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Direct method for solving nonlinear strain wave equation in microstructure solids

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The modeling of wave propagation in microstructure materials should be able to account for the various scales of microstructure. In this paper, the extended trial equation method was modified to construct the traveling wave solutions of the strain wave equation in microstructure solid. Some new different kinds of traveling wave solutions was gotten as, hyperbolic functions, trigonometric functions, Jacobi elliptic functions and rational functional solutions for the nonlinear strain wave equation when the balance number is positive integer. The balance number of this method is not constant and changes by changing the trial equation. These methods allow us to obtain many types of the exact solutions. By using the Maple software package, it was noticed that all the solutions obtained satisfy the original nonlinear strain wave equation.

Key words: Strain wave equation, extended trial equation method, exact solutions, balance number, soliton solutions, Jacobi elliptic functions.

INTRODUCTION

Nonlinear evolution equations (NLEEs) are very important model equations in mathematical physics and engineering for describing diverse types of physical mechanisms of natural phenomena in the field of applied sciences and engineering. The search for exact traveling wave solutions to nonlinear evaluation equations plays very important role in the study of these physical phenomena. In recent years, the exact solutions of nonlinear partial differential equation have been investigated by many authors (Ablowitz and Clarkson, 1991; Rogers and Shadwick, 1982; Matveev and Salle, 1991; Li and Chen, 2003; Conte and Musette, 1992; Ebaid and Aly, 2012; Gepreel, 2014; Cariello and Tabor, 1991; Fan, 2000; Fan, 2002; Wang and Li, 2005; Abdou, 2007; Wu and He, 2006; Wu and He, 2008; Li and Wang, 2007; Zheng, 2012; Triki and Wazwaz, 2014; Bibi and Mohyud-Din, 2014; Yu-Bin and Chao, 2009; Zayed and Gepreel, 2009; He, 2006; Gepreel, 2011; Adomian, 1988; Wazwaz, 2007; Liao, 2010; Gepreel and Mohamed, 2013; Wang et al., 2008; Yan, 2003a) who are interested in nonlinear physical phenomena. Many powerful methods have been presented by authors such as the

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inverse scattering transform (Ablowitz and Clarkson, 1991), the Backlund transform (Rogers and Shadwick, 1982), Darboux transform (Matveev and Salle, 1991), the generalized Riccati equation (Li and Chen, 2003; Conte and Musette, 1992), the Jacobi elliptic function expansion method (Ebaid and Aly, 2012; Gepreel, 2014), Painlev´e expansions method (Cariello and Tabor, 1991), the extended Tang-function method (Fan, 2000; Fan, 2002), the F-expansion method (Wang and Li, 2005; Abdou, 2007), the ex-function expansion method (Wu and He, 2006; 2008), the sub-ODE method (Li and Wang, 2007; Zheng, 2012), the extended sinh-cos and sine-cosine methods (Triki and Wazwaz, 2014; Bibi and Mohyud-Din, 2014), the (G´/G) -expansion method (Yu-Bin and Chao, 2009; Zayed and Gepreel, 2009), etc. Also, there are many methods for finding the analytic approximate solutions for nonlinear partial differential equations such as the homotopy perturbation method (He, 2006; Gepreel, 2011), a domain decomposition method (Adomian, 1988), variation iteration (Wazwaz, 2007) and homotopy analysis method (Liao, 2010; Gepreel and Mohamed, 2013). There are many other methods for solving the nonlinear partial differential equations (Wang et al., 2008; Yan, 2003a; 2003b; 2008; 2009; Zayed and Al-Joudi, 2009; Zayed, 2009; Zhang, 2009; Jang, 2009). Bulut et al. (2013); Bulut and Pandir (2013) and Baskonus et al. (2014) have used the modified trial equation method to find some new exact solutions for nonlinear evolution equations in mathematical physics.

