, 2Q^{(2)}\psi = \psi g_k, -2Q^{(1)}\psi = \psi\widehat{g}_k, 2J^{(2)}\psi = \psi\widehat{g}_0$$

$$g_0 = <2\psi|J^{(1)}|\ \psi> b^2, g_k = <2\psi|Q^{(2)}|\ \psi> b^2,$$

$$2J_k\psi = \psi\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, 2J^{(3)}\ \psi = \psi\begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}, <\psi|\ \psi> b^2 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}$$

The generalized Dirac equation is invariant under inversion of $\widehat{P}$, $\widehat{I}$ and the solution $\Psi = \Psi^C$ for a charge-conjugate particle satisfies the complex conjugate Equation 18:

$$\overline{Q}^{(n)})_k = -Q^{(n)})_k, (\overline{P} - q\overline{A}/c) = -(P + qA/c), J^{(k)} = -J^{(k)}, \Psi^C = (\overline{\psi}\overline{C}) = (\psi C^C),$$

Where A is the vector-potential, q/h is the charge of a particle. For example, if j = 1/2, then $\overline{\psi} = i\psi g_2, C^C = ig_2 C$, (Flügge, 1974).

For spin j = ½, there exists 2 = 2j + 1 Lorentz-invariant states $\Psi^{(\pm)} = C^{(\pm)}\xi^{(\pm 1.2)}$. The generalized Weyl equations for the right- or left-handed neutrino have the form $N^{(\pm)}\Psi^{(\mp)} = 0$ or $(-P_0 c \pm \mathbf{P\sigma})C^{\pm} = 0$, (Akhiezer, 1959). For the anti neutrino and neutrino, the balance between left and right orientations is violated because of asymmetry of equations $J^{(3)}\Psi^{(\pm)} = \pm\Psi^{(\pm)}/2$.

THE GENERALIZED MAXWELL EQUATIONS

The Maxwell equations describing the state of the vector field E, H, have the following form:

$$[\mathbf{PE}] = P_0\mathbf{H}/c, (\mathbf{PH}) = 0,$$
$$[\mathbf{PH}] = -P_0\mathbf{E}/c - i4\pi/c\mathbf{I}, (\mathbf{PE}) = -i4\pi/cI_0 \quad (20)$$

Where **I** = (I_1, I_2, I_3) is the density of electric current and I_0 is the density of electric charge. Let

$$\mathbf{F}^{(1)} = \mathbf{E}, \mathbf{F}^{(2)} = \mathbf{H}, \mathbf{F}^{3)} = \mathbf{A}, b = \sqrt{(2j+1)}/(4\pi).$$

Consider the generalized Maxwell's equations which imply the standard Maxwell equations for E and H which are not just a new representation, but a tool for an expanded description of the states of electromagnetic fields with spin. To this end, we introduced the wave function Ψ. The Lorentz-invariant of the wave function Ψ_C for charges and currents has the following form:

$$\Psi_C = (I_0 X^{(0)}{}_0 + I X^{(3)}/c), I_0 = 1/(4\pi)^2 < \Psi_C | X^{(3)}{}_k >, I_k = b^2 < \Psi_C | X^{(3)}{}_k >.$$

Let us describe the state Ψ on the basis of eigen functions of $J_3, \mathbf{J}^2, J^{(3)}, j = 1, \mathbf{J}^2\Psi = 2\Psi$ and consider all three cases $J^{(3)}\Psi = \pm\Psi, J^{(3)}\Psi = 0$.

This Ψ is composed of 3=2j +1, (j=1) Lorentz-invariants of the electromagnetic field written in the form of the two complex-conjugate invariants $\Psi^{(\pm)} = C^{(\pm)}\xi^{(\pm 1)} = \mathbf{F}^{(\mp)}\mathbf{X}^{(\pm)}$, the two real invariants of $\Psi = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}, \Psi_{AN} = \mathbf{E}\,\mathbf{X}^{(2)} - \mathbf{H}\,\mathbf{X}^{(1)}$ and the invariant $\Psi_A = \mathbf{A}\,\mathbf{X}^{(3)} + A_0 X^{(0)}{}_0$ for the vector-potential $\mathbf{A}, A_0, J^{(3)}\Psi_A = 0$. The states $J^{(3)}\Psi^{(\pm)} = \pm\Psi^{(\pm)}$ are called the right and the left vector states, respectively.

