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Full Length Research Paper

A class of fixed denominator rational integrators

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Existing methods such as Niekerk (1987) and Fatunla (1982) are effective but could not handle problem whose initial value is zero. In this paper, we designed and implemented a class of free denominator rational integrators for the solution of initial value problems (IVPs) in ordinary differential equations (ODES), particularly for the case of Stiff and singular problems. The class of integration obtained was found to be consistent and convergent. Our study of the stability characteristics reveal that the integrators are A-stable when m = 0, 1, 2. Their Region of Absolute Stability (RAS) decreases with increasing value of m. Experiments carried out and analyzed with the computer shows encouraging computational results and also show that the rational integrator copes well with all kinds of problem.

Key words: Fixed denominator, rational integrators, Jordan curves, A-stable Region of Absolute stability, consistency and convergence.

INTRODUCTION

The problem is to solve the initial value problem (IVP)

$\underline{y}^{(1)} = \underline{f}(x,\underline{y}), \ y(a) = y_0$

Where

$\underline{y}, \ \underline{f} \in C^m$ (R), $x \in [a.b]$, a, b $\in$ R and f satisfies the lipschitz condition $\| \underline{f}(x,y_2) - \underline{f}(x,y_1) \| \leq L/y_2 - y_1/$,

L is the Lpschitz constant. Fatunla (1982) proposed a method which approximates the theoretical solution by the rational form

$$F_k(x) = \frac{A}{1 + \sum_{r=1}^{k} a_r x^n}$$

Where

A and the polynomial coefficient a_r are real variable parameters. This method of Fatunla (1982) is applicable only to the cases where the initial value $y_0 \neq 0$. Rational integrators that emerge after Fatunla (1982) include Neikerk (1987), Veldhulzien (1987), Aashikpelokhai and Fatunla (1994), Ikhile (2001, 2002) and Aashikpelokhai and Momodu (2008). These later methods can handle problems whose initial values $y_0 = 0$ as well as those with $y_0 \neq 0$.

Our focus is in Neikerk (1987)) where there has been a problem created and directly unresolved in the Niekerk (1987). This is what motivated this research. Niekerk (1987) attempted making an execution of his method:

$$y_{n+1} = a_{n+1} + \frac{b_{n+1}}{1+C_{n+1}x_{n+1}}$$

to $$y_{n+1} = a_{n+1} + \frac{b_{n+1}}{1+C_{n+1}x_{n+1} + d_{n+1}x_{n+1}^2}$$

and ran into problem with C_{n+1}. The identity

$$y_{n+1} = a_{n+1} + b_{n+1} [1 + C_{n+1} x_{n+1} + d_{n+1} \ x_{n+1}^2]^{-1}$$

gave rise to a quadratic equation in C_{n+1} whose determinants sign depends on the values of $y_n^{(r)}$, r = 0, 1, 2, 3. The determinants could have negative sign under the root sign at some of the interpolation points. He aban-

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doned the move. Bearing in mind that a_{n+1}, b_{n+1}, d_{n+1} and dues are read variable parameters, we re-write

$$y_{n+1} = a_{n+1} + \frac{b_{n+1}}{1+C_{n+1}x_{n+1} + d_{n+1}x_{n+1}^2} \qquad (1.1)$$

as

$$y_{n+1} = \frac{(a_{n+1} + b_{n+1}) + a_{n+1}C_{n+1}x_{n+1} + a_{n+1}d_{n+1}x_{n+1}^2}{1+C_{n+1}x_{n+1} + d_{n+1}x_{n+1}^2},$$

resulting in $y_{n+1} = \dfrac{P_0 + P_1 x_{n+1} + P_2 x_{n+1}^2}{1+q_1 x_{n+1} + q_2 x_{n+1}}$ (1.2)

Where

$P_0 = a_{n+1} + b_{n+1}$ (1.3)
$P_r = a_{n+1} q_r$, $r = 1, 2$ (1.4)
$q_1 = c_{n+1}$ (1.5)
$q_2 = d_{n+1}$ (1.6)

are all real variable parameters.

