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Full Length Research Paper

Quasi-radical operation on the submodules in a module

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All rings are commutative with identity and all modules are unital. The purpose of this paper is to introduce interesting and useful properties of quasi-radical operation on the submodules in a module.

Key words: Prime submodules, qausi-radical operation.

INTRODUCTION

Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let R be a ring and M be a unital R-module. For any submodule N of M,we define $(\mathrm{N}:\mathrm{M}) = \{\mathrm{r} \in \mathrm{R}: \mathrm{rM} \subseteq \mathrm{N}\}$. A submodule N of M is called prime if $N \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in N$, $m \in N$ or $r \in (N:M)$. Let $PSpec(M)$ denote the collection of all prime submodules. Note that some modules have no prime submodules (that is, $PSpec(M)$=∅). In recent years, prime submodules have attracted a good deal of attention (Lu, 1984; John, 1978; James and Patrick, 1992; Shahabaddin, 2004). An R-module M is called a multiplication module provided for each submodule N of M there exists an ideal I of R such that $N = IM$. We say that I is a presentation ideal of N. Let N and K be submodules of a multiplication module M with $N = I_1 M$ and $K = I_2 M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1 I_2 M$. Then, the product of N and K is independent of presentation of N and K (Reza, 2003, Theorem 3.4). Moreover, for $a, b \in M$, by ab we mean the product of Ra and Rb. Let M be a nonzero multiplication R-module. Then, every proper submodule of M is contained in maximal submodule of M (Zeinab and Patrick, 1988, Theorem 2.5). Let N be a submodule of M. Then, the radical of N denoted by $\sqrt{N}$ is defined to be intersection of all prime submodules of M containing N. If N is not contained in any prime submodule of M, then $\sqrt{N}$ =M. Let N be a submodule of a multiplication R-module M. Then

$$\sqrt{N} = \{m \in M: m^k \subseteq N for\ some\ k > 0\},$$

(Reza, 2003, Theorem 3.13).

In this paper, we generalize some properties of quasi-radical operation on the ideals in a ring to quasi-radical operation on the submodules in a module (Magnus, 2004).

Definition 1

Let M be an R-module. An operation F on the submodules of M is a correspondence that to every submodule N in M associates a submodule $F(N)$ in M.

Definition 2

(i) Let M be an R-module. Let F be an operation on the submodules of M, and let N be a submodule in M. We say that $F(N)$ is the F-radical of N.
(ii) Let M be an R-module. We say that N is F-radical if $F(N) = N$. A prime submodule N is called F-prime if it is F-radical.

Definition 3

Let M be an R-module and F an operation on the submodules of M. We define F-prime spectrum of M as:

$$Spec(M) = \{F - prime\ submodules\ N \subset M\}.$$

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Definition 4

Let M be an R-module. Let F be an operation on the submodules in M. We say that M satisfies the ascending chain condition (acc) for F-radical submodules if for every chain $\{N_i\}_{i\in\mathbb{N}}$ of F-radical submodules we have that $N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$ stabilizes.

Relations 1

Let M be an R-module and F be an operation on the submodules of M. It is natural to ask if F satisfies the following relations for any submodules N, K and $\{N_i\}_{i\in I}$ in M:

(a) $N \subseteq F(N)$,
(b) $F(F(N)) = F(N)$,
(c) $F(N \cap K) = F(N) \cap F(K)$,
$(c)^*$ If M is a multiplication R-module then $F(N \cap K) = F(N) \cap F(K) = F(NK)$,
(d) $F(\sum_{i\in I} N_i) = F(\sum_{i\in I} F(N_i))$,
(e) $\sqrt{N} \subseteq F(N)$ if M is a multiplication R-module,
(f) $N \subseteq K$ implies $F(N) \subseteq F(K)$,
(g) $F(\cup_{i\in I} N_i) = F(\cup_{i\in I} F(N_i))$ if $\{N_i\}_{i\in I}$ is ordered family.

Proposition 1

Let M be an R-module. Let F be an operation on the submodules in M. The following assertions hold for $(a)-(g)$ of Relations 1.