The Lorentz-invariance of the $\Psi^{(+)}$ implies the well-known Lorentz transformations of the fields (Pauli, 1991): $F^{(-)\prime}{}_- = F^{(-)}{}_- \exp(-\chi), F^{(-)\prime}{}_3 = F^{(-)}{}_3, F^{(-)\prime}{}_+ = F^{(-)}{}_+ \exp(\chi)$.

For the state $\Psi^{(+)}$, we have $\mathbf{F}^{(-)}\mathbf{X}^{(+)} = (F^{(-)}{}_+ X^{(+)}{}_- + 2F^{(-)}{}_3 X^{(+)}{}_3 + F^{(-)}{}_- X^{(+)}{}_-)/2 = C_R\xi^{(1)}$, $\overline{D}^1{}_{\pm 1,1} = \pm X^{(+)}{}_\pm, \overline{D}^1{}_{0,1} = -X^{(+)}{}_3/\sqrt{2}$. The basis $\xi^{(1)} = (\overline{D}^1{}_{1,1}, \overline{D}^1{}_{0,1}, \overline{D}^1{}_{-1,1})$ and the amplitudes $C^{(+)} = (F^{(-)}{}_+, 2F^{(-)}{}_3, F^{(-)}{}_-)$ correspond to the projections of the spin (1, 0,−1) on the x_3 axis. We expand similarly $\Psi = \Psi^{(-)}$.

The spins of the right and left vector states are collinear or anticollinear to the Poynting vector. The proof follows from the identities $b^2 < \Psi^{(\pm)} | \mathbf{J} | \Psi^{(\pm)} >= \pm 4[\mathbf{EH}]$.

The Poynting vector and the density of the electromagnetic energy $\mathbf{s} = c[\mathbf{EH}]/(4\pi), s_4 = (E^2 + H^2)/(8\pi)$ for $\Psi = \Psi^{(\pm)}$ are written in the form:

$$2\mathbf{s} = -icb^2 < \Psi | \mathbf{Q}^{(3)} | \Psi > /(8\pi), 2s_4 = b^2 < 2\Psi | \Psi > /(8\pi).$$

The mixed state is the sum of the right and the left vector states, currents and fields:

$$\Psi = \Psi^{(+)} + \Psi^{(-)}, E = E_R + E_L, H = H_R + H_L, I = I_R + I_L, I_0 = I_{0R} + I_{0L}, \Psi_C = \Psi_{CR} + \Psi_{CL},$$

Proposition 4

The spin (the energy, the Poynting vector) of the mixed state is the sum of the spins (the energy, the Poynting vector) of the right and the left states.

Proof follows from the identity, similarly for $Q^{(3)}$: $< \Psi | \mathbf{J} | \Psi >=< \Psi^{(+)} | \mathbf{J} | \Psi^{(+)} > + < \Psi^{(-)} | \mathbf{J} | \Psi^{(-)} >$. The generalized Maxwell equations describing just right state or only left state with spin 1 have the following form:

$$N^{(1)}\Psi^{(\pm)} = 4\pi\Psi^c, \text{or}, N^{(2)}\Psi^{(\pm)} = \mp i4\pi\Psi^c \quad (21)$$

We rewrite Equations 21 as a pair of complex-conjugate Equations 22 for the wave functions either $\Psi^{(+)}$ or $\Psi^{(-)}$, as $N^{(\pm)}\Psi^{(\pm)} = 0$. Each of these equations describes just the right or the left vector spin-1 states:

$$N^{(-)}\Psi^{(+)} = 8\pi\Psi^c, N^{+)}\Psi^{-)} = 8\pi\Psi^c, \quad (22)$$

Each of the Equations 22 agrees with the Maxwell equations either for the right components E_R,H_R, I_R, I_{0R} or for the left components. This implies that the total components E, H, I, I_0 satisfy the Maxwell equations. Maxwell Equation 20 follow from Equations 21, 10 and identity 23 for j = 1, where G(x_1, x_2, x_3) is an arbitrary vector:

$$(\mathbf{Q}^{(i)}\mathbf{P})(\mathbf{G}\mathbf{X}^{(k)}) = i\,\delta_{ik}\,(\mathbf{PG})X^{(0)}{}_0 + i\,\varepsilon_{ikn}\,([\mathbf{PG}]X^{(n)}). \quad (23)$$