To overcome the problem, we allow the polynomial parameters P_0, P_1, P_2, q_1, q_2 to be arbitrary real parameters to be determined. In this case, P_1 and P_2 do not necessarily need to satisfy (1.4). By this approach which we highlight in section 2 of this work, we would have the problem of determining values for the first parameters instead of the four in Niekerk [1987]. Since the parameter values depend on $y_n^{(r)}$, $r = 0, 1, 2$, if a problem exists such that $P_1 = a_{n+1} q_1$ and $P_2 = a_{n+1} q_2$, then such a resulting interpolant reduces to the Neikerk form without the attendant problem. This work therefore makes Neikerk's of our work whenever $m = 2$ and $P_r = a_{n+1} q_r$, $r = 1, 2$

DEVELOPMENT OF THE NEW INTEGRATOR

Let U: R → R be the real rational interpolant defined by the identity

$$U(x).[1 + q_1x + q_2x^2] = \sum_{s=0}^{m} P_r x^2 \qquad (2.1)$$

Subject to the integration constants

$$U(x_{n+1}) = \begin{cases} y(x_{n+i}), & i = 0 \\ y_{n+i}, & i = 0, 1 \end{cases} \qquad (2.2)$$

when $m = 2$ and (1.4) is satisfied, we obtain Neikerk (1987). The m+3 unknown parameter $[P_0, P_1, P_2, \ldots P_m]$ and $[q_1, q_2]$ are to be determined by considering $x_n = nh$, (2.1) and (2.2) to yield

$$Y_{n+1} = \left[\sum_{s=0}^{m} P_r x_{n+1}^r\right][1 + q_1x_{n+1} + q_2x_{n+1}^2]^{-1} \qquad (2.3)$$

Where $n = 0, 1, 2, \ldots\ldots$

Writing

$Y_{n+1} = \sum_{r=0}^{\infty} \dfrac{h^r y_n^{(r)}}{r!}$ in power series, relation (2.3) becomes

$$\sum_{r=0}^{\infty} \frac{h^r y_n^{(r)}}{r!} \equiv \left[\sum_{s=0}^{m} P_r x_{n+1}^r\right]\left[\sum_{t=0}^{\infty}(-1)^t (q_1x_{n+1} + q_2x_{n+1}^2)\right]^{-1} \qquad (2.4)$$

Since our unknown parameter to be determined are m+3, we may re-write (2.4) as

$$\sum_{r=0}^{m+2} \frac{h^r y_n^{(r)}}{r!} \equiv \left[\sum_{s=0}^{m} P_r x_{n+1}^r\right]\left[\sum_{t=0}^{m+2}(-1)^t (q_1x_{n+1} + q_2x_{n+1}^r)^t\right]^{-1} \qquad (2.5)$$

Equating term by term, (2.5) then yield the following results:
For the first terms, $y_n = P_0$ and a simple re-arrangement yields

$P_0 = y_n$

The second terms yield $hy_n^{(1)} = [P_1 - P_0 q_1] x_{n+1}$ (2.6)

Substituting for P_0 from (2.6) and re-arranging, we obtain

$$P_1 = \frac{hy_n^{(1)}}{x_{n+1}} + y_n q_1 \qquad (2.7)$$

For the third terms, we have

$$\frac{h^2 y_n^{(2)}}{2!} = [P_2 - P_1q_1 - P_0q_2 + P_0 q_1^2]x_{n+1}$$

Writing this result as a combination of the preceeding results in (2.6) and (2.7), we obtain

$$P_2 = \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} + \frac{hy_n^{(1)}}{x_{n+1}} q_1 + y_n q_2 \qquad (2.8)$$

The fourth terms yield

$$\frac{h^3 y_n^{(3)}}{3!} = [P_3 - P_2q_1 - P_1q_2 + P_1 q_1^2 - 2P_0 q_1 q_2 - P_0 q_1^3]\, x_{n+1}^3$$
$$= [P_3 - (P_1P_0q_1)q_2 - \{P_2 - P_0 q_2 - (P_1 - P_0 q_1)q_1\}q_1]x_{n+1}^3$$

Employing (2.6) – (2.8) and re-arranging we obtain

$$P_3 = \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} + \frac{h y_n^{(1)}}{1!x_{n+1}} q_2 \qquad (2.9)$$

Continuing in this manner of writing the results on-hand in terms of the preceeding ones we get

$$P_r = \frac{h^r y_n^{(r)}}{r! x_{n+1}^r} + \frac{h^{r-1} y_n^{(r-1)}}{(r-1)! x_{n+1}^{r-1}} q_1 + \frac{h^{r-2} y_n^{(r-2)}}{(r-2)! x_{n+1}^{r-1}} q_2 \qquad (2.10)$$

Where r = 2, 3, …., m.

