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Full Length Research Paper

Several new iterative methods for solving nonlinear algebraic equations incorporating homotopy perturbation method (HPM)

M. M. Sehati[1]*, S. M. Karbassi[1], M. Heydari[2] and G. B. Loghmani[2]

[1]Department of Mathematics, Islamic Azad University, Yazd Branch, Yazd, Iran.
[2]Faculty of Mathematics, Yazd University, Yazd, Iran.

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In this paper, several new iterative methods for solving nonlinear algebraic equations are presented. The iterative formulas are based on the He's homotopy perturbation method (HPM). It is shown that the new methods lead to eight algorithms which are of fifth, seventh, tenth and fourteenth order convergence. These methods result in real or complex simple roots of certain nonlinear equations. The merit of the new algorithms is that, in case the nonlinear equation have complex roots, it can give complex solutions even when the initial approximation is chosen real. Several examples are presented and compared to other methods, showing the accuracy and fast convergence of the presented method.

Key words: Iterative methods, homotopy perturbation method, nonlinear algebraic equations, efficency index, convergence order.

INTRODUCTION

Finding the roots of nonlinear algebraic equations are common yet an important problem in science and engineering. Analytical methods for solving such equations are difficult or almost non-existent. Therefore, it is only possible to obtain approximate solutions by numerical techniques based on iteration procedures.

Recently, due to the development of various computer software and hardware many iterative methods have been developed to approximate a solution of nonlinear equation

$$f(x) = 0. \tag{1}$$

We present some new modifications in the last nine years for Newton-Raphson method, also we derived some new methods with higher order convergence for solving nonlinear Equations (1) which are presented in this paper.

Iterative methods are based on the idea of successive approximations, that is, starting with one or more initial approximations to the root, we obtain a sequence of approximations or iterates $\{x_k\}_{k=0}^{\infty}$, which in the limit converges to the root. The methods give only one root at a time.

Definition 1

A sequence of iterates $\{x_k\}_{k=0}^{\infty}$ is said to converge to the root α, if $\lim_{k\to\infty} |x_k - \alpha| = 0$ or $\lim_{k\to\infty} x_k = \alpha$. In practice, except in rare cases, it is not possible to find α which satisfies the given equation exactly.

Definition 2 (Order of convergence)

Assume that a sequence of iterates $\{x_k\}_{k=0}^{\infty}$ converges to α and $e_k = x_k - \alpha$ for $k \geq 0$. If two positive constants $M \neq 0$ and $q > 0$ exist, and

*Corresponding author. E-mail: mmsehati@yahoo.com, mehdikarbassi@gmail.com. Tel/Fax: +98 351 8120695.

$$\lim_{k\to\infty}\frac{|e_{k+1}|}{|e_k|^q}=M,$$

Then the sequence is said to converge to α with order of convergence q. The number M is called asymptotic error constant.

In recent years much attention has been given to develop several iterative type methods for solving non-linear equations. Here we present the methods, but we focus on our new methods.

The most popular and widely method for finding zeros of non-linear equations (1) is Newton's method for simple root which is defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},$$

and Newton's method for multiple roots is defined by

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)},$$

where m is the multiplicity of the root. These are important and basic methods, which converge quadratically.

During the last decade, many methods have been developed for solving nonlinear equations. These methods are developed using Taylor series, Adomian decomposition method (Abbasbandy, 2003; Babolian and Biazar, 2002; Jafari and Gejji, 2006), homotopy perturbation method (Golbabai and Javidi, 2007; He, 2003; Javidi, 2009) and other techniques (Basto et al., 2006; Chun, 2006, 2005; Noor, 2007; Noor et al., 2006).

The homotopy perturbation method (HPM), first proposed by He in 1998, was developed and improved by He (1999, 2000, 2003a, b). Very recently, the new interpretation and new development of the homotopy perturbation methods have been given and well addressed by He (2006a, b). The He's homotopy perturbation method is a novel and effective method, which can solve various nonlinear equations.

In this study, a numerical method based on He's homotopy perturbation method is proposed for solving the nonlinear algebraic equations. For this purpose, we use HPM and Taylor's expansion of order two for the equation. We also prove the convergence of the proposed method. Several numerical examples are given to illustrate the performance of the new iterative methods.