For $\Psi^{(+)} = \mathbf{F}^{(-)}\mathbf{X}^{(+)}, \mathbf{G} = \mathbf{F}^{(-)}$ we obtain the following identity:

$$(|Q^{(2)}P - iJ^{(1)}P_0/c)(\mathbf{F}^{(-)}\mathbf{X}^{(+)}) = -i([\mathbf{PF}^{(-)}] + iP_0\mathbf{F}^{(-)}/c)\mathbf{X}^{(3)} + (\mathbf{PF}^{(-)})X^{(0)}{}_0$$

The generalized Maxwell Equation 24 for states with zero projection of spin and its analogue $< \Psi | J_i | \Psi >= 0, < \Psi | J^{(k)} | \Psi >= 0$ correspond to the real-valued state $\Psi = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}$. Since $\Psi, N^{(2)}$ is the real one, then $C_L = C_R$ and Equations 22 have the following form:

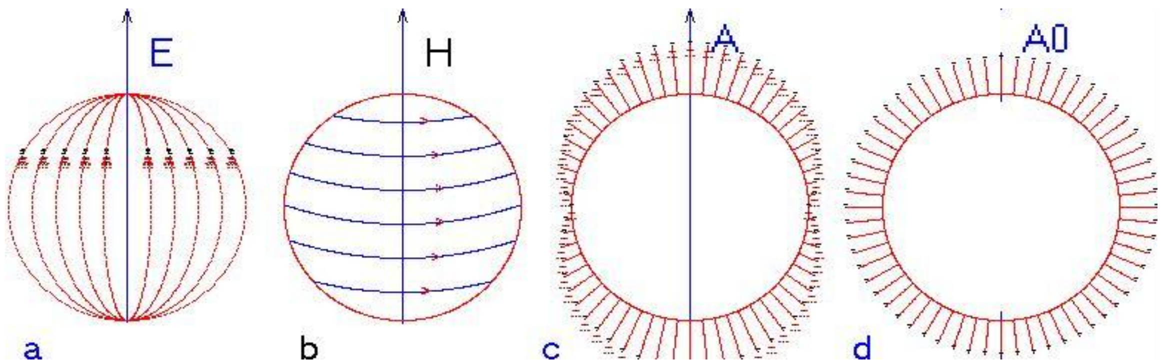

Figure 3. Visual model of the electric, magnetic fields: $\sin(\theta)\mathbf{i}_\theta$, $\sin(\theta)\mathbf{i}_\varphi$, $\cos(\theta)\mathbf{i}_r$, $\mathbf{i}_r$

$$\mathrm{N}^{(1)}\Psi = 4\pi\Psi^{c}, \mathrm{N}^{(2)}\Psi = 0 \quad (24)$$

The transformation properties of Equation 15 of the basis $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ and the **E, H** of electric and magnetic fields are dual to each other, so that $\mathbf{E} = \mathrm{b}^2 < \Psi|\mathbf{X}^{(1)} >, \mathbf{H} = \mathrm{b}^2 < \Psi|\mathbf{X}^{(2)} >$.

The matrix representation of Equations 24 on the basis $\mathrm{X}^{(k)}{}_i$ k coincides with the Maxwell Equation 20. Model of the electric field, magnetic field and the field of magnetic vector potential is represented in the form $\mathrm{X}^{(1)}{}_3, \mathrm{X}^{(2)}{}_3, \mathrm{X}^{(3)}{}_3, \mathrm{X}^{(0)}{}_0$; it is composed from D-Wigner functions for spin j = 1, after averaging over the angle γ (Figure 3). Vector is the set of its projections.

Theorem 3

The generalized Dirac equation $\mathrm{N}^{(1)}\Psi = -m_e c/(2h)\ \Psi$ for a particle with mass m_e, spin j = 1, and zero projection of the spin on any axis X_i and its analogue on any axis $X^{(k)}$ for the amplitudes on the basis $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \mathrm{X}^{(0)}{}_0$ are equivalent to the Maxwell and Proca equations, respectively, for the 1-spin particle of mass m_e (heavy vector virtual photon).

Proof

We represent the solution as a sum of Lorentz invariants

$$\Psi = \Psi_{EH} + \Psi_A/\Lambda, \Psi_{EH} = \mathbf{E}\,\mathbf{X}^{(1)} + \mathbf{H}\,\mathbf{X}^{(2)}, \Psi_A == \mathbf{A}\,\mathbf{X}^{(3)} + \mathrm{A}_0\,\mathrm{X}^{(0)}{}_0,$$

Where $\Lambda = (2h)/(m_e c)$.

