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*Full Length Research Paper*

# Homotopy perturbation method for solving a system of third-order boundary value problems

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**In this paper, we use the homotopy perturbation method for solving a system of third-order boundary value problem. We show that numerical results obtained by this method are highly accurate. Some examples are given to illustrate the efficiency and implementation of the homotopy perturbation method.**

**Key words:** Variation iteration method, homotopy perturbation method, obstacle problem, third-order boundary value problem.

## INTRODUCTION

In recent years, much attention have been given to study the system of boundary value problems, which are associated with obstacle, unilateral, contact and moving boundary value problems. Several numerical methods are being developed to solve these problems due to their importance. Motivated and inspired by the research activities in these fields, we consider the following system of third-order nonlinear boundary value problem of the type:

$$u'''(x)=\begin{cases} f(x,u(x))+u(x)g(x)+r, & a\le x<c,\\ f(x,u(x)), & c\le x<d,\\ f(x,u(x))+u(x)g(x)+r, & d\le x\le b,\end{cases} \tag{1}$$

with the boundary conditions:

$$u(a)=\beta_0,\quad u'(a)=u'(b)=\beta_1, \tag{2}$$

and the continuity conditions of $u(x)$, $u'(x)$ and $u''(x)$ at $c$ and *d*. Here $\beta_0$ and $\beta_1$ are finite real constants and $f(x,u(x))\,f(u)$, $g(x)$ are continuous functions on $[a,\ b]$. Systems of the type (1) arise in the mathematical modeling of contact, obstacle, unilateral, moving and free boundary value problems and have important applications in pure and applied sciences (Noor, 1998, 2000, 2004, 2009; and Noor et al., 1993, 1994, 2003, 2010, 2010a, 2011, 2011a). It is well known that these obstacle problems can be studied in the general framework of variational inequalities. The equivalent variational inequalities approach is used to study the existence of the solution of these obstacle boundary value problems. We would like to point out that these numerical methods can not be extended and applied for solving these obstacle problems. However, one can use the penalty function technique of Lewy et al. (1969) to characterize these obstacle problems as system of variational equations. In the case of the known obstacle, this sytem of variational equations can be written as a system of boundary value of the type (1). Al-Said (2000, 2002), Al-Said et al. (1994, 1998, 2003), Gao et al. (2006), Geng et al. (2010), Noor et al. (2003, 1994) and Islam et al. (2005) have considered the systems of boundary value problems by using various numerical and analytical techniques. One of these techniques is called the homotopy perturbation technique (He, 1999, 2000, 2006, 2006a; Moyud-Din et al., 2009). The homotopy perturbation method can be considered as an effective and reliable analytical method for solving a wide class of initial and boundary value problems arising in various

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branches of pure and applied sciences. The homotopy perturbation method is preferred over Adomian decomposition method in Wazwaz (2002). To implement the Adomian method, one has to calculate the Adomian polynomials which are difficult to find. In this paper, we use the homotopy perturbation methods for solving a system of third-order boundary value problems of the type (1). Some examples are given to illustrate the implementation and efficiency of this method.

## HOMOTOPY PERTURBATION METHOD

To illustrate the basic concept of this technique, we consider the following general nonlinear differential equation:

$$A(u(x)) - f(x) = 0, \; x \in \Omega, \tag{3}$$

with the boundary conditions:

$$B\left(u(x), \frac{\partial u(x)}{\partial n}\right) = 0, \; x \in \Gamma, \tag{4}$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(x)$ is a known analytic function, and $\Gamma$ is the boundary of the domain $\Omega$. The operator $A$ can be divided into two parts, $L$ and $N$, where $L$ is a linear and $N$ is a nonlinear operator. Equation (3) can be rewritten as follows:

$$L(u(x)) + N(u(x)) - f(x) = 0. \tag{5}$$

By the homotopy technique of He (1999, 2000, 2006, 2006a), we can construct a homotopy $H(v(x), p): \Omega \text{ x } [0,1] \rightarrow \mathrm{R}$, which satisfies:

$$H(v(x), p) = (1-p)[L(v(x)) - L(u_0(x))] + p[L(v(x)) + N(v(x) - f(x)] = 0, \tag{6}$$

or equivalently,

$$H(v(x), p) = L(v(x)) - L(v(x)) + p[L(v(x)) + N(v(x)) - f(x)] = 0, \tag{7}$$

where $p \in [0, 1]$ is an embedding parameter, $u_0$ is an initial approximation of Equation (3), which satisfies the boundary conditions (4). Obviously, from Equation (7), we have:

$$H(v(x),0) = L(v(x)) - L(u_0(x)) = 0,$$
$$H(v(x),1) = L(v(x)) + N(v(x)) - f(x) = 0.$$