Recently, Gurefe et al. (2013) have presented a direct method, namely, the extended trial equation method for solving the nonlinear partial differential equations. Demiray et al. (2016; 2015a; 2015b); Demiray and Bulut (2015) and Bulut et al. (2014) have successively applied the extended trial method for solving the nonlinear partial differential equations. The governing nonlinear equation of the strain waves in microstructure solid is given by (Alam et al., 2014; Samsonov, 2001):

$$u_{tt}-u_{xx}-\beta\lambda_1(u^2)_{xx}-\gamma\lambda_2 u_{xxt}+\delta\lambda_3 u_{xxxx} -(\delta\lambda_4-\gamma^2\lambda_7)u_{xxtt}+\gamma\delta(\lambda_5 u_{xxxxt}+\lambda_6 u_{xxttt})=0, \quad (1)$$

where β accounts for elastic strains, δ characterizes the ratio between the microstructure size and the wavelength, γ characterizes the influence of dissipation and $\lambda_i\ (i=1,..,6)$ are constants. The balance between nonlinearity and dispersion takes place when $\delta=O(\beta)$. If $\gamma=0$ is set, then we have the non-dissipative case and governed by the double dispersive Equation 45 and 46 as follows:

$$u_{tt}-u_{xx}-\beta(\lambda_1(u^2)_{xx}-\lambda_3 u_{xxxx}+\lambda_4 u_{xxtt})=0\,. \quad (2)$$

Previous models were derived using the assumption of the homogeneity of microstructure. This is the case for the example of functionally graded materials which are made up of two or more material combined in solid state (Mahamood et al., 2012; Birman and Byrd, 2007). The main objective of this paper is to use the modified extended trial equation method to find a series of new analytical solutions to the strain wave Equation 2 for many different type of the roots of the trial equation.

DESCRIPTION OF THE EXTENDED TRIAL EQUATION METHOD

Suppose we have a nonlinear partial differential equation in the following form:

$$F(u,u_t,u_x,u_{tt},u_{xt},u_{xx},.....)=0, \quad (3)$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u=u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps for solving equation (3) using the extended trial equation method as (Gurefe et al., 2013; Demiray et al., 2016; 2015a; 2015b; Demiray and Bulut, 2015; Bulut et al., 2014; Ekici et al., 2013):

Step 1. The traveling wave variable:

$$u(x,t)=u(\xi)\,,\qquad \xi=x+Vt, \quad (4)$$

where V is a nonzero constant, Equation 4 permits reducing equation (3) to the following ODE:

$$P(u,Vu',u',V^2u'',Vu'',u'',......)=0, \quad (5)$$

where P is a polynomial of $u(\xi)$ and its total derivatives.

Step 2. Suppose the solution of Equation 5 takes the form:

$$u(\xi)=\sum_{i=0}^{\delta_1}\tau_i Y^i, \quad (6)$$

where $Y(\xi)$ satisfies the following nonlinear trial differential equation:

$$(Y')^2=\Lambda(Y)=\frac{\Phi(Y)}{\Psi(Y)}=\frac{\xi_\theta Y^\theta+\xi_{\theta-1}Y^{\theta-1}+...+\xi_1 Y+\xi_0}{\zeta_\varepsilon Y^\varepsilon+\zeta_{\varepsilon-1}Y^{\varepsilon-1}+...+\zeta_1 Y+\zeta_0}, \quad (7)$$

where ξ_i,ζ_j are constants to be determined later. Using Equations 6 and 7, we have

$$u''(\xi)=\frac{\Phi'(Y)\Psi(Y)-\Phi(Y)\Psi'(Y)}{2\Psi^2(Y)}\left(\sum_{i=0}^{\delta_1} i\ \tau_i Y^{i-1}\right) +\frac{\Phi(Y)}{\Psi(Y)}\left(\sum_{i=0}^{\delta_1} i\ (i-1)\tau_i Y^{i-2}\right), \quad (8)$$

where $\Phi(Y), \Psi(Y)$ are polynomials in Y.

Step 3. Balancing the highest order derivative with the nonlinear terms, we can find the relations between δ_1, θ and ε. We can calculate some values of δ_1, θ and ε.

Step 4. Substituting Equations 6 to 8 into Equation 5 yields a polynomial $\Omega(y)$ of $Y(\xi)$ as follows:

$$\Omega(y) = \rho_s Y^s + ... + \rho_1 Y + \rho_0 = 0. \quad (9)$$

Step 5. Setting the coefficients of this polynomial $\Omega(y)$ to be zero, we yield a set of algebraic equations:

$$\rho_i = 0, \qquad i = 0, ..., s. \quad (10)$$

Solve this system of algebraic equations to determine the values of $\xi_\theta, \xi_{\theta-1}, ..., \xi_1, \xi_0, \zeta_\varepsilon, \ \zeta_{\varepsilon-1}, ..., \zeta_1, \zeta_0$ and $\tau_{\delta_1}, \tau_{\delta_1-1}, ..., \tau_1, \tau_0$.