Zero values of the spin projection and its counterpart are equivalent to the real values of the amplitudes of E,H, A,A_0. The generalized Dirac equation splits into two equations:

$$N^{(1)}\Psi_A = -\Psi_{EH}, or, \mathbf{H} = i[\mathbf{PA}], \mathbf{E} = iP_0\mathbf{A}/c - i\mathbf{P}A_0,$$
$$N^{(1)}\Psi_{EH} = -\Psi_A/\Lambda^2, or, i(\mathbf{PE}) + A_0/\Lambda^2 = 0, iP_0\mathbf{E}/\mathrm{c} + i[\mathbf{PH}] + \mathbf{A}/\Lambda^2 = 0$$

The Maxwell equations imply the continuity equation for I_s which coincides with the Lorentz calibration of the vector potential of magnetic field, and the first London equation, which describes the Meissner effect of Equation 25, (London and London, 1935), where Λ is the London penetration depth, **A** is the vector potential, I_s is the superconducting component of the electric current.

$$\mathbf{A} = -\mathbf{I}_s 4\pi\Lambda^2/\mathrm{c}, \quad \mathrm{A}_0 = -4\pi\Lambda^2\mathrm{I}_{0s}, \quad [\mathbf{P}\,\mathbf{I}_s] = \mathrm{i}\,\mathbf{H}\mathrm{c}/(4\pi\Lambda^2) \quad (25)$$

The quaternion forms of these equations are equivalent to the equation $(\mathbf{A}, \mathrm{A}_0) = -4\Lambda^2(\mathbf{I}_s/\mathrm{c}, \mathrm{I}_{0s})$. Using the equation $\lozenge(\mathbf{A}, \mathrm{A}_0) = 4\pi^2(\mathbf{I}/\mathrm{c}, \mathrm{I}_{0s}), \mathbf{I}_s = \mathbf{I}$, we obtain the equation $((\lozenge + 1/\Lambda^2)(\mathbf{A}, \mathrm{A}_0) = 0$, written as the Proca equation (Ginzburg, 1979), where $\lozenge$ is the d'Alembert operator. The above equations can describe the superconductivity phenomena because the acquisition of mass by photons is associated with the losses of long-range interactions. This can reduce the energy losses by radiation.

RESULTS AND DISCUSSION

The reasons for the above proposed generalizations are related to the fact that fields E, H are not sufficient enough to describe the electromagnetic fields with spin; this is because the spin part is closely related to the wave function Ψ. By virtue of this, we introduce the wave function $\Psi = \Psi^{(+)} + \Psi^{(-)}$ which is equal to the sum of the left and right vector states with spin 1. The contribution of the left and right states $\Psi^{(+)}$ and $\Psi^{(-)}$ to the wave function is independent of the Maxwell equations. The spin of electromagnetic field E, H, Ψ is equal to the sum of spins of the left and the right states of the field. Any electromagnetic field E,H related to a stationary state can possess a spin. The generalized Maxwell equation admits spin states, but does not describe completely their variation. The spin of electromagnetic fields E,H corresponds to either the right or left of the vector. Spins of the left and right vector states E,H are collinear and anticollinear to the Poynting vector.

We propose the following mechanism for the transition of a conductor to superconducting state. The low-temperature superconductivity corresponds to a spontaneous transition to the state with zero spin projection and its analogues $< \Psi|\mathrm{J}_i|\ \Psi >= 0, < \Psi|\mathrm{J}^{(k)}|\ \Psi >= 0$, without changing the electromagnetic field E,H.

The high-temperature superconductivity can be considered as a transition to the state with spin 1. Besides, state Ψ must possess an analogue of spin 1, whose projection to $X^{(3)}$ axis is equal to $\pm 1, J^{(3)}\Psi = \pm\Psi$.

Our conjecture is the following: there exists just three generations of leptons (electrons, muons, tau-leptons), because the generalized spin group possesses at the same time just the three different spinor representations of the Lorentz group of Equation 7. The three representations of the Poincaré group derived by cyclic permutations of superscripts p = 1,2,3 correspond to the three representations of the Dirac equation. From a mathematical point of view, they are equivalent. The spinor representation of Poincaré group p = 3 associated with the first (stable in the decay) generation of leptons (electron) as well as the generator of the Lorentz transformation $Q^{(3)}{}_3$ is independent of the angles φ,γ.