Changing the process of $p$ from zero to unity is just that change of $H(v(x), p)$ from $L(v(x)) - L(u_0(x)) = 0$ to $L(v(x)) + N(v(x)) - f(x)$. this is called homotopy and $L(v(x)) - L(u_0(x))$ and $L(v(x)) + N(v(x)) - f(x)$ are called homotopic. According to the homotopy perturbation method, the solution of Equation (6) and (7) can be written in the form of a power series in *p:*

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + ... \tag{8}$$

Setting $p = 1$, results in the approximate solution of Equation (5):

$$u(x) = \lim_{p \to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + ... \tag{9}$$

The convergence of the homotopy perturbation method is given in Biazar et al. (2009).

## NUMERICAL RESULTS

Here, we give two examples of a system of third-order linear boundary value problems to illustrate the implementation and efficiency of the homotopy perturbation method.

### Example 1

We consider the system of third-order boundary value problem (1) and (2) by taking $f(x, u(x)) = 0, \; g(x) = 1, \; r = -1,$ and in this case the system of third- order boundary value problem (1) reduces to:

$$u'''(x) = \begin{cases} u - 1, & 0 \le x < \frac{1}{4}, \\ 0, & \frac{1}{4} \le x < \frac{3}{4}, \\ u - 1, & \frac{3}{4} \le x \le 1, \end{cases} \tag{10}$$

with the boundary conditions

$$u(0) = 0, \;\; u'(0) = u'(1) = 0, \tag{11}$$

and the continuity condition of $u(x)$, $u'(x)$ and $u''(x)$ at $\frac{1}{4}$ and $\frac{3}{4}$. The exact solution of the system of third-order boundary value problem (10) is:

$$u(x) = \begin{cases} 1 + C_1 e^x + e^{\frac{-x}{2}}[C_2 \cos(\frac{\sqrt{3}}{2}x) + C_3 \sin(\frac{\sqrt{3}}{2}x)], & 0 \le x < \frac{1}{4}, \\ C_4 + C_5 x + C_6 \frac{x^2}{2}, & \frac{1}{4} \le x < \frac{3}{4}, \\ 1 + C_7 e^x + e^{\frac{-x}{2}}[C_8 \cos(\frac{\sqrt{3}}{2}x) + C_9 \sin(\frac{\sqrt{3}}{2}x)],, & \frac{3}{4} \le x \le 1. \end{cases} \tag{12}$$

where:

$$C_1 = \text{-}0.250371, \;\; C_2 = \text{-}0.749629, \;\; C_3 = \text{-}0.143696,$$

$C_4 = \text{-}0.000627,\ C_5 = 0.031161,\ C_6 = \text{-}0.002595,$
$C_7 = \text{-}0.149478,\ C_8 = \text{-}0.784090,\ C_9 = \text{-}0.562439.$

We now solve the system of third-order boundary value problems (10) by the homotopy perturbation method by constructing homotopy equations in Cases I and II following:

***Case I***

We now consider:

$$u'''(x) = u - 1, \qquad 0 \le x < \frac{1}{4},\ \frac{3}{4} \le x \le 1. \tag{13}$$

We construct the homotopy equation for Equation (13) as:

$$L(v(x)) - L(u_0(x)) + p[L(u_0(x)) - v(x) + 1] = 0, \tag{14}$$

where:

$$L = \frac{d^3}{dx^3}. \tag{15}$$

Suppose the solution of Equation (10) has the form as defined in (8), substituting (8) into (14) and comparing the terms with identical powers of *p*, leads to:

$$\begin{aligned}
&p^0 : v_0'''(x) - u_0'''(x) = 0,\\
&p^1 : v_1'''(x) = -v_0'''(x) + v_0(x) - 1,\\
&p^2 : v_2'''(x) = v_1(x),\\
&p^3 : v_3'''(x) = v_2(x),\\
&\vdots\\
&p^n : v_n'''(x) = v_{n-1}(x),\ n = 2,3,4,\ldots
\end{aligned} \tag{16}$$

For the domain $0 \le x < \frac{1}{4}$, we assume $v_0(x) = u_0(x) = a_0 \frac{x^2}{2!} + a_1 x + a_2$, as an initial approximation to the solution of Equation (10) and using this initial approximation into the recurrence relation (16), we obtain the following relation:

$$\begin{aligned}
&v_0(x) = a_0 \frac{x^2}{2!} + a_1 x + a_2,\\
&v_1(x) = a_0 \frac{x^5}{5!} + a_1 \frac{x^4}{4!} + a_2 \frac{x^3}{3!} - \frac{x^3}{3!},\\
&v_2(x) = a_0 \frac{x^8}{8!} + a_1 \frac{x^7}{7!} + a_2 \frac{x^6}{6!} - \frac{x^6}{6!},\\
&v_3(x) = a_0 \frac{x^{11}}{11!} + a_1 \frac{x^{10}}{10!} + a_2 \frac{x^9}{9!} - \frac{x^9}{9!},\\
&\vdots
\end{aligned} \tag{17}$$