DERIVATION OF THE METHODS USING HPM

Consider the nonlinear algebraic equation (Equation 1). We assume that α is a simple root of Equation (1) and λ an initial guess sufficiently close to it. We can rewrite Equation (1) using the Taylor's series as:

$$f(\lambda)+(x-\lambda)f'(\lambda)+\frac{1}{2!}(x-\lambda)^2f''(\lambda)\approx 0. \qquad (2)$$

We construct a homotopy $H: R\times[0,1]\to R$ which satisfies

$$H(x,p)=pf(x)+(1-p)\{f(\lambda)+(x-\lambda)f'(\lambda)+\frac{1}{2!}(x-\lambda)^2f''(\lambda)\}, \qquad (3)$$

where p is an embedding parameter. Hence, it is obvious that

$$H(x,0)=f(\lambda)+(x-\lambda)f'(\lambda)+\frac{1}{2!}(x-\lambda)^2f''(\lambda)\approx 0, \qquad (4)$$

$$H(x,1)=f(x)=0, \qquad (5)$$

and the changing process of p from 0 to 1, refers to $H(x,p)$ from $H(x,0)$ to $H(x,1)$. In topology, this is called deformation, $H(x,0)$ and $H(x,1)$ are call homotopic. Applying the perturbation technique (Nayfeh, 1985), due to the fact that $0\le p\le 1$ can be considered as a small parameter, we can assume that the solution of (3) can be expressed as a series in p

$$x=x_0+x_1p+x_2p^2+x_3p^3+\cdots, \qquad (6)$$

when $p\to 1$, (3) corresponds to (1) and (6) becomes the approximate solution of Equation (1), that is

$$\tilde{x}=\lim_{p\to 1}x=x_0+x_1+x_2+\cdots. \qquad (7)$$

The convergence of the series (7) has been proved by He in his famous paper (He, 1999).

For the application of homotopy perturbation method to Equation (1) we can write (3) by expanding $f(x)$ into a Taylor series around x_0 as follows:

$$p\{f(x_0)+(x-x_0)\frac{f'(x_0)}{1!}+(x-x_0)^2\frac{f''(x_0)}{2!}+\cdots\}$$
$$+(1-p)\{f(\lambda)+(x-\lambda)f'(\lambda)+\frac{1}{2!}(x-\lambda)^2f''(\lambda)\}=0. \qquad (8)$$

Substitution of (6) into (8) yields:

$$p\{f(x_0)+(x_1p+x_2p^2+x_3p^3+\cdots)\frac{f'(x_0)}{1!}+(x_1p+x_2p^2+x_3p^3+\cdots)^2\frac{f''(x_0)}{2!}+\cdots\}$$
$$+(1-p)\{f(\lambda)+(x_0+x_1p+x_2p^2+x_3p^3+\cdots-\lambda)f'(\lambda)$$
$$+\frac{1}{2!}(x_0+x_1p+x_2p^2+x_3p^3+\cdots-\lambda)^2f''(\lambda)\}=0. \qquad (9)$$

By equating the terms with identical powers of p, we have:

$$p^0: f(\lambda)+(x_0-\lambda)f'(\lambda)+\frac{1}{2!}(x_0-\lambda)^2 f''(\lambda)=0, \quad (10)$$

$$p^1: f(x_0)+x_1 f'(\lambda)+x_1(x_0-\lambda)f''(\lambda)-f(\lambda)-(x_0-\lambda)f'(\lambda)-\frac{1}{2!}(x_0-\lambda)^2 f''(\lambda)=0, \quad (11)$$

$$p^2: x_1 f'(x_0)+x_2 f'(\lambda)+\frac{1}{2}x_1^2 f''(\lambda)+x_2(x_0-\lambda)f''(\lambda)-x_1 f'(\lambda)-x_1(x_0-\lambda)f''(\lambda)=0. \quad (12)$$

By solving Equation (10), we obtain

$$x_0=\lambda+\frac{-f'(\lambda)+\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}{f''(\lambda)}, \quad (13)$$

where $\beta=\pm 1$. So, from Equation (11) and (12) we have:

$$x_1=\frac{-f(x_0)}{\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}, \quad (14)$$

$$x_2=\frac{x_1\{f'(\lambda)-f'(x_0)+(x_0-\lambda-\frac{1}{2}x_1)f''(\lambda)\}}{\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}. \quad (15)$$