1. If F satisfies $(a), (b)$ and (f) then F satisfies (d).
2. If F satisfies (c) then F satisfies (f).
3. Let M be a multiplication R-module. If F satisfies (a) and $(c)^*$ then F satisfies (e).
4. If F satisfies (d) then F satisfies (b).
5. If F satisfies (a) and (d) then F satisfies (f) and (g).
(Let M be a multiplication R-module. The relations: (a), (b) and $(c)^*$ imply the relations (d), (e), (f) and (g)).

Proof

1. We have from (a) that $N_i \subseteq F(N_i)$ for each $i \in I$. It follows that $\sum_{i\in I} N_i \subseteq \sum_{i\in I} F(N_i)$. Consequently, we see by (f) that $F(\sum_{i\in I} N_i) \subseteq F(\sum_{i\in I} F(N_i))$. Conversely, since $N_j \subseteq \sum_{i\in I} N_i$ for each $j \in I$, we have by (f) that $F(N_j) \subseteq F(\sum_{i\in I} N_i)$ for each $j \in I$. Thus since $F(\sum_{i\in I} N_i)$ is an submodule we see that $\sum_{i\in I} F(N_i) \subseteq F(\sum_{i\in I} N_i)$. This implies, again by (f), that $F(\sum_{i\in I} F(N_i)) \subseteq F(F(\sum_{i\in I} N_i))$. Now since from (b) $F(F(N)) = F(N)$ for any submodule N in M we get that $F(\sum_{i\in I} F(N_i)) \subseteq F(\sum_{i\in I} N_i)$. This shows that $F(\sum_{i\in I} N_i) = F(\sum_{i\in I} F(N_i))$, that is (d) holds.
2. Assume (f) is not true. There exist N, K such that $N \subseteq K$ but $F(N) \nsubseteq F(K)$. This implies $F(N \cap K) = F(N) \neq F(N) \cap F(K)$ which contradicts (c). Thus $N \subseteq K$ implies $F(N) \subseteq F(K)$ for any submodules $N, K \subseteq M$, that is (f) holds.
3. From the relation $(c)^*$ we get $F(t^2) = F((t)) \cap F((t)) = F((t))$ for every $t \in M$. By induction on n, we obtain $F(t^n) = F((t))$ for all positive integer n. Let N be a submodule of M and $t \in \sqrt{N}$. Then $t^n \subseteq N$ for some positive integer n. We have that and from relation (a) that $t \in F((t))$. Hence $t \in F(N)$ and we have proved that $\sqrt{N} \subseteq F(N)$, that is (e) holds.
4. If $F(N) \neq F(F(N))$,then $F(\sum_{i\in I} N_i) \neq F(\sum_{i\in I} F(N_i))$ for $I = 1$ and $N_1 = N$ that is we get a contradiction of (d). Thus (b) is satisfied.
5. If $N \subseteq K$ does not imply that $F(N) \subseteq F(K)$ then there exist submodules $N \subseteq K$ in M such that $F(N) \nsubseteq F(K)$. Then $F(K) \subset F(N) + F(K)$ so we have by (a) that $F(N + K) = F(K) \neq F(N) + F(K) \subseteq F(F(N) + F(K))$, which contradicts (d). Thus (f) satisfied. If $\{N_i\}_{i\in I}$ is an ordered family then it is clear that $\cup_{i\in I} N_i = \sum_{i\in I} N_i$ Since from (f) we have that $N \subseteq K$ implies $F(N) \subseteq F(K)$; it follows that $\{F(N_i)\}_{i\in I}$ is an ordered family of submodules as well. This implies that $\cup_{i\in I} F(N_i) = \sum_{i\in I} F(N_i)$. Thus (d), that is $F(\sum_{i\in I} N_i) = F(\sum_{i\in I} F(N_i))$ implies $F(\cup_{i\in I} N_i) = F(\cup_{i\in I} F(N_i))$, that is (g) holds.

Lemma 1

Let M be an R- module. Let N be a prime submodule in M and let F be an operation on the submodules in M satisfying (a) and (f) of Relations 1. The following two conditions are equivalent.