Using (24) in (11), we obtain the approximate solution of Equation (13) for this domain as:

$$\begin{aligned}
u(x) = \sum_{j=0}^{\infty} v_j = a_0\left\{\frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots\right\} + a_1\left\{x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots\right\} + a_2\left\{1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots\right\}\\
- \left\{\frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots\right\}, \qquad 0 \le x < \frac{1}{4},
\end{aligned} \tag{18}$$

For the domain $\frac{3}{4} \le x \le 1$, we assume $v_0(x) = u_0(x) = a_6 \frac{x^2}{2!} + a_7 x + a_8$, as an initial approximation to the solution of Equation (10) and using this initial approximation into the recurrence relation (17), we obtain the following relation:

$$\begin{aligned}
u(x) = \sum_{j=0}^{\infty} v_j = a_6\left\{\frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots\right\} + a_7\left\{x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots\right\} + a_8\left\{1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots\right\}\\
- \left\{\frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots\right\}, \qquad \frac{3}{4} \le x \le 1.
\end{aligned} \tag{19}$$

***Case II***

In this case, we consider:

$$u'''(x) = 0, \qquad \frac{1}{4} \le x < \frac{3}{4}. \tag{20}$$

We construct the homotopy equation for Equation (20) as:

$$L(v(x)) - L(u_0(x)) + pL(u_0(x)) = 0, \tag{21}$$

where the operator *L* is defined in (15).

$$\begin{aligned}
&p^0 : v_0'''(x) - u_0'''(x) = 0,\\
&p^1 : v_1'''(x) = u_0'''(x),\\
&p^2 : v_2'''(x) = 0,\\
&p^3 : v_3'''(x) = 0,\\
&\vdots\\
&p^n : v_n'''(x) = 0,\ n = 2,3,4,\ldots
\end{aligned}$$

**Table 1.** Numerical results for Example 1.

| ***x*** | **Absolute error** |
|---|---|
| 0 | 0 |
| 0.1 | 1.31269E-37 |
| 0.2 | 5.25136E-37 |
| 0.3 | 1.18191E-36 |
| 0.4 | 2.10189E-36 |
| 0.5 | 3.28508E-36 |
| 0.6 | 4.73149E-36 |
| 0.7 | 6.44112E-36 |
| 0.8 | 8.42298E-36 |
| 0.9 | 1.06878E-35 |
| 1 | 1.31479E-35 |

We assume $v_0(x) = u_0(x) = a_3 \frac{x^2}{2!} + a_4 x + a_5$, as an initial approximation to the solution of Equation (20). Solutions of all the components $v_n(x),\ n = 0,1,2,3,\cdots$ are same as $v_0(x)$, which is the exact solution of Equation (20), so:

$$u(x) = a_3 \frac{x^2}{2!} + a_4 x + a_5, \qquad \frac{1}{4} \le x < \frac{3}{4}. \tag{22}$$

Here $a_0, a_1, a_2, a_3, \cdots a_8$ are unknown constants and would be determined by the given boundary conditions as defined in (13) and continuity conditions of $u(x),\ u'(x) \text{ and } u''(x)$ at $\frac{1}{4}$ and $\frac{3}{4}$. Combining Equation (18), (19) and (22), we obtain the approximate solution of the system of third- order boundary value problems (10) as:

$$u(x) = \begin{cases} a_0\left\{\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots\right\}+a_1\left\{x+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\cdots\right\}+a_2\left\{1+\frac{x^3}{3!}+\frac{x^6}{6!}+\cdots\right\}- & \\ \left\{\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots\right\}, & 0 \le x < \frac{1}{4}, \\ a_3\frac{x^2}{2}+a_4x+a_5, & \frac{1}{4} \le x < \frac{3}{4}, \\ a_6\left\{\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots\right\}+a_7\left\{x+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\cdots\right\}+a_8\left\{1+\frac{x^3}{3!}+\frac{x^6}{6!}+\cdots\right\}- & \\ \left\{\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots\right\}, & \frac{3}{4} \le x \le 1. \end{cases} \tag{23}$$

where:

$a_0 = 0.248888$, $a_1 = 0$, $a_2 = 0$, $a_3 = -6.26594267E{-}4$, $a_4 = 0.031161$,
$a_5 = -0.002595$, $a_6 = 0.729653$, $a_7 = -0.244520$, $a_8 = 0.0664319$.