Step 6. Reduce Equation 7 to the elementary integral form:

$$\pm(\xi - \eta_0) = \int \frac{dY}{\sqrt{\Lambda(y)}} = \int \sqrt{\frac{\Psi(Y)}{\Phi(Y)}} dY. \quad (11)$$

where η_0 is an arbitrary constant. Using a complete discrimination system for the polynomial to classify the roots of $\Phi(Y)$, we solve Equation 11 with the help of software package such as Maple or Mathematica and classify the exact solutions to Equation 5. In addition, we can write the exact traveling wave solutions to Equation 3, respectively.

Remark 1. The difference between the modified trial expansion method, extended trial expansion method and modified extended trial method:

(i) In the modified trial method, the trial equation is taking the following form:

$$Y' = \frac{\Phi(Y)}{\Psi(Y)} = \frac{\xi_\theta Y^\theta + \xi_{\theta-1} Y^{\theta-1} + ... + \xi_1 Y + \xi_0}{\zeta_\varepsilon Y^\varepsilon + \zeta_{\varepsilon-1} Y^{\varepsilon-1} + ... + \zeta_1 Y + \zeta_0} \quad (12)$$

and the reduced elementary integral takes the following form:

$$\pm(\eta - \eta_0) = \int \frac{\Psi(Y)}{\Phi(Y)} dy \quad (13)$$

(ii) In the extended trial method, the trial equation is taking the following form:

$$Y' = \sqrt{\frac{\Phi(Y)}{\Psi(Y)}} = \sqrt{\frac{\xi_\theta Y^\theta + \xi_{\theta-1} Y^{\theta-1} + ... + \xi_1 Y + \xi_0}{\zeta_\varepsilon Y^\varepsilon + \zeta_{\varepsilon-1} Y^{\varepsilon-1} + ... + \zeta_1 Y + \zeta_0}} \quad (14)$$

and the reduced elementary integral takes the following form:

$$\pm(\eta - \eta_0) = \int \sqrt{\frac{\Psi(Y)}{\Phi(Y)}} dy \quad (15)$$

(iii) In the modified extended trial expansion method, it seems to the reader as extended trial expansion method. But in the extended trial equation, there is no connection between the roots of the right side of Equation 11 α_i and the coefficients of the solutions τ_i and ξ_i. Many papers have used the extended trial equation without making the connection between the root α_i and the coefficients of the solutions τ_i and ξ_i. So all the solutions in these papers does not satisfy the original equations. Then, this response was searched for, the authors which used the extended trial equation must be related between the roots of right side of Equation 11 and the solution coefficients τ_i and the trial equation coefficients ξ_i. For this, we call the modified extended trial expansion method.

MODIFIED EXTENDED TRIAL EQUATION METHOD FOR THE STRAIN WAVE EQUATION

Here, the modified extended trial equation method was used to find the traveling wave solutions to the following nonlinear strain wave differential equation:

$$u_{tt} - u_{xx} - \beta(\lambda_1 (u^2)_{xx} - \lambda_3 u_{xxxx} + \lambda_4 u_{xxtt}) = 0. \quad (16)$$

Porubov and Pastrone (2004) studied the propagation and attenuation or amplification of bell-shaped and kink-shaped waves, whose parameters are defined in an explicit form through the parameters of the microstructured medium. Also, Alam et al. (2014) used the generalized (G′/G)-expansion method to find an exact traveling wave solution of nonlinear strain wave differential equation. The traveling wave variable:

$$u(x,t) = u(\xi), \qquad \xi = x - Vt \ , \quad (17)$$

where V is the speed of the traveling wave, permitting us to convert Equation 16 into the following ODE:

$$(V^2 - 1)u'' - \beta\lambda_1 (u^2)'' + \beta(\lambda_3 - \lambda_4 V^2) u^{(4)} = 0. \quad (18)$$

Integrating Equation 18 twice with respect to ξ, we have:

$$(V^2 - 1)u - \beta\lambda_1 u^2 + \beta(\lambda_3 - \lambda_4 V^2) u'' + k = 0, \quad (19)$$

where k is the integral constant. We suppose the traveling wave solution of the Equation 19 into the following form:

$$u(\xi) = \sum_{i=0}^{\delta_1} \tau_i Y^i, \quad (20)$$

where Y satisfies Equation 7 and δ_1 is an arbitrary positive integer. Balancing the highest order derivative u'' with the nonlinear term u^2 in Equation 19, we have:

$$\delta_1 = \theta - \varepsilon - 2. \tag{21}$$

Equation 21 has infinitely many solutions, consequently, we suppose some of these solutions as the following cases.

