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Full Length Research Paper

Travelling wave solution for non-linear Klein- Gordon equation

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In this work, we construct explicit exact solutions for the non-linear Klein-Gordon equation by using a $(\frac{G'}{G})$ -expansion method. By means of the method, many new exact travelling wave solutions for the non-linear Klein- Gordon equation are successfully obtained.

Key words: Non-linear Klein-Gordon equation, $(\frac{G'}{G})$ -expansion method.

INTRODUCTION

The investigation of exact travelling wave solutions to nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. In recent years, the exact solutions of non-linear PDEs have been investigated by many authors and Many powerful methods have been presented by those authors such as the homogeneous balance method (Wang, 1995; Zayed, 2004), the hyperbolic tangent expansion method (Yang, 2001; Zedan, 2004), the trial function method (M. Inc, 2004), the tanh-method (Abdou, 2007; Fan, 2000; Malfliet, 1992; Parkes, 1996; Wang, 2005), the non-linear transform method (Hu, 2004), the inverse scattering transform (Ablowitz, 1991), the Backlund transform (Miura, 1978; Rogers, 1982), the Hirota's bilinear method (Hirota, 1973; Hirota, 1981) and so on. The objective of this paper is to use a new method which is called the $(\frac{G'}{G})$ -expansion method (Bekir, 2008; Wang, 2008; Zhang, 2008; Zhang, 2008). The $(\frac{G'}{G})$ -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$ and that $G = G(x)$ satisfies a second order linear ordinary differential equation (ODE).

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Description of the $(\frac{G'}{G})$ - expansion method

Considering the nonlinear partial differential equation in the form

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx}, ...) = 0 \quad (1)$$

Where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and it has various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G'}{G})$ -expansion method.

Step 1: Combining the independent variables x and t into one variable $x = x - vt$, we suppose that

$$u(x,t) = u(x), \qquad x = x - vt \quad (2)$$

The travelling wave variable (2) permits us to reduce

Equation (1) to an ODE for $G = G(\xi)$, namely

$$P(u, -vu', u', v^2u'', -vu'', u'', ...) = 0 \tag{3}$$

Step 2: Suppose that the solution of ODE (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = a_m(\frac{G'}{G}) + ..., \tag{4}$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{5}$$

$a_m, ..., \lambda$ and μ are constants to be determined later $a_m \neq 0$, the unwritten part in (4) is also a polynomial in $(\frac{G'}{G})$, but the degree of which is generally equal to or less than $m-1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

Step 3: By substituting (4) into Equation (3) and using the second order linear ODE (5), collecting all terms with the same order $(\frac{G'}{G})$ together, the left-hand side of Equation (3) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_m, ..., \lambda$ and μ.

Step 4: Assuming that the constants $a_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting $a_m, ..., v$ and the general solutions of Equation (5) into (4) we have more travelling wave solutions of the nonlinear evolution Equation (1).

Non-linear Klein-Gordon equation

In this section we consider the non-linear Klein-Gordon equation in the following form

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0 \tag{6}$$

The travelling wave variable below

$$u(x,t) = u(\xi), \qquad \xi = x - vt \tag{7}$$

Permits us converting Equation (7) into an ODE for $G = G(\xi)$

$$(v^2 - 1)u'' + \alpha u + \beta u^3 = 0 \tag{8}$$

Suppose that the solution of ODE (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = a_m(\frac{G'}{G}) + ..., \tag{9}$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{10}$$

a_1, a_0, v and μ are to be determined later.

By using (9) and (10) and considering the homogeneous balance between u'' and u^3 in Equation (8) we required that $2m = m + 1$ then $m = 1$. So we can write (9) as

$$u(\xi) = a_1(\frac{G'}{G}) + a_0 \tag{11}$$

Therefore

$$u^2 = \alpha_1^2(\frac{G'}{G})^2 + 2\alpha_1\alpha_0(\frac{G'}{G}) + \alpha_0$$

$$u^3 = \alpha_1^3(\frac{G'}{G})^3 + 3\alpha_1^2\alpha_0(\frac{G'}{G})^2 + 3\alpha_1\alpha_0^2(\frac{G'}{G}) + \alpha_0^3$$

$$u'' = 2\alpha_1(\frac{G'}{G})^3 + 3\alpha_1\lambda(\frac{G'}{G})^2 + (\alpha_1\lambda^2 + 2\alpha_1\mu)(\frac{G'}{G}) + \alpha_1\lambda\mu$$

By substituting Equation (11) and relations above into Equation (8) we have

$$\beta a_1^3 + 2(v^2 - 1)a_1 = 0$$
$$3\beta a_1^2 a_0 + 3(v^2 - 1)a_1\lambda = 0$$
$$3\beta a_1 a_0^2 + (v^2 - 1)(a_1\lambda^2 + 2a_1\mu) + \alpha a_1 = 0$$
$$\beta a_0^3 + (v^2 - 1)a_1\lambda\mu + \alpha a_0 = 0$$