Numerical results obtained with the sum of first 10 terms of the series solution (23) are given in Table 1. We now consider the system of nonlinear third-order boundary value problem which is due to Noor et al. (2011a).

## Example 2

We give another example of a system of third-order nonlinear boundary value problem by taking $f(x, u(x)) = 2u^3,\ g(x) = 1,\ \ r = -1,$ then the system of third-order boundary value problem (1) and (2) reduces to:

$$u'''(x) = \begin{cases} 2u^3 + u - 1, & -1 \le x < \frac{-1}{2}, \\ 2u^3, & \frac{-1}{2} \le x < \frac{1}{2}, \\ 2u^3 + 1 - 1, & \frac{1}{2} \le x \le 1, \end{cases} \tag{24}$$

with the boundary conditions:

$u(-1) = u(1) = 0,\ u'(-1) = 1.$

and the continuity condition of $u(x),\ u'(x) \text{ and } u''(x)$ at:

$\frac{-1}{2}$ and $\frac{1}{2}$.

### *Case I*

$-1 \le x < -\frac{1}{2}$. In this case, we implement the convex homotopy as follows:

$$u_0''' + pu_1''' + p^2u_2''' + \ldots\ldots = p\left[2\left(u_0 + pu_1 + p^2u_2 + \cdots\right)^3 + \left(u_0 + pu_1 + p^2u_2 + \cdots\right) - 1\right].$$

Comparing the co-efficient of like powers of $p$ :

$$p^0 : u_0(x) = c_1 + c_2 x + c_3\frac{x^2}{2},$$
$$p^1 : u_1(x) = c_1 + \frac{1}{2}c_3 x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 c_2 + \frac{1}{60}x^6 c_2^3,$$
$$p^2 : u_2(x) = \frac{1}{6}c_1 x^3 + \frac{1}{5}\left(\frac{1}{2}c_2^2 c_1 + \frac{1}{24}c_3\right)x^5 - \frac{1}{720}x^6 + \frac{1}{7}\left(\frac{1}{10}c_2^2 c_3 + \frac{1}{720}c_2\right)x^7 - \frac{1}{336}c_2^2 x^8$$
$$+ \frac{1}{1890}c_2^3 x^9 + \frac{1}{9900}c_2^5 x^{11},$$
$$\vdots$$

***Case II***

For $-\dfrac{1}{2} \le x < \dfrac{1}{2}$. In this case, we have:

$$u_0''' + pu_1''' + p^2u_2''' + \ldots\ldots = 2p\left(u_0 + pu_1 + p^2u_2 + \cdots\right)^3.$$

Comparing the co-efficient of like powers of $p$ :

$$p^0 : u_0(x) = c_4 + c_5 x + c_6\frac{x^2}{2},$$
$$p^1 : u_1(x) = c_4 + \frac{1}{2}x^2 c_6 + \frac{1}{60}x^6 c_5^3,$$
$$p^2 : u_2(x) = \frac{1}{10}c_4 c_5^2 x^5 + \frac{1}{70}c_5^2 c_6 x^7 + \frac{1}{9900}c_5^5 x^{11}$$
$$\vdots$$

***Case III***

For $\dfrac{1}{2} \le x \le 1$. In this case, we have:

$$u_0''' + pu_1''' + p^2u_2''' + \ldots\ldots = p\left[2\left(u_0 + pu_1 + p^2u_2 + \cdots\right)^3 + \left(u_0 + pu_1 + p^2u_2 + \cdots\right) - 1\right].$$

Comparing the co-efficient of like powers of $p$ :

$$p^0 : u_0(x) = c_7 + c_8 x + c_9\frac{x^2}{2},$$
$$p^1 : u_1(x) = c_7 + \frac{1}{2}c_9 x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 c_8 + \frac{1}{60}x^6 c_8^3,$$
$$p^2 : u_2(x) = \frac{1}{6}c_7 x^3 + \frac{1}{5}\left(\frac{1}{2}c_8^2 c_7 + \frac{1}{24}c_9\right)x^5 - \frac{1}{720}x^6 + \frac{1}{7}\left(\frac{1}{10}c_8^2 c_9 + \frac{1}{720}c_8\right)x^7 - \frac{1}{336}c_8^2 x^8$$
$$+ \frac{1}{1890}c_8^3 x^9 + \frac{1}{9900}c_8^5 x^{11},$$
$$\vdots$$