By substituting (13), (14) and (15) into (7), we can obtain the zero of Equation (1) as follows:

$$\tilde{x}=\lambda+\frac{-f'(\lambda)+\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}{f''(\lambda)}+\frac{-f(x_0)}{\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}$$
$$+\frac{x_1\{f'(\lambda)-f'(x_0)+(x_0-\lambda-\frac{1}{2}x_1)f''(\lambda)\}}{\beta\sqrt{f'^2(\lambda)-2f(\lambda)f''(\lambda)}}+\cdots. \quad (16)$$

Now by substituting $\beta=1$, these formulations allow us to suggest the following iterative methods for solving the nonlinear algebric Equation (1).

Algorithm 1

For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme:

$$\begin{cases} y_n=x_n+\dfrac{-f'(x_n)+\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}{f''(x_n)}, \\ x_{n+1}=y_n-\dfrac{f(y_n)}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}. \end{cases} \quad (17)$$

Algorithm 2

For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme:

$$\begin{cases} y_n=x_n+\dfrac{-f'(x_n)+\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}{f''(x_n)}, \\ z_n=-\dfrac{f(y_n)}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}, \\ x_{n+1}=y_n+z_n+\dfrac{z_n\{f'(x_n)-f'(y_n)+(y_n-x_n-\frac{1}{2}z_n)f''(x_n)\}}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}. \end{cases} \quad (18)$$

Algorithm 3

For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme:

$$\begin{cases} y_n=x_n+\dfrac{-f'(x_n)+\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}{f''(x_n)}, \\ z_n=y_n-\dfrac{f(y_n)}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}, \\ x_{n+1}=z_n-\dfrac{f(z_n)}{f'(z_n)}. \end{cases} \quad (19)$$

Algorithm 4

For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme:

$$\begin{cases} y_n=x_n+\dfrac{-f'(x_n)+\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}{f''(x_n)}, \\ z_n=-\dfrac{f(y_n)}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}, \\ w_n=y_n+z_n+\dfrac{z_n\{f'(x_n)-f'(y_n)+(y_n-x_n-\frac{1}{2}z_n)f''(x_n)\}}{\sqrt{f'^2(x_n)-2f(x_n)f''(x_n)}}, \\ x_{n+1}=w_n-\dfrac{f(w_n)}{f'(w_n)}. \end{cases} \quad (20)$$

It is clear that formulations to obtain higher order convergence can be continued in the same manner but the formulas get more and more complicated. Obviously, if the second derivative is not known exactly, it may be replaced by its numerical approximations. In this case more initial points are required. Indeed this would effect the order of convergence, as in the case of secant method compared with the Newton-Raphson method. The criteria assumed for the initial guess is once again same as the case of Newton-Raphson method.

CONVERGENCE ANALYSIS

In this section, the convergence of Algorithms 1 - 4 is considered. The approach is similar to that of (Javidi, 2009).

Definition 1

Let $e_n = x_n - \alpha$ be the truncation error in the nth iterate. If there exists a number $k \geq 1$ and a constant $\eta \neq 0$ such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^k} = \eta \tag{21}$$

Then k is called the order of convergence of the method (Gautschi, 1997).

Theorem 2

Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Equation (17), the convergence is at least of order five.

Proof

Let α be a simple zero of $f(x)$. Since, $f(x)$ is sufficiently differentiable, by expanding $f(x_n), f'(x_n)$ and $f''(x_n)$ around α, we get

$$\begin{aligned} f(x_n) &= f'(\alpha)[e_n + d_2e_n^2 + d_3e_n^3 + d_4e_n^4 + d_5e_n^5 + \cdots], \\ f'(x_n) &= f'(\alpha)[1 + 2d_2e_n + 3d_3e_n^2 + 4d_4e_n^3 + 5d_5e_n^4 + 6d_6e_n^5 + \cdots], \\ f''(x_n) &= f'(\alpha)[2d_2 + 6d_3e_n + 12d_4e_n^2 + 20d_5e_n^3 + 30d_6e_n^4 + 42d_7e_n^5 + \cdots], \end{aligned} \tag{22}$$

where $d_n = \frac{1}{n!}\frac{f^{(n)}(\alpha)}{f'(\alpha)}$, $n = 1,2,3,\cdots$ and $e_n = x_n - \alpha$.