Case 1. In the special case, if $\varepsilon = 0$ and $\theta = 3$, we get $\delta_1 = 1,$ then Equations 6 to 11 lead to:

$$u(\xi) = \tau_0 + \tau_1 Y,$$
$$(u')^2 = \frac{\tau_1^2(\xi_3 Y^3 + \xi_2 Y^2 + \xi_1 Y + \xi_0)}{\zeta_0},$$
$$u'' = \frac{\tau_1(3\xi_3 Y^2 + 2\xi_2 Y + \xi_1)}{2\zeta_0}. \tag{22}$$

Substituting equations (22) into Equation 19 we get a system of algebraic equations which can be solved by using the Maple software package to obtain the following results:

$$\xi_1 = -\frac{4\zeta_0^2\lambda_1(V^2\tau_0 - \tau_0 - \beta\lambda_1\tau_0^2 + k)}{3\beta\,\xi_3(\lambda_4^2V^4 - 2\lambda_4V^2\lambda_3 + \lambda_3^2)},\ \xi_2 = \frac{\zeta_0(-1+V^2 - 2\beta\lambda_1\tau_0)}{\beta(\lambda_4V^2 - \lambda_3)},\ \tau_1 = -\frac{3\xi_3(\lambda_4V^2 - \lambda_3)}{2\lambda_1\zeta_0}, \tag{23}$$

where ζ_0, ξ_0, ξ_3 and τ_0 are arbitrary constants. Substituting Equation 5 into Equations 7 and 9, we have

$$\pm(\xi - \eta_0) = L\int \frac{dY}{\sqrt{Y^3 + \frac{\xi_2}{\xi_3}Y^2 + \frac{\xi_1}{\xi_3}Y + \frac{\xi_o}{\xi_3}}}, \tag{24}$$

where $L = \sqrt{\frac{\zeta_0}{\xi_3}}$. Now we will discuss the roots of the following equation:

$$Y^3 + \frac{\zeta_0(-1+V^2 - 2\beta\lambda_1\tau_0)}{\beta(\lambda_4V^2 - \lambda_3)\xi_3}Y^2 - \frac{4\zeta_0^2\lambda_1(V^2\tau_0 - \tau_0 - \beta\lambda_1\tau_0^2 + k)}{3\beta\,\xi_3^2(\lambda_4^2V^4 - 2\lambda_4V^2\lambda_3 + \lambda_3^2)}Y + \frac{\xi_0}{\xi_3} = 0, \tag{25}$$

to integrate Equation 24 as the following families:

Family 1. If Equation 25 has three equal repeated roots α_1, consequently we can write Equation 25 in the following form:

$$Y^3 + \frac{\zeta_0(-1+V^2 - 2\beta\lambda_1\tau_0)}{\beta(\lambda_4V^2 - \lambda_3)\xi_3}Y^2 - \frac{4\zeta_0^2\lambda_1(V^2\tau_0 - \tau_0 - \beta\lambda_1\tau_0^2 + k)}{3\beta\,\xi_3^2(\lambda_4^2V^4 - 2\lambda_4V^2\lambda_3 + \lambda_3^2)}Y + \frac{\xi_0}{\xi_3} = (Y - \alpha_1)^3. \tag{26}$$

From equating the coefficients of Y to both sides of Equation 26, we get a system of algebraic equations:

$$-\zeta_0 + \xi_3 = 0,$$
$$\xi_0 + \alpha_1^3\zeta_0 = 0,$$
$$-1 + V^2 + 3\alpha_1\beta(\lambda_4V^2 - \lambda_3) - 2\lambda_1\tau_0\beta = 0,$$
$$\frac{4\zeta_0\lambda_1\tau_0}{3\beta\xi_3} + \frac{4\zeta_0\lambda_1^2\tau_0^2}{3\xi_3} - \frac{4\zeta_0\lambda_1 k}{3\beta\xi_3} - 3\alpha_1^2(\lambda_4^2V^4 - 2\lambda_4V^2\lambda_3 + \lambda_3^2) - \frac{4\zeta_0\lambda_1V^2\tau_0}{3\beta\xi_3} = 0. \tag{27}$$