We conjecture that there exists just three colors of quarks (red, green, blue), since the generalized spin Lorentz group possesses at the same time just the three different transposed spinor representations of the Lorentz group.

Minkowski space (non-locally isotropic in the presence of particles and fields) has three additional independent dimensions φ,θ,ν, which fully describe the degree of freedom of the spin.

Note that physical systems normally can be represented precisely in terms of purely complex generalized spin Lorentz groups, but not just by the Lorentz group in the spinor representation, due to the existence of an additional degree of freedom. Generalized spin Lorentz groups consist of the three Lorentz groups in spinor representations, and the three transposed Lorentz groups in spinor representations.

Similarly, spin of the particle, electromagnetic field (in state E, H, A) can be regarded as a consequence of the presence of internal degree of freedom (analog of spin 1 for isovector space) of Minkowski space.

Similarly (6,7), replacement of $\mathbf{L},\mathbf{K}^{(3)}$ to $\mathbf{J},\mathbf{Q}^{(3)}$,we introduce analog Pauli-Lubanski Spin Operator: $\widehat{W}_0 = (\mathbf{P}\,\mathbf{J}), \widehat{\mathbf{W}} = P_0\,\mathbf{J} - [\mathbf{P}\mathbf{Q}^{(3)}]$.

Conclusion

In this paper, spinors and their transformation properties are described as the properties of the rotation of local observer, but not in terms of rotation of the ordinary (meter) observer. This permutation of the local and ordinary observers agrees with the general theory of relativity. To this end, we introduce one-component wave functions $\Psi(x_1, x_2, x_3, t, \varphi, \theta, \nu)$, depending on the position and the orientation of a local observer in the Minkovski space-time determined by the three Euler angles φ,θ,ν. The seven variables $x_1, x_2, x_3, t, \varphi, \theta, \nu$ are independent, but their transformation properties are bound between them. For an arbitrary integer or half-integer spin j, the Poincaré group in angular (spinor) representation is described explicitly.

The matrix representation of the algebra $(\mathbf{J}, \mathbf{Q}^{(3)}, \mathbf{Q}^{(-)}, i\mathbf{J}^{(-)})$ coincides with the Poincaré group in spinor representation. A matrix representation of 16 generators of the generalized spin Lorentz group on the basis of j = 1/2 coincides with 16 basic elements of the Clifford algebra formed by the Dirac gamma-matrices (Flügge, 1974). Momentum operator $\mathbf{Q}^{(-)}, i\mathbf{J}^{(-)}$ in the Lie algebra of the angular variables of the Poincaré group is always a complex operator. Analogs of discrete operator $\widehat{P}$ (reverse the orientation of space) are included in the generalized spin Lorentz group as continuous operator $\widehat{I} = \exp(i\,\pi J^{(2)}), \widehat{I}Q^{(3)} = -Q^{(3)}$. Discrete operator $\widehat{P}\widehat{T}$ ($\widehat{T}$ - reverse the time) corresponds to the operator of complex conjugation for P, P_0, Ψ. This corresponds to continuous operator $\bar{\xi}^{(\pm j)} = -\exp(-i\,\pi J^{(2)} - i\,\pi J_2)\xi^{(\pm j)}$.

The Lorentz group O(1, 3) has four connected components (Ohnuki, 1988). The elements in each component are characterized by whether or not they reverse the orientation of space and/or time. Generalized spin Lorentz group and the Lorentz group O(1, 3) have four connected components.

Also are derived in uniform manner the generalized Dirac and Maxwell equations for Ψ with arbitrary spin j in terms of scalar product of two four-vector momentum operators (generators the Poincaré group) in the angular representation and coordinate representation. The matrix representations of the corresponding operator equations for spin j = 1/2, 1 particles coincide with the Dirac, Maxwell, or Proca equations.