Now from (22) and $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$, we have

$$\begin{aligned} \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)} &= f'(\alpha)\sqrt{1 - [6d_3e_n^2 + (4d_2d_3 + 16d_4)e_n^3 + \cdots]} \\ &= 1 - \frac{1}{2}\{6d_3e_n^2 + (4d_2d_3 + 16d_4)e_n^3 + \cdots\} - \frac{1}{8}\{6d_3e_n^2 + (4d_2d_3 + 16d_4)e_n^3 + \cdots\}^2 + \cdots. \end{aligned} \tag{23}$$

Also from (17), (22) and (23) we have

$$y_n - \alpha = -d_3e_n^3 - 3d_4e_n^4 - (6d_5 + 3d_3^2)e_n^5 - (10d_6 + 17d_3d_4 + d_2d_3^2)e_n^6 + \cdots. \tag{24}$$

Expanding $f(y_n)$ about α and using (24), we get

$$\begin{aligned} f(y_n) &= f(\alpha) + (y_n - \alpha)f'(\alpha) + \frac{1}{2!}(y_n - \alpha)^2 f''(\alpha) + \cdots \\ &= f'(\alpha)\{-d_3e_n^3 - 3d_4e_n^4 - (6d_5 + 3d_3^2)e_n^5 - (10d_6 + 17d_3d_4)e_n^6 + \cdots\}. \end{aligned} \tag{25}$$

From (23) and (25) we obtain

$$\begin{aligned} \frac{f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}} &= -d_3e_n^3 - 3d_4e_n^4 - (6d_3^2 + 6d_5)e_n^5 \\ &- (34d_3d_4 + 2d_2d_3^2 + 10d_6)e_n^6 + \cdots, \end{aligned} \tag{26}$$

So, from (17), (24) and (26) we obtain

$$e_{n+1} = 3d_3^2e_n^5 + (d_2d_3^2 + 17d_3d_4)e_n^6 + O(e_n^7), \tag{27}$$

which shows that Algorithm 1 is at least a fifth-order convergent method; the required result.

Theorem 3

Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Equation (18), the convergence is at least of order seven.

Proof

Expanding $f'(y_n)$ about α and using (24), we get

$$\begin{aligned} f'(y_n) &= f'(\alpha) + (y_n - \alpha)f''(\alpha) + \frac{1}{2!}(y_n - \alpha)^2 f'''(\alpha) + \cdot \\ &= f'(\alpha)\{1 - 2d_2d_3e_n^3 - 6d_2d_4e_n^4 - (12d_2d_5 + 6d_2d_3^2)e_n^5 + \cdots\}. \end{aligned} \tag{28}$$

From (26) we have

$$z_n = d_3e_n^3 + 3d_4e_n^4 + (6d_3^2 + 6d_5)e_n^5 + (34d_3d_4 + 2d_2d_3^2 + 10d_6)e_n^6 + \cdots. \tag{29}$$

Also from (22) and (23) we get

$$y_n - x_n = -e_n - d_3e_n^3 - 3d_4e_n^4 - (6d_5 + 3d_3{}^2)e_n^5 + \cdots. \tag{30}$$

So, form (18), (23), (24), (28), (29) and (30) we obtain

$$e_{n+1} = -12d_3^3e_n^7 - 9d_3^2(d_2d_3 + 11d_4)e_n^8 + O(e_n^9), \quad (31)$$

which shows that Algorithm 2 is at least of a seventh-order convergent method, the required result.

Theorem 4

Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Equation (19), the convergence is at least of order ten.

Proof

From (19), (24) and (26) we obtain

$$z_n - \alpha = 3d_3^2e_n^5 + (d_2d_3^2 + 17d_3d_4)e_n^6 + (6d_2d_3d_4 + 24d_4^2 + 33d_3d_5 + 24d_3^3)e_n^7 + \cdots. \quad (32)$$

By expanding $f(z_n)$ and $f'(z_n)$ about α and using (32), we get

$$f(z_n) = f(\alpha) + (z_n - \alpha)f'(\alpha) + \frac{1}{2!}(z_n - \alpha)^2 f''(\alpha) + \cdots \quad (33)$$

$$f'(z_n) = f'(\alpha) + (z_n - \alpha)f''(\alpha) + \frac{1}{2!}(z_n - \alpha)^2 f'''(\alpha) + \cdots$$
$$= f'(\alpha)\{1 + 6d_2d_3^2e_n^5 + 2d_2d_3(d_2d_3 + 17d_4)e_n^6 + \cdots\}. \quad (34)$$

So, from (19), (32), (33) and (34) we obtain

$$e_{n+1} = 9d_2d_3^4e_n^{10} + 6d_3^3d_2(d_2d_3 + 17d_4)e_n^{11} + O(e_n^{12}), \quad (35)$$

which shows that Algorithm 3 is at least a tenth-order convergent method; the required result.