where $c_i, i = 1, 2 \ldots 9,$ are real constants and will be determined further by using boundary conditions and continuity conditions. This enables us to find the series solution as:

$$u(x) = \begin{cases} c_1 + c_2 x + \frac{1}{2}c_3 x^2 + \left(\frac{1}{6}c_1 - \frac{1}{6}\right)x^3 + \frac{1}{24}c_2 x^4 + \frac{1}{5}\left(\frac{1}{2}c_2^2 c_1 + \frac{1}{24}c_3\right)x^5 \\ + \left(\frac{1}{60}c_2^3 - \frac{1}{720}\right)x^6 + \frac{1}{7}\left(\frac{1}{10}c_2^2 c_3 + \frac{1}{720}c_2\right)x^7 - \frac{1}{336}c_2^2 x^8 + \frac{1}{1890}c_2^3 x^9 + \frac{1}{9900}c_2^5 x^{11}, \\ \qquad -1 \le x < -\frac{1}{2}, \\ c_4 + c_5 x + \frac{1}{2}c_6 x^2 + \frac{1}{60}x^6 c_5^3 + \frac{1}{10}c_4 c_5^2 x^5 + \frac{1}{70}c_5^2 c_6 x^7 + \frac{1}{9900}c_5^5 x^{11}, \quad -\frac{1}{2} \le x < \frac{1}{2}, \\ c_7 + c_8 x + \frac{1}{2}c_9 x^2 + \left(\frac{1}{6}c_7 - \frac{1}{6}\right)x^3 + \frac{1}{24}c_8 x^4 + \frac{1}{5}\left(\frac{1}{2}c_8^2 c_7 + \frac{1}{24}c_9\right)x^5 \\ + \left(\frac{1}{60}c_8^3 - \frac{1}{720}\right)x^6 + \frac{1}{7}\left(\frac{1}{10}c_8^2 c_9 + \frac{1}{720}c_8\right)x^7 - \frac{1}{336}c_8^2 x^8 + \frac{1}{1890}c_8^3 x^9 + \frac{1}{9900}c_8^5 x^{11}, \\ \qquad \frac{1}{2} \le x \le 1. \end{cases} \tag{25}$$

Now we use boundary conditions and continuity conditions at $x = -\dfrac{1}{2}$ and $x = \dfrac{1}{2}$, and we obtain system of nonlinear equations. In order to solve system of nonlinear equations, we use Newton's method. Hence we have the following values of unknown constants:

$$c_1 = .5249671409, \quad c_2 = -.0523372245, \quad c_3 = -1.3277592800, \quad c_4 = .5362933239,$$
$$c_5 = .0140953602, \quad c_6 = -1.0719942040, \quad c_7 = .5476022140, \quad c_8 = -.0519640851,$$
$$c_9 = -.8198949221. \tag{26}$$

By using values of unknowns from (26) into (25), we have following analytic solution of system of third-order nonlinear boundary value problems (24):

$$u(x) = \begin{cases} .5249671409 - .0523372245x - .6638796400x^2 - .07917214321x^3 \\ -.00218071768x^4 - .0109208624x^5 - .0013912782x^6 - 6.23412041\times10^5\,x^7 \\ -8.15233651\times10^6\,x^8 - 7.58525629\times10^7\,x^9 - 3.96659851\times10^{11}\,x^{11}, \quad -1 \le x < -\frac{1}{2}, \\ .5362933239 + .0140953602x - .5359971020x^2 + 1.065503174\times10^5\,x^5 \\ +4.667424325\times10^8\,x^6 - 3.04261326\times10^6\,x^7 + 5.62012141\times10^{14}\,x^{11}, \quad -\frac{1}{2} \le x < \frac{1}{2}, \\ .5476022140 - .0519640851x - .4099474610x^2 - .07539963103x^3 \\ -.002165170213x^4 - .006684590512x^5 - .001391227503x^6 \\ -4.193797002\times10^5\,x^7 - 8.036506369\times10^5\,x^8 - 7.424172460\times10^8\,x^9 \\ -3.827200653\times10^{11}\,x^{11}, \qquad \frac{1}{2} \le x \le 1. \end{cases}$$

## Conclusion

In this paper, we have used the homotopy perturbation methods for finding the approximate solutions of systems of third-order boundary value problems. Results show that the approximate solution obtained by this method converges rapidly to its exact solution. Results show that the homotopy perturbation method is highly effective, reliable and efficient for finding the approximate solution of a system of third-order boundary value problems. Results obtained in this paper may stimulate further research activities in this field and related areas. It is an interesting problem to consider the homotopy perturbation technique for solving the variational inequalities and related optimization problems.

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