We use the Maple software package to solve the system (equation 27) in $k, \zeta_0, \xi_0, \xi_3, \tau_0$ and α_1. We get the following results:

$$\xi_0 = -\alpha_1^3\zeta_0,\ \ \xi_3 = \zeta_0,\ \ \tau_0 = \frac{-1+V^2 + 3\alpha_1\beta\lambda_4V^2 - 3\alpha_1\beta\lambda_3}{2\beta\lambda_1},\ k = -\frac{1 - 2V^2 + V^4}{4\beta\lambda_1}. \tag{28}$$

Equations (27), (23) and (24) lead to:

$$\xi_1 = 3\alpha_1^2\zeta_0, \quad \xi_2 = -3\alpha_1\zeta_0, \quad \tau_1 = -\frac{3(\lambda_4V^2 - \lambda_3)}{2\lambda_1}, \tag{29}$$

where ζ_0 is an arbitrary constant and

$$\pm(\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)^{3/2}} = \frac{-2}{\sqrt{Y - \alpha_1}},$$

or

$$Y = \alpha_1 + \frac{4}{(x - Vt - \eta_0)^2}. \tag{30}$$

Substituting Equations 30, 28 and 27 into Equation 22, we get the traveling wave solution of nonlinear strain wave Equation 16 takes the following form:

$$u_1(\xi) = \frac{-1+V^2 + 3\alpha_1\beta\lambda_4V^2 - 3\alpha_1\beta\lambda_3}{2\beta\lambda_1} - \frac{3(\lambda_4V^2 - \lambda_3)}{2\lambda_1}\left\{\alpha_1 + \frac{4}{(x - Vt - \eta_0)^2}\right\}. \tag{31}$$

Family 2. If Equation 25 has two equal repeated roots α_1 and the third root is α_2 and $\alpha_1 \neq \alpha_2$, consequently we can write Equation 25 in the following form:

$$Y^3 + \frac{\zeta_0(-1+V^2-2\beta\lambda_1\tau_0)}{\beta(\lambda_4 V^2-\lambda_3)\xi_3}Y^2 - \frac{4\zeta_0^2\lambda_1(V^2\tau_0-\tau_0-\beta\lambda_1\tau_0^2+k)}{3\beta\,\xi_3^2(\lambda_4^2V^4-2\lambda_4V^2\lambda_3+\lambda_3^2)}Y + \frac{\xi_0}{\xi_3} = (Y-\alpha_1)^2(Y-\alpha_2) \quad (32)$$

From equating the coefficients of Y to both sides of Equation 32, we get a system of algebraic equations in k, ζ_0, ξ_0, ξ_3 and τ_0 which can be solved by using the Maple software package to get the following results:

$$k = \frac{[\beta(\alpha_1-\alpha_2)(\lambda_4V^2-\lambda_3)]^2 - 2V^2 - V^4 - 1}{4\beta\lambda_1},$$

$$\xi_0 = -\alpha_1^2\alpha_2\xi_3, \quad \zeta_0 = \xi_3, \quad \tau_0 = \frac{\beta(\alpha_2+2\alpha_1)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1}. \quad (33)$$

Equations 33, 23 and 24 lead to:

$$\xi_1 = \alpha_1(\alpha_1+2\alpha_2)\xi_3, \quad \xi_2 = -(2\alpha_1+\alpha_2)\xi_3, \quad \tau_1 = -\frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}, \quad (34)$$

where ξ_3 is an arbitrary constant. In this family, the solution of Equation 24, when $\alpha_2 > \alpha_1$ takes the following form:

$$\pm(\xi-\eta_0) = \int \frac{dY}{(Y-\alpha_1)\sqrt{Y-\alpha_2}} = \frac{2}{\sqrt{\alpha_2-\alpha_1}}\tan^{-1}\left[\frac{\sqrt{Y-\alpha_2}}{\sqrt{\alpha_2-\alpha_1}}\right], \quad \alpha_2 > \alpha_1, \quad (35)$$

or

$$Y = \alpha_2 + (\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x-Vt-\eta_0)\right], \qquad \alpha_2 > \alpha_1. \quad (36)$$