Theorem 5

Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Equation (20), the convergence is at least of order fourteen.

Proof

From (20), (23), (24), (28), (29) and (30) we get

$$w_n - \alpha = -12d_3^3e_n^7 - 9d_3^2(11d_4 + d_2d_3)e_n^8$$
$$- d_3(164d_3^3 + 78d_2d_3d_4 + 190d_3d_5 + 2d_2^2d_3^2 + 271d_4^2)e_n^9 + \cdots. \quad (36)$$

By expanding $f(w_n)$ and $f'(w_n)$ about α and using (36), we get

$$f(w_n) = f(\alpha) + (w_n - \alpha)f'(\alpha) + \frac{1}{2!}(w_n - \alpha)^2 f''(\alpha) + \cdots$$
$$= f'(\alpha)\{-12d_3^3e_n^7 - 9d_3^2(d_2d_3 + 11d_4)e_n^8 + \cdots\}, \quad (37)$$

$$f'(w_n) = f'(\alpha) + (w_n - \alpha)f''(\alpha) + \frac{1}{2!}(w_n - \alpha)^2 f'''(\alpha) + \cdots$$
$$= f'(\alpha)\{1 - 24d_2d_3^3e_n^7 - 18d_2d_3^2(11d_4 + d_2d_3)e_n^8 + \cdots\}. \quad (38)$$

So, from (20), (36), (37) and (38) we obtain

$$e_{n+1} = 144d_2d_3^6e_n^{14} + 216d_2d_3^5(d_2d_3 + 11d_4)e_n^{15} + O(e_n^{16}), \quad (39)$$

which shows that Algorithm 4 is at least a fourteenth-order convergent method; once again the required result.

Remark 1

In case $\beta = -1$ is chosen, similar Algorithms to 1 - 4 are obtained, but for the sake of clarity are denoted by Algoritms 5 - 8 in this paper.

Remark 2

We consider the definition of efficiency index (Gautschi, 1997) as $I = p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we assume that all the evaluations have the same cost as function one, then the methods (17), (18), (19) and (20) have the efficiency indices $I = \sqrt[4]{5} \approx 1.495$, $I = \sqrt[5]{7} \approx 1.476$, $I = \sqrt[6]{10} \approx 1.468$ and $I = \sqrt[7]{14} \approx 1.458$, respectively, which are just better than $I = \sqrt[2]{2} \approx 1.414$ of Newton's method.

NUMERICAL EXAMPLES

In this section, Algorithms 1 to 4 (AL 1-AL 4) and Algorithms 5 to 8 (AL 5-AL 8) are employed to solve some nonlinear algebraic equations and are compared with Newton's method (NM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

Table 1. Numerical results for $f_1(x) = x^3 + 4x^2 - 10$.

Method	x_0	IT	x_n	$\lvert f(x_n)\rvert$	COC
NM	1	6	1.36523001341409	3.98e−43	2.0000000
NR2	1	4	1.36523001341409	2.36e−60	3.0000001
CM	1	4	1.36523001341409	7.42e−96	3.9999999
AL 1	1	3	1.36523001341409	6.59e−115	4.9999997
AL 2	1	2	1.36523001341409	1.22e−41	6.9971385
AL 3	1	2	1.36523001341409	9.50e−90	9.9999584
AL 4	1	2	1.36523001341409	1.55e−167	13.9973331
AL 5	1	5	−2.68261500670704+0.35825935992404i	6.43e−65	4.9999891
AL 6	1	4	−2.68261500670704+0.35825935992404i	1.28e−25	6.9180714
AL 7	1	3	−2.68261500670704+0.35825935992404i	1.26e−33	10.1389082
AL 8	1	3	−2.68261500670704+0.35825935992404i	3.50e−63	14.1021150