(1). $F(N) = N$
(2). $A \subseteq N$ implies $F(A) \subseteq N$ for each submodule A in M.

Proof

Assume (1) does not hold, that is by (a) we have that $N \subset F(N)$ then condition (2) with $A = N$ does not hold. Thus (2) implies (1). Conversely, assume that (2) does not hold. Then there is a submodule A in M such that $A \subseteq N$ but $F(A) \nsubseteq N$. From (f) we get that $F(A) \subseteq F(N)$. Thus by (a) we see that $N \subset F(N)$ that is condition (1) does not hold. This shows that (1) implies (2).

Definition 5

Let M be a multiplication R-module. A quasi-radical operation F on the submodules in M is defined as an operation on the submodules in M such that for all submodules A and B in M the following conditions hold:

(a) $A \subseteq F(A)$

(b) $F\big(F(A)\big) = F(A)$
(c) $F(A \cap B) = F(A) \cap F(B) = F(AB)$

Remark 1

From Proposition 1 we see that any quasi-radical operation satisfies $(a) - (g)$ of Relations 1.

Proposition 2

Let M be a multiplication R-module. A quasi-radical operation F on the submodules in M satisfies $F(N) = \sqrt{F(N)} = F(\sqrt{N})$ for any submodules $N \subseteq M$.

Proof

It is clear that $\mathrm{F(N)} \subseteq \sqrt{\mathrm{F(N)}}$. Conversely, let $\mathrm{m} \in \sqrt{\mathrm{F(N)}}$. Then $m^n \subseteq F(N)$ for some positive integer n. Therefore $F(m^n) \subseteq F(F(N))$ and so $m \in F((m)) \subseteq F(N)$. Hence, $\sqrt{\mathrm{F(N)}} \subseteq \mathrm{F(N)}$. Thus we have that $F(N) = \sqrt{F(N)}$. Since F is quasi-radical operation it is satisfies Relations 1 (b), (e) and (f). This implies that $F(N) \subseteq F\big(\sqrt{N}\big) \subseteq F\big(F(N)\big) = F(N)$. Thus $F(N) = F\big(\sqrt{N}\big)$ and we have proved the proposition.

Proposition 3

Let M be a multiplication R-module. Let F be a quasi-radical operation on the submodules in M. Then for each submodule A in M the following holds:

$$F(A) = \bigcap_{F(A)\subseteq N,\ N \text{ a prime submodule}} N$$

Proof

We have that:

$$F(A) = \sqrt{F(A)} = \bigcap_{F(A)\subseteq N,\ N \text{ a prime submodule}} N.$$

By Proposition 2, we get first equality. The second equality is clear.

Theorem 1

Let M be a multiplication R-module. Let F be a quasi-radical operation on the submodules in M. If M satisfies the acc for F-radical submodules, then any F-radical submodule is the intersection of a finite number of F-prime submodules.

Proof

Let T be the set of F-radical submodules which are not intersection of a finite number of F-prime submodules. Assume that $T \neq \emptyset$. Then T admits a maximal element N, because the acc for F-radical submodules holds. Then N is F-radical and can not be prime. Take $m \notin N$ and $r \notin (N\!:M)$ such that $rm \in N$, then $N \subset N + Rm$ and $N \subset N + rM$. Since N is maximal in T these two new modules are not in T. From (a) we get $N \subset N + Rm \subseteq F(N + Rm)$and $N \subset N + rM \subseteq F(N + rM)$. Thus the submodules $F(N + Rm)$ and $F(N + rM)$ are F-radical by (b) but are not in T and therefore expressible as a finite intersection of F-prime submodules. By (c) we have:

$$N \subseteq F(N + Rm) \cap F(N + rM) = F\big((N + Rm)(N + rM)\big)$$
$$= F(N^2 + rN + mN + rmM) \subseteq F(N) = N$$

So, $N = F(N + Rm) \cap F(N + rM)$ and thus, a finite intersection of F-prime submodules, which contradicts the assumption that N is in T. Thus $T = \emptyset$.

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