Substituting Equations 36, 34 and 33 into Equation 22, we get the traveling wave solution of nonlinear strain wave Equation 16 taking the form:

$$u_2(\xi) = \frac{\beta(\alpha_2+2\alpha_1)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1} - \frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}\left\{\alpha_2 + (\alpha_2-\alpha_1)\tan^2\left[\frac{\sqrt{\alpha_2-\alpha_1}}{2}(x-Vt-\eta_0)\right]\right\}. \quad (37)$$

Also when $\alpha_1 > \alpha_2$, the solution of Equation 24 has the form:

$$Y = \alpha_1 + (\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x-Vt-\eta_0)\right], \qquad \alpha_1 > \alpha_2, \quad (38)$$

Substituting Equations 38, 34 and 33 into Equation 22, we get the traveling wave solution of nonlinear strain wave Equation 16 takes the form:

$$u_3(\xi) = \frac{\beta(\alpha_2+2\alpha_1)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1} - \frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}\left\{\alpha_1 + (\alpha_1-\alpha_2)\operatorname{csch}^2\left[\frac{\sqrt{\alpha_1-\alpha_2}}{2}(x-Vt-\eta_0)\right]\right\}. \quad (39)$$

Family 3. If Equation 25 has three different roots α_1, α_2 and α_3, $\alpha_1 \neq \alpha_2 \neq \alpha_3$, consequently we can write Equation 25 in the following form:

$$Y^3 + \frac{\zeta_0(-1+V^2-2\beta\lambda_1\tau_0)}{\beta(\lambda_4V^2-\lambda_3)\xi_3}Y^2 - \frac{4\zeta_0^2\lambda_1(V^2\tau_0-\tau_0-\beta\lambda_1\tau_0^2+k)}{3\beta\,\xi_3^2(\lambda_4^2V^4-2\lambda_4V^2\lambda_3+\lambda_3^2)}Y + \frac{\xi_0}{\xi_3} - (Y-\alpha_1)(Y-\alpha_2)(Y-\alpha_3) = 0. \quad (40)$$

From equating the coefficients of Y to both sides of Equation 40, we get a system of algebraic equations in k, ζ_0, ξ_0, ξ_3 and τ_0 which can be solved by using the Maple software package to get the following results:

$$\xi_0 = -\alpha_1\alpha_2\alpha_3\xi_3, \quad \zeta_0 = \xi_3, \quad \tau_0 = \frac{\beta(\alpha_1+\alpha_2+\alpha_3)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1},$$

$$k = \frac{\beta^2(\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_1\alpha_3-\alpha_1\alpha_2-\alpha_2\alpha_3)(\lambda_4V^2-\lambda_3)^2+2V^2-V^4-1}{4\beta\lambda_1}. \quad (41)$$

Equations 41, 23 and 24 lead to:

$$\xi_1 = (\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\xi_3, \; \xi_2 = -(\alpha_1+\alpha_2+\alpha_3)\xi_3, \tau_1 = -\frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}, \quad (42)$$

where ξ_3 is an arbitrary constant. In this family, the solution of Equation 24 has the form:

$$\pm(\xi-\eta_0) = \int \frac{dY}{\sqrt{(Y-\alpha_1)(Y-\alpha_2)(Y-\alpha_3)}} = \frac{2}{\sqrt{\alpha_3-\alpha_1}} EllipticF\left[\frac{\sqrt{Y-\alpha_1}}{\sqrt{\alpha_2-\alpha_1}}, \sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right], \quad (43)$$

or

$$Y = \alpha_1 + (\alpha_2-\alpha_1)sn^2\left[\frac{\sqrt{\alpha_3-\alpha_1}}{2}(x-Vt-\eta_0), \sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right], \quad (44)$$