Noor's method (Noor et al., 2006) (NR2)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = -\frac{(y_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{(y_n + z_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)},$$

and Chun's method (Chun, 2006) (CM)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - 2\frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'(x_n)^2}.$$

All computations are done by Maple 13 with 1500 digits precision. We used $\varepsilon = 10^{-25}$. The following stopping criteria were used in computer programs:

$$(i)\ \ |x_{n+1} - x_n| < \varepsilon \quad and \quad (ii)\ \ |f(x_{n+1})| < \varepsilon.$$

Throughout this section we have used the following test functions (Abbasbandy, 2003; Babolian, and Biazar, 2002; Bi et al., 2009a, b; Chun, 2006; Golbabai and Javidi, 2007, 2009; Jisheng et al., 2007a, b; Noor et al., 2006):

$$f_1(x) = x^3 + 4x^2 - 10,$$
$$f_2(x) = x^4 + x^2 + 1,$$
$$f_3(x) = e^x - 3x^2,$$
$$f_4(x) = \sin^2(x) - x^2 + 1,$$
$$f_5(x) = e^{-x} + \cos(x)\ ,$$
$$f_6(x) = e^{-x^2+x+2} - 1\ ,$$

The number of iterations to approximate the zero (IT), the absolute values of the function ($|f(x_n)|$) and the computational order of convergence (COC) are also shown in Tables 1 to 6. Here, COC is defined by

$$\rho \approx \frac{\ln|(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}. \quad (40)$$

The test results in Tables 1 to 6 show that for most of the functions we tested, the algorithms introduced in the present presentation for numerical tests have equal or better performance compared to the other methods; besides these algorithms have the ability to calculate the complex and real zeroes of nonlinear algebraic equations with real initial approximations.

In Table 1, the methods NM, NR2, CM and the algorithms 1 to 4 all produce the real root with the same initial approximation; however, the algorithms 5 to 8 result in the complex root with the same initial approximation.

In Table 2, the methods NM, NR2 and CM with the same real starting approximation are divergent, but Algorithms 1 to 4 give one complex root while Algorithms 5 to 8 produce the other complex roots of the equation.

In Table 3, Algorithms 1 to 4 give a different root from Algorithms 5 to 8 with the same initial approximation.

In Table 4, Algorithms 1 to 4 give a different root from Algorithms 5 to 8 with the same initial approximation. In Table 5, Algorithms 1 to 4 give a different root from Algorithms 5 to 8 with the same initial approximation.

Table 6 shows that the methods NM, NR2 and CM fail in the first iteration, while the new algorithms produce two

Table 2. Numerical results for $f_2(x) = x^4 + x^2 + 1.$

Method	x_0	IT	x_n	$\vert f(x_n)\vert$	COC
NM	-1	4	Divergent	---	---
NR2	-1	4	Divergent	---	---
CM	-1	4	Divergent	---	---
AL 1	-1	4	−0.500000000000000+0.86602540378443i	4.62e-58	5.0002958
AL 2	-1	4	−0.500000000000000+0.86602540378443i	5.77e−170	6.9999858
AL 3	-1	3	−0.500000000000000+0.86602540378443i	8.66e−74	9.9816452
AL 4	-1	3	−0.500000000000000+0.86602540378443i	3.30e−166	13.9992665
AL 5	-1	6	+0.500000000000000+0.86602540378443i	1.62e−42	4.9990634
AL 6	-1	5	+0.500000000000000+0.86602540378443i	1.21e−25	6.9414508
AL 7	-1	4	+0.500000000000000−0.86602540378443i	7.52e−28	9.6534574
AL 8	-1	4	+0.500000000000000−0.86602540378443i	9.03e−38	13.3464835