By recalling results (2.4) and not its subset (2.5) and then observing that

$$\sum_{s=0}^{m} P_r x^r$$

Is a polynomial of degree arbitrary positive integer m meant that one could write 0 = P_r for every r = m+1, m+2, m+3, ...

These enable us to have the next two consecutive terms which are $(m+1)^{th}$ and $(m+2)^{th}$ terms yielding the simultaneous linear algebraic equation in q_1 and q_2. That is

$$0 = \frac{h^{m+1} y_n^{(m+1)}}{(m+1)! x_{n+1}^{m+1}} + \frac{h^m y_n^{(m)}}{(m)! x_{n+1}^{m}} q_1 + \frac{h^{m-1} y_n^{(m-1)}}{(m-1)! x_{n+1}^{m-1}} q_2 \qquad (2.11)$$

$$0 = \frac{h^{m+2} y_n^{(m+2)}}{(m+2)! x_{n+1}^{2}} + \frac{h^{m+1} y_n^{(m+1)}}{(m+1)! x_{n+1}^{m+1}} q_1 + \frac{h^m y_n^{(m)}}{m! x_{n+1}^{m}} q_2 \qquad (2.12)$$

Solving the simultaneous linear algebraic equation (2.11) and (2.12), we obtain

$$q_1 = \frac{NUM\ Q_1(M, y_n)}{DENOM\ (M, y_n)} \qquad (2.13)$$

$$q_2 = \frac{NUM\ Q_2\ (M, y_n)}{DENOM\ (M, y_n)} \qquad (2.14)$$

Where

DENOM ((M, y_n)

$$= \frac{(m+1)\ \{y_n^{m}\}^2\ - mY_n^{(m-1)} Y_n^{(m)}}{m!\ (m+1)!} \qquad (2.15)$$

NUM Q1 (M, y_n)

$$= \frac{(mY_n^{(m-1)}\ Y_n^{(m+2)}\ -\ (m+2) Y_n^{(m)}\ Y_n^{(m+1)}}{m!\ (m+2)!} \qquad (2.16)$$

NUM Q2 (M, y_n)

$$= \frac{(m+2)\ \left\{y_n^{(m+1)}\right\}^2 - (m+1)\ y_n^{(m)}\ y_n^{(m+2)}}{(m+1)!\ (m+2)!} \qquad (2.17)$$

Consequently,

$q_1 x_{n+1}$ = h[NUMQ1 (M,y_n) DENOM^{-1} (m,y_n)] (2.18)

$q_2 x^2_{n+1}$ = h^2[NUMQ2 (M,y_n) DENOM^{-1} (m,y_n)] (2.19)

$P_1 x_{n+1}$ = h[$y_n^{(1)}$ + y_n NUMQ1(m,y_n) DENOM^{-1} (m,y_n)] (2.20)

$P_r x_{n+1}{}^r = \frac{h^r y_n^{(r)}}{r!} + \frac{h^{r-1} y_n^{(r-1)}}{(r-1)!}$ h[NUMQ1 (M,y_n) DENOM^{-1} (m,y_n)] + $\frac{h^{r-2} y_n^{(r-2)}}{(r-2)!}$ h^2[NUMQ2 (M,y_n) DENOM^{-1} (m,y_n)] (2.21)

Where r = 2, 3, ………, m

we could at this stage have stopped and rightly state that our rational integrator is given by (2.3) where [$q_r\ x_{n+1}^r$, r = 1(1)m] and [$P_r\ x_{n+1}^r$, r = 0(1)m] are given respectively by (2.15) – (2.19), (2.6), (2.20) and (2.21). However the need for reduced computational workload at implementation stages motivates us to substitute the work-out values of these parameters at this design stage into results (2.3).