Solving the algebraic equations above, yields

Case 1: If

$$a_1 = \frac{2\sqrt{b\left(4\mu - \lambda^2\right)\alpha}}{b\left(4\mu - \lambda^2\right)}$$

Then

$$a_0 = \frac{\sqrt{b\left(4\mu - \lambda^2\right)\alpha}\,\lambda}{b\left(4\mu - \lambda^2\right)},$$

$$v = \pm \frac{\sqrt{\left(4\mu - \lambda^2\right)\left(-2\alpha + 4\mu - \lambda^2\right)\gamma}}{4\mu - \lambda^2}$$

By using above, expression (11) can be written as

$$u(\xi) = \frac{2\sqrt{b\left(4\mu - \lambda^2\right)\alpha}}{b\left(4\mu - \lambda^2\right)}\left(\frac{G'}{G}\right) + \frac{\sqrt{b\left(4\mu - \lambda^2\right)\alpha}\,\lambda}{b\left(4\mu - \lambda^2\right)} \qquad (12)$$

and $\xi = x \mp \frac{\sqrt{\left(4\mu - \lambda^2\right)\left(-2\alpha + 4\mu - \lambda^2\right)\gamma}}{4\mu - \lambda^2}t.$

Eq (12) is the formula of a solution of Equation (8). Substituting the general solutions of Equation (10) as follows

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\times \left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right) - \frac{\lambda}{2}$$

Into (12) we have three types of travelling wave solutions of the non-linear Klein-Gordon Equation (6) as follows:

When $\lambda^2 - 4\mu > 0$

$$u(\xi) = \frac{i\sqrt{b\alpha}}{b}\,\times \left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right) + \frac{\sqrt{b\left(4\mu - \lambda^2\right)\alpha}\,\lambda}{b\left(4\mu - \lambda^2\right)} - \frac{\lambda}{2}$$

Where $\xi = x \mp \frac{\sqrt{\left(4\mu - \lambda^2\right)\left(-2\alpha + 4\mu - \lambda^2\right)\gamma}}{4\mu - \lambda^2}t$.

C_1, and C_2, are arbitrary constants.

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, u, become

$$u(\xi) = \frac{i\sqrt{b\alpha}}{b}tgh\frac{1}{2}\lambda\,\xi + \frac{i\sqrt{b\alpha}}{b} - \frac{\lambda}{2}$$

When $\lambda^2 - 4\mu < 0$

$$u(\xi) = \frac{i\sqrt{b\alpha}}{b}\,\times \left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right) + \frac{\sqrt{b\left(4\mu - \lambda^2\right)\alpha}\,\lambda}{b\left(4\mu - \lambda^2\right)} - \frac{\lambda}{2}$$

When $\lambda^2 - 4\mu = 0$

$$u(\xi) = \frac{2\sqrt{b\left(4\mu - \lambda^2\right)\alpha}\,C_2}{(C_1 + C_2\xi)\,b\left(4\mu - \lambda^2\right)},$$

$$\xi = x \mp \frac{\sqrt{\left(4\mu - \lambda^2\right)\left(-2\alpha + 4\mu - \lambda^2\right)\gamma}}{4\mu - \lambda^2}t$$

Where C_1 and C_2 are arbitrary constants.

Case 2: If

$$a_1 = -\frac{2\sqrt{b\left(4m- l^2\right)a}}{b\left(4m- l^2\right)}$$

Then

$$a_0 = -\frac{\sqrt{b\left(4m- l^2\right)a l}}{b\left(4m- l^2\right)},$$

$$v = \pm\frac{\sqrt{\left(4m- l^2\right)\left(-2a + 4m- l^2\right)}l}{4m- l^2}$$

By using the expression above and general solution of Equation (10) we have three type solution of Equation (6) as above, for example

When $l^2 - 4m \text{ p } 0$

$$u(x) = -\frac{i\sqrt{ba}}{b},$$

$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4m- l^2}x + C_2 \cos\frac{1}{2}\sqrt{4m- l^2}x}{C_1 \cos\frac{1}{2}\sqrt{4m- l^2}x + C_2 \sin\frac{1}{2}\sqrt{4m- l^2}x}\right)$$

$$-\frac{\sqrt{b\left(4m- l^2\right)a l}}{b\left(4m- l^2\right)} - \frac{l}{2}$$

Conclusion

The $(\frac{G'}{G})$ -expansion method has been successfully used to seek exact solutions of the non-linear Klein-Gordon equation. As a result, abundant new exact explicit solutions are obtained. It is shown that this method provides a very effective and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

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