These generalized Dirac and Maxwell equations imply the following conclusions:

(1) Free fixed electron and positron have analogues of the spin projection ±1/2, $J^{(1)}\,\Psi = \pm\Psi/2$ along $\mathbf{X}^{(1)}$ axis because the term, depending on the momentum P in (18), k=2, vanishes. In the standard representation (Weinberg, 2003), this determines our choice of the basis: $\eta = \xi^{(1/2)} \pm \xi^{(-1/2)}$;

(2) Neutrinos and antineutrinos, left and right photons moving along x_3 are considered as states with spin projections on the axis x_3; analogues of the spin projections on the axis of $X^{(3)}$ are equal to ±1/2 and ±1, respectively. For neutrinos and antineutrinos, we have $j = 1/2,\ J^{(3)}\,\Psi = \pm\Psi/2, J_3\,\Psi = \mp\Psi/2$, where $\Psi = \psi_2, \Psi = \psi_3$; for photons $j = 1,\ J^{(3)}\,\Psi = \pm\Psi, J_3\,\Psi = \mp\Psi$, where $\Psi = X^{(\pm)}{}_{\mp}$. The Lorentz transformation along x_3 does not change form Ψ for neutrino, antineutrino and photons, since Ψ is an eigen function of generator Lorentz transformations $Q^{(3)}{}_3$.

The sum of the left and right photons $X^{(+)}_{-} + X^{(-)}_{+} = 2(X^{(1)}_{1} - X^{(2)}_{2})$ corresponds to the plane polarized photon;

(3) The generalized Dirac equation for a massive particle with j =1, zero spin projection and its analogue $J^{(k)}$, and the assumptions on implementation of the Maxwell equation imply the first London's equation for superconductivity which produces the Meissner effect and the possibility of creation of massive photons.

Using the known representation for the transposed spinor representation of rotation group (4) and spinor representation (7) of the Lorentz group of angular variables, which are unique for p = 3, we obtain the generalized spin Lorentz spin group (4),(12) corresponding to the unification of the spinor representation of the Lorentz group and its transposed representation. Unambiguous form of generalized spin Lorentz group follows from the principle of the special relativity theory on equivalence of coordinate systems with respect to permutation of local observer and ordinary observer: $Q^{(k)}_{n} = (-1)^{k+n} W Q^{(n)}_{k}$.

Conflict of Interests

The authors have not declared any conflict of interests.

ACKNOWLEDGMENTS

The authors are thankful to the doctors of physical and mathematical Sciences, professors: Chebotarev A. M., Yu.S. Vladimirov, Frolov B. N. for scientific support of this work and for stimulating discussion.

REFERENCES

Akhiezer AI, Berestetskii VB (1965). Quantum electrodynamics , Nauka, Moscow, 1959, Interscience Publishers John Wiley and Sons, Inc., New York, London, Sydney, pp. 31:52-53,97.

Biedenharn LC, Louck JD (1984). Angular momentum in quantum physics. Encyclopedia of Mathematics and its Applications, 8 (Addison-Wesley Publishing Co., Reading, Mass., 1981; Mir, Moscow, 1984). Russian 1:34, 50, 62, 64, 69, 49, 137.

Flügge S (1974). Practical Quantum Mechanics (Springer, New York, 1974; Mir, Moscow, pp. 196-214.

Fushchich VI, Nikitin AG (1994). Symmetries of Equations of Quantum Mechanics (Nauka, Moscow, 1990; Allerton Press, Inc., New York). 16:28-30.

London F, London H (1935). ЛОНДОНОВ УРАВНЕНИЕ (Londonov Equation). http://dic.academic.ru/dic.nsf/enc_physics/3725 1935.

Ohnuki Y (1988). Unitary Representations of the Poincaré Group and Relativistic Wave Equations. World Scientific Pub. Co. Inc. P.24. http://dx.doi.org/10.1142/0537

Rumer. JB, Fet AI (1997). Group theory and quantized fields (Nauka, Moscow, 1977), in Russian, 14:25.

Teschl G (2009). Mathematical methods in quantum mechanics: With applications to Schrodinger operations, by the American Mathematical Society, pp. 173-178. http://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf

Varadarajan VS (1989). The geometry of quantum theory. Springer, Berlin. pp.303-309

Varshalovich DA, Moskalev AN, Khersonskii VK (1975). Quantum Theory of Angular Momentum, Nauka, Leningrad, 1975, World Scientific Publishing Co., Inc., Teaneck, NJ, 1975.

Weinberg S (2003). The quantum theory of fields. Vol. I. Foundations, Cambridge University Press, Cambridge, 1996; Fizmatlit, Moscow, P. 283.