Employing (2.1) – (2.2), (2.6) and (2.18) – (2.21), our desired fixed denominator rational integrator becomes

$$Y_{n+1} = \frac{\sum_{r=0}^{m} P_r x_{n+1}^r}{1+q_1 x_{n+1} + q_2 x^2_{n+1}}$$

$$= \frac{P_0 + P_1 x_{n+1} + \sum_{r=2}^{m} \Pr x_{n+1}^r}{1+q_1 x_{n+1} + q_2 x^2_{n+1}} = \frac{TOP\ (m.h.y_n)}{BOT\ (m.h.y_n)}$$

Where

$$TOP\ (m, h, y_n) = \left[\sum_{r=0}^{m} \frac{h^r y_n^{(r)}}{r!} DENOM\ (m, y_n) + h \sum_{r=1}^{m} \frac{h^{r-1} y_n^{(r-1)}}{(r-1)!}\ NUM\ Q1\ (m, y_n) + h^2 \sum_{r=2}^{m} \frac{h^{r-2} y_n^{(r-2)}}{(r-2)!}\ NUM\ Q2\ (m, y_n)\ DENOM^{-1}(m, y_n)\right] \qquad (2.22)$$

BOT (m,h,y_n) = [DENOM (m,y_n) + hNUM Q1(m,y_n) + h^2NUM Q2 (m,y_n)]DENOM^{-1}(m,y_n) (2.23)

Our final expression for the integrator is given by

$$y_{n+1} = \left[\sum_{r=0}^{m} \frac{h^r y_n^{(r)}}{r!} DENOM\,(m, y_n) + h \sum_{r=1}^{m} \frac{h^{r-1} y_n^{(r-1)}}{(r-1)!} NUM\ Q1\,(m, y_n) + h^2 \sum_{r=2}^{m} \frac{h^{r-2} y_n^{(r-2)}}{(r-2)!} NUM\ Q2\,(m, y_n)\right]$$

divided by

$[DENOM\ (M,y_n) + hNUMQ1\ (m,y_n) + h^2NUMQ_2(m,y_n)$ (2.24)

Observation

Users of integrators and who are not fully of sound familiarity with the use of sigma $(\sum)notation$, in particular the applicability of their lower and upper limit, we state that m=0 is not applicable to $\sum_{r=1}^{m}$ and $\sum_{r=2}^{m}$, m = 1 is not applicable to $\sum_{r=2}^{m}$. we demonstrate this usage below.

Case m = 0

$$y_{n+1} = \frac{y_n DENOM\ (0, y_n)}{DENOM\ (o, y_n) + hNUMQ1(o, y_n) + h^2NumQ2(o,y_n);\ y_n \neq 0}$$

with the available values of h, y_n and its derivatives, we compute DENOM (o, y_n), NUM Q1 (o,y_n) and NUMQ2 (o, y_n); the results for y_{n+1}, n = 0, 1, 2, …. follows.

Case m = 1

$$y_{n+1} = \frac{\left(y_n + hy_n^{(1)}\right) DENOM\ (1_1 y_n) + hy_n\ NUM\ Q2(1, y_n)}{DENOM\ (1, y_n) + hNUM\ Q1(1, y_n) + h^2 NUM\ Q2(1, y_n)}$$

By similar computation as in case m = 0, we obtain y_{n+1}, n = 0, 1, 2, 3,…. For all the cases m $\geq$ 2, we do not need any demonstration here.

CONSISTENCY AND CONVERGENCE

Our remarks here are basically definitions.

Definition 1. Lambert (1976)

Every one-step numerical integration is convergent if and only if it is consistent.

Definition 2. Lambert (1973, 1976, 2000).

Every one step numerical integrator is consistent if the potential function which is the integrator expression representing $(y_{n+1} - y_n)/h$ tends to $y_n^{(1)}$ in the limit h tends to zero.

Theorem 1

From our introductory remarks here, all that is needed is to prove that the integration yield the result

$$Lt\ \left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)} \qquad h \to 0$$

By writing and direct simplification, the integrator (2.4) becomes

$$y_{n+1} - y_n = \left[\sum_{r=1}^{m} \frac{h^r y_n^{(r)}}{r!} DENOM\,(m, y_n) + h \sum_{r=2}^{m} \frac{h^{r-1} y_n^{(r-1)}}{(r-1)!} NUM\ Q1\,(m_1 y_n) + h^2 \sum_{r=3}^{m} \frac{h^{r-2} y_n^{(r-2)}}{(r-2)!} NUM\ Q2\,(m_1 y_n)\right] \text{divided by}$$