Table 3. Numerical results for $f_3(x) = e^x - 3x^2.$

Method	x_0	IT	x_n	$\vert f(x_n)\vert$	COC
NM	0.5	6	0.91000757248871	3.91e−29	2.0000000
NR2	0.5	4	0.91000757248871	6.12e−26	3.0016218
CM	0.5	4	0.91000757248871	1.67e−29	3.9936715
AL 1	0.5	3	−0.45896226753695	7.61e−66	5.0004537
AL 2	0.5	3	−0.45896226753695	4.16e−163	6.9999450
AL 3	0.5	2	−0.45896226753695	1.12e−46	10.2982094
AL 4	0.5	2	−0.45896226753695	5.82e−86	14.2875589
AL 5	0.5	3	0.91000757248871	1.98e−95	4.9999858
AL 6	0.5	2	0.91000757248871	7.14e−35	6.8776326
AL 7	0.5	2	0.91000757248871	1.91e−75	9.9013972
AL 8	0.5	2	0.91000757248871	3.52e−139	13.8856974

Table 4. Numerical results for $f_4(x) = \sin^2(x) - x^2 + 1.$

Method	x_0	IT	x_n	$\vert f(x_n)\vert$	COC
NM	2	6	1.40449164821534	2.26e−32	2.0000000
NR2	2	4	1.40449164821534	8.64e−63	3.0000003
CM	2	4	1.40449164821534	1.45e−91	3.9999998
AL 1	2	4	−1.40449164821534	1.77e−32	5.0598509
AL 2	2	4	−1.40449164821534	1.71e−165	6.9997716
AL 3	2	4	−1.40449164821534	3.11e−234	9.9963148
AL 4	2	3	−1.40449164821534	5.80e−159	13.8973368
AL 5	2	3	1.40449164821534	4.45e−73	4.9990932
AL 6	2	2	1.40449164821534	9.18e−26	7.1089119
AL 7	2	2	1.40449164821534	4.38e−55	10.1005348
AL 8	2	2	1.40449164821534	1.85e−99	14.1019285

Table 5. Numerical results for $f_5(x) = e^{-x} + \cos(x)$.

Method	x_0	IT	x_n	$\lvert f(x_n) \rvert$	COC
NM	2.5	6	1.74613953040801	1.59e−47	2.0000000
NR2	2.5	4	1.74613953040801	3.94e−30	3.0000104
CM	2.5	4	1.74613953040801	7.75e−90	3.9999988
AL 1	2.5	3	4.70332375945224	3.20e−53	4.9999823
AL 2	2.5	3	4.70332375945224	4.85e−135	6.9999998
AL 3	2.5	2	4.70332375945224	1.85e−65	9.1684262
AL 4	2.5	2	4.70332375945224	1.40e−105	13.0055833
AL 5	2.5	3	1.74613953040801	1.93e−68	4.9999904
AL 6	2.5	3	1.74613953040801	8.32e−170	6.9999983
AL 7	2.5	2	1.74613953040801	5.26e−60	9.8635147
AL 8	2.5	2	1.74613953040801	3.23e−105	13.8324853

Table 6. Numerical results for $f_6(x) = e^{-x^2+x+2} - 1$.

Method	x_0	IT	x_n	$\lvert f(x_n) \rvert$	COC
NM	0.5	1	Fail	---	---
NR2	0.5	1	Fail	---	---
CM	0.5	1	Fail	---	---
AL 1	0.5	4	−1.00000000000000	5.09e−117	4.9999990
AL 2	0.5	3	−1.00000000000000	1.51e−34	6.6457287
AL 3	0.5	3	−1.00000000000000	4.54e−103	9.9740136
AL 4	0.5	3	−1.00000000000000	3.21e−219	13.9862945
AL 5	0.5	4	2.00000000000000	5.09e−117	4.9999990
AL 6	0.5	3	2.00000000000000	1.51e−34	6.6457287
AL 7	0.5	3	2.00000000000000	4.54e−103	9.9740136
AL 8	0.5	3	2.00000000000000	3.21e−219	13.9862945

different roots of the equation.

Conclusions

The present paper suggests several new algorithms for solving nonlinear algebraic equations which have the fifth, seventh, tenth and fourteenth order convergence. Examples given in this study do not only demonstrate the comparison of our results with those of Newton's method (NM), Noor's method (NR2) and Chun's method (CM) but also show the merits of the new algorithms. These new algorithms did not fail or became divergent in any of the examples while resulted in different roots (real or complex) with the same real initial approximations. Obviously, these algorithms are more complicated than most of the existing methods, but instead they can produce different simple roots; even complex roots of nonlinear algebraic equations with the same real initial approximations. The only disadvantage of these algorithms lies in the complexity of the formulas and also in case the second derivative is not known exactly.

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