$[DENOM\ (m,y_n) + hNumQ1(m,y_n) + h^2\ NUMQ2\ (m,y_n)]$

Further simplification and re-arrangement lead us to

$$\frac{y_{n+1} - y_n}{h} = y_n^{(1)} + h\left[\sum_{r=2}^{m} \frac{h^{r-2} y_n^{(r)}}{r!} DENOM\,(m, y_n) + h \sum_{r=3}^{m} \frac{h^{r-3} y_n^{(r-1)}}{(r-1)!} NUM\ Q1\,(m, y_n) + h^2 \sum_{r=4}^{m} \frac{h^{r-4} y_n^{(r-4)}}{(r-2)!} NUM\ Q2\,(m, y_n)\right]$$

divided by

$[DENOM\ (m,y_n) + NumQ1(m,y_n) + NUMQ2\ (m,y_n)]$

Hence

$$Limit\left(\frac{y_{n+1} - y_n}{h}\right) = y_r^{(1)} \qquad h \to 0$$

We are done. Hence the integrators are consistent and convergent.

STABILITY CONSIDERATION

Theorem. The stability function of the class of rational integrators (2.24) are given by

$$\bar{s}(\bar{h}) = \frac{(m+1)(m+2)\sum_{r=0}^{m} \frac{\bar{h}^r}{r!} - 2(m+1)\bar{h} \sum_{r=1}^{m} \frac{\bar{h}^{r-1}}{(r-1)!} + \bar{h}^2 \sum_{r=2}^{m} \frac{\bar{h}^{r-2}}{(r-2)!}}{(m+1)(m+2) - 2(m+1)\bar{h} + \bar{h}^2} \quad (5.1)$$

Proof

To obtain the stability function, we subject the integrator (2.24) to the usual meshsize-eigenvalue relationship $\bar{h} = \lambda h$ and the normal test equation

$Y^{(1)} = \lambda y$; these results in

$[DENOM\ (m,y_n)$ becoming $\frac{\lambda^{2m} y_n^2}{m!(m+1)!}$ (5.2) hNUMQ1 (m,y_n) becoming $-2\bar{h}\left(\frac{\lambda^{2m}y_n^2}{m!(m+2)!}\right)$ (5.3)

and

h^2 NUMQ2 (m,y_n) becoming $\bar{h}^2\left(\frac{\lambda^{2m}y_n^2}{(m+1)!(m+2)!}\right)$ (5.4)

Therefore

$$\sum_{r=0}^{m}\frac{h^r y_n^{(r)}}{r!}\ DENOM\ (m, y_n) + h\sum_{r=1}^{m}\frac{h^{r-1}y_n^{(r-1)}}{(r-1)!}\ NUMQ1(m, y_n) + h^2\sum_{r=2}^{m}\frac{h^{r-2}y_n^{r-2}}{(r-2)!}\ NUMQ2\ (m, y_n)$$

becomes

$$\frac{\lambda^{2m}\ y_n^3[(m+1)(m+2)\sum_{r=0}^{m}\frac{h^r}{r!} - (2m+1)\ \bar{h}\ \sum_{r=1}^{m}\frac{h^r-1}{(r-1)!} + h^2\sum_{r=2}^{m}\frac{\bar{h}^{r-2}}{(r-2)!}}{(m+1)!\,(m+2)!}$$

While

DENOM (m,y_n) + hNumQ1(m,y_n) +h^2 NUMQ2 (m,y_n)] becomes

$$\frac{\lambda^{2m}y_n^2[(m+1)(m+2)-2(m+1)\bar{h}+\bar{h}^2]}{(m+1)!(m+2)!} \quad (5.6)$$

These results lead us to

$$y_{n+1}(\bar{h}) = \frac{\left[(m+1)(m+2)\sum_{r=0}^{m}\frac{\bar{h}^r}{r!} - 2(m+1)\bar{h}\sum_{r=1}^{m}\frac{\bar{h}^{r-1}}{(r-1)!} + \bar{h}^2\sum_{r=2}^{m}\frac{\bar{h}^{r-2}}{(r-2)!}\right]y_n(\bar{h})}{(m+1)!(m+2)-2(m+1)\bar{h}+h^2} \quad (5.7)$$

For one step methods, the stability functions are defined by the ratio $\frac{y_{n+1}(\bar{h})}{y_n(\bar{h})}$ which we designate here by S ($\bar{h}$).

Conclusively the required stability is given by

$$S(\bar{h}) = \frac{(m+1)(m+2)\sum_{r=0}^{m}\frac{\bar{h}^r}{r!} - 2(m+1)\bar{h}\sum_{r=1}^{m}\frac{\bar{h}^{r-1}}{(r-1)!} + \bar{h}^2\sum_{r=2}^{m}\frac{\bar{h}^{r-2}}{(r-2)!}}{(m+1)(m+2)-2(m+1)\bar{h}+\bar{h}^2} \quad (5.8)$$

Which is what we are required to establish.

Demonstrations on the decreasing RAS

We examine the region in which the stability function S($\bar{h}$) satisfies /S(u, v)/ $\leq$ 1, where $\bar{h}$ = u + iv, i = $\sqrt{-1}$. Our work here is restricted to the class where m = 0, 1, 2. In order to reduce our workload we state herein an elementary relationship we employed, and this is :

For all real a_1, a_2 ….., a_n

$$\left[\sum_{1\leq r\leq n} a_r\right] = \sum_{1\leq r\leq n} a_r^2 + 2\sum a_r a_s$$

Case 1: m = 0

$$S(\bar{h}) = \frac{2}{2-2\bar{h}+\bar{h}^2} = \frac{2}{(2-2U+U^2-V^2)+i(2UV-V)}$$

Hence /S(u,v)/ $\leq$1 if and only if $U^4 + 2U^2V^2 + V^2 - 4U^3 - 4UV^2 + 8U^2 - 8U \geq 0$

That is

S(u,v)/ $\leq$1 if and only if $(U^2+V^2)^2 - 4U(U^2+V^2) + 8U^2 - 8U \geq 0$

But then

$(U^2 + V^2)^2 - 4U\,(U^2 + V^2) + 8U^2 - 8U^2 - 8U \geq 0$

Whenever U $\leq$ 0 or U $\geq$ 2. Consequently, for m = 0,

/S(u,v)/ $\leq$1 whenever U $\leq$0 or U $\geq$ 2

Hence the RAS for the integrator

= {u+iv:u □ QUOTE □ □□□ 0 or u □ QUOTE □ □□□ 2, i = □ QUOTE □ □□□ }

for the entire left-half of the U-V complex plane.

To make clearer the RAS, we employ polar co-ordinates (R,$\emptyset$) by setting u = RCos$\emptyset$ and v = R Sin $\emptyset$. The result is the Jordan curve.

$R^3 - 4R^2Cos\emptyset + 8Rcos^2\emptyset - 8Cos\emptyset = 0$

Whose exterior

{(R, $\emptyset$): $R^3 - 4R^2Cos\emptyset + 8RCos^2\emptyset - 8Cos\emptyset > 0$}

is the RAS of the integrator.

The Extremities of the Jordan curve in the (R, $\emptyset$) plane are (0, 90°) and R(2,0°). These correspond to the origin (0,0) and the point (2,0) in the U-V complex plane. The region of instability (RIS) for this case where m = 0 is therefore given by

{(R, $\emptyset$): $R^3 - 4R^2Cos\emptyset + 8RCos^2\emptyset - 8Cos\emptyset < 0$}

Case 2: m = 1

$$S(\bar{h}) = \frac{6+2\bar{h}}{6-4\bar{h}+\bar{h}^2} = \frac{(6+2u)+i2v}{(6-4u+u^2-v^2)+i(2uv-4v)}$$

By similar analysis, we obtain

$(u^2 + v^2)^2 - 8u(u^2+ v^2) + 24u^2 – 72u \geq 0$ whenever

$U \leq 0$ or $U \geq 6$.

Consequently, for m = 1,

$/\bar{S}(u, v)/$ whenever u<0

Hence the RAS for the integrator = {u+iv: $u \leq 0$ or $u \geq 6$, $i = \sqrt{-1}$ } for the entire left-half of the U-V complex plane.

The corresponding Jordan curve is given by:

$R^3 – 8R^2Cos\emptyset + 24Rcos^2\emptyset - 72Cos\emptyset = 0$

has its exterior (RAS), the region

$\{(R, \emptyset): R^3 – 8R^2Cos\emptyset + 24RCos^2\emptyset - 72Cos\emptyset > 0\}$

and its interior (RIS), the region

$\{(R, \emptyset): R^3 – 8R^2Cos\emptyset + 24RCos^2\emptyset - 72Cos\emptyset < 0\}$.

The extremities of the Jordan curve are $(0, 90^o)$ and (6,0). These correspond to the origin (0,0) and the point (6,0) in the U-V complex plane.

Relationship

Observe that for m = 0, 1

$\{(R, \emptyset): R^3 – 4R^2Cos\emptyset + 8RCos^2\emptyset - 8Cos\emptyset < 0\}$.

$\subsetneq \{R, \emptyset): R^3 – 8R^2Cos\emptyset + 24Cos^2\emptyset - 72Cos\emptyset < 0\}$ respectively.

Hence $\{(R, \emptyset): R^3 – 8R^2Cos\emptyset + 24Cos^2\emptyset - 72Cos\emptyset > 0\}$ $\subsetneq \{(R, \emptyset): R^3 – 4R^2Cos\emptyset + 8Cos^2\emptyset - 8Cos\emptyset > 0\}$

Hence the RAS of the integrator with m = 1 is a proper subregion of the RAS of the integration with m = 0; a decrease in size as we move from the case m = 0 to the case m = 1.

Case 3: m = 2

$$S(\bar{h}) = \frac{12+6\bar{h}+\bar{h}^2}{12-6\bar{h}+\bar{h}^2}$$

This lead us to having $/S(u,v)/ \leq 1$ if and only if $u^3+uv^2 +12u \leq 0$, that is, $u(u^2+v^2+12) \leq 0$ whenever $u \leq 0$. Conclusively $/S(u,v)/ \leq 1$ whenever $u \leq 0$, In this case, the RAS is simply the region

{u+iv: $u \leq 0$, $i = \sqrt{-1}$ } which is the entire left half of the U-V complex plane

RESULTS AND IMPLEMENTATION

A numerical implementation of this problem is demonstrated by the problem below.

Problem 1

$Y^{(1)}$ $1+y^2$, y(0) = 1.0 $0 \leq x \leq 1$

Theoretical solution in $y_0 = \tan(x+\frac{\pi}{4})$, h = -.05

Table 1 shows the performance of the new integrator over Niekerk and also the performance against the theoretical solution.

Problem 2

$Y^{(1) =\ 1+y\text{-}(2)}$, y(0) = 0, y = tan x

In this case, m = 0 cannot handle this kind of problem as can be seen in the Table 2.

Problem 3

$Y^{1\,=}$ -100(y-Sin x), y(0) = 0.0, $0 \leq x \leq 3$

TSOL

$$Y(x) = \frac{\text{Sin } x – 0.01 \text{ Cos } x + 0.01 \exp(-100x)}{1.0001}$$

In this case, the new integrator with m = 1, 2 copes favourably well compared to Runge-Kutta of order 4 (Table 3).

Conclusion

The integrators are A-stable for the cases m = 0, 1, 2 and hence has the required L-stability properties.

From the analysis in the paper, we have

RAS for m = 2 $\subsetneq$ RAS for m =1 $\subsetneq$ RAS for m = 0.

i. But the denominator of the integrator is fixed, hence we expect that as m increases, then the RAS of the integrator decreases.

ii. The integrators considered has the required A-stability properties and they cover the left half of the complex plane and hence highly recommended for users who are

Table 1. Theoretical solution in $y_0 = \tan(x+\frac{\pi}{4})$, h = -.05.

Errors in [0, 1]	New integration m = 2 (p = 3)	Niekerk I	Niekerk II
Min(e) = 7.897(-6)	0.0	1E-4	2E-6
Max(e) = -6.256(-2)	5E-6	5E-1	2E-2

Table 2. Integrator at m = 0

Error in 0 (x (1	The Integrator	Niekerk I
Min (e)	0.0	0.0
Max (e)	0.0	1.2 E-4

Table 3. Integrator at m = 1 and 2

Error in 0 (x (3	The Integrator		Rk(4)
	M = 1, h = 0.003	M = 2, h = 0.003	
Min e = 1.897(6)	1E-6	2E-6	6.7 E-11
Max e = 6.356(-2)	5E-2	6E-2	1.9 E-3

working in this area of research.

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