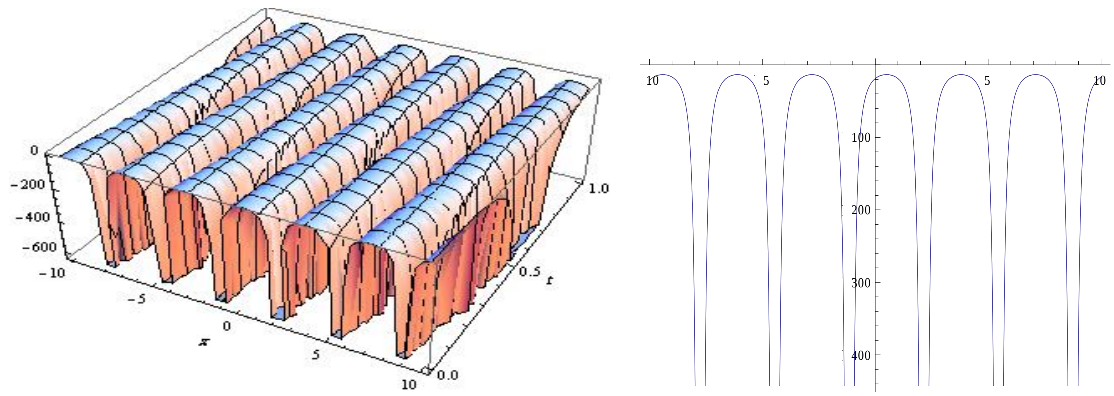
Substituting Equations 44, 42 and 41 into Equation 22, we get the traveling wave solution of nonlinear strain wave Equation 16 takes the form:

$$u_4(\xi) = \frac{\beta(\alpha_1+\alpha_2+\alpha_3)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1} - \frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}\{\alpha_1 + (\alpha_2-\alpha_1)sn^2\left[\frac{\sqrt{\alpha_3-\alpha_1}}{2}(x-Vt-\eta_0), \sqrt{\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}}\right]\}. \quad (45)$$

The behavior of the exact Solution 45 has been illustrated in Figure 1.

Family 4. If Equation 25 has one real roots α_1 and two imaginary roots $\alpha_2 = N_1 + i\,N_2$, $\alpha_3 = N_1 - i\,N_2$, where N_1, N_2 are real numbers, consequently we can write Equation 25 in the following form:

$$Y^3 + \frac{\zeta_0(-1+V^2-2\beta\lambda_1\tau_0)}{\beta(\lambda_4V^2-\lambda_3)\xi_3}Y^2 - \frac{4\zeta_0^2\lambda_1(V^2\tau_0-\tau_0-\beta\lambda_1\tau_0^2+k)}{3\beta\,\xi_3^2(\lambda_4^2V^4-2\lambda_4V^2\lambda_3+\lambda_3^2)}Y + \frac{\xi_0}{\xi_3}$$



(a) (b)

Figure 1. The traveling wave solution Equation 45 and its projection at $t=0$ when the parameters take special values $\alpha_1=5,\ \alpha_2=1,\ \alpha_3=1.5,\ \lambda_1=2,\ \lambda_3=-3,\ \lambda_4=1,\ V=2,\ \beta=-2.5$ and $\eta_0=5$.

$$-(Y-\alpha_1)(Y^2-2N_1Y+N_1^2+N_2^2)=0. \quad (46)$$

From equating the coefficients of Y to both sides of Equation 46, we get a system of algebraic equations in k,ζ_0,ξ_0,ξ_3 and τ_0 which can be solved by using the Maple software package to get the following results:

$$\xi_0=-(N_1^2+N_2^2)\alpha_1\xi_3,\quad \zeta_0=\xi_3,\quad \tau_0=\frac{\beta(\alpha_1+2N_1)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1},$$
$$k=\frac{\beta^2([\alpha_1-N_1]^2-3N_2^2)(\lambda_4V^2-\lambda_3)^2+2V^2-V^4-1}{4\beta\lambda_1}. \quad (47)$$

Equations 47, 28 and 24 lead to:

$$\xi_1=(N_1^2+N_2^2+2\alpha_1N_1)\xi_3,\quad \xi_2=-(\alpha_1+2N_1)\xi_3,\quad \tau_1=-\frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}, \quad (48)$$

where ξ_3 is an arbitrary constant. In this family, the integration of Equation 24 takes the following form:

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{(Y-\alpha_1)(Y^2-2N_1Y+N_1^2+N_2^2)}}$$
$$=\frac{2}{\sqrt{N_1+iN_2-\alpha_1}}EllipticF\left[\frac{\sqrt{Y-\alpha_1}}{\sqrt{N_1-iN_2-\alpha_1}},\sqrt{\frac{N_1-iN_2-\alpha_1}{N_1+iN_2-\alpha_1}}\right], \quad (49)$$

or

$$Y=\alpha_1+(N_1-iN_2-\alpha_1)sn^2\left[\frac{\sqrt{N_1+iN_2-\alpha_1}}{2}(x-Vt-\eta_0),\sqrt{\frac{N_1-iN_2-\alpha_1}{N_1+iN_2-\alpha_1}}\right]. \quad (50)$$

Substituting Equations 50, 48 and 47 into Equation 22, we get the traveling wave solution of nonlinear strain wave Equation 16 has the form:

$$u_5(\xi)=\frac{\beta(\alpha_1+2N_1)(\lambda_4V^2-\lambda_3)+V^2-1}{2\beta\lambda_1}$$

$$-\frac{3(\lambda_4V^2-\lambda_3)}{2\lambda_1}\{\alpha_1+(N_1-iN_2-\alpha_1)sn^2\left[\frac{\sqrt{N_1+iN_2-\alpha_1}}{2}(x-Vt-\eta_0),\sqrt{\frac{N_1-iN_2-\alpha_1}{N_1+iN_2-\alpha_1}}\right]\}. \quad (51)$$

The behavior of the exact Solution 51 has been illustrated in Figure 2.

Case 2. In the special case, if $\varepsilon=0$ and $\theta=4$, we get $\delta=2$, then Equations 6 to 11 lead to:

$$u(\xi)=\tau_0+\tau_1Y+\tau_2Y^2,$$
$$(u')^2=\frac{(\tau_1+2\tau_2Y)^2(\xi_4Y^4+\xi_3Y^3+\xi_2Y^2+\xi_1Y+\xi_0)}{\zeta_0},$$
$$u''=\frac{\tau_1(4\xi_4Y^3+3\xi_3Y^2+2\xi_2Y+\xi_1)}{2\zeta_0}+\frac{\tau_2}{\zeta_0}(6\xi_4Y^4+5\xi_3Y^3+4\xi_2Y^2+3\xi_1Y+2\xi_0), \quad (52)$$

Substituting Equation 51 into Equation 19, we get a system of algebraic equations which can be solved to obtain the following results:

$$k=\frac{\xi_4\tau_2\{(-6\tau_1^2+48\tau_0\tau_2)(V^2-1)\}+\beta\lambda_1\{(-\xi_4\tau_1^4+16\tau_2^4\xi_0)+(\xi_4\tau_0\tau_2)(12\tau_1^2-48\tau_0\tau_2)\}}{48\tau_2^2\xi_4}.$$
$$\xi_1=-\frac{\tau_1\xi_4[6\tau_2(V^2-1)+\beta\lambda_1(-12\tau_0\tau_2+\tau_1^2)]}{4\tau_2^3\beta\lambda_1},\quad \zeta_0=-\frac{6\xi_4(\lambda_4V^2-\lambda_3)}{\lambda_1\tau_2},\ \xi_3=\frac{2\xi_4\tau_1}{\tau_2},$$
$$\xi_2=-\frac{3\xi_4[2\tau_2(V^2-1)-\beta\lambda_1(4\tau_0\tau_2+\tau_1^2)]}{4\tau_2^2\beta\lambda_1}, \quad (53)$$

where $\xi_0,\xi_4,\tau_0,\tau_1$ and τ_2 are arbitrary constants. Substituting Equation 53 into Equations 7 and 11, we have:

$$\pm(\xi-\eta_0)=L\int\frac{dY}{\sqrt{Y^4+\frac{\xi_3}{\xi_4}Y^3+\frac{\xi_2}{\xi_4}Y^2+\frac{\xi_1}{\xi_4}Y+\frac{\xi_o}{\xi_4}}}, \quad (54)$$

where $L=\sqrt{\frac{\zeta_0}{\xi_4}}$. Now we will discuss the roots of the following equation:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3-\frac{3[2\tau_2(V^2-1)-\beta\lambda_1(4\tau_0\tau_2+\tau_1^2)]}{4\tau_2^2\beta\lambda_1}Y^2-\frac{\tau_1[6\tau_2(V^2-1)+\beta\lambda_1(-12\tau_0\tau_2+\tau_1^2)]}{4\tau_2^3\beta\lambda_1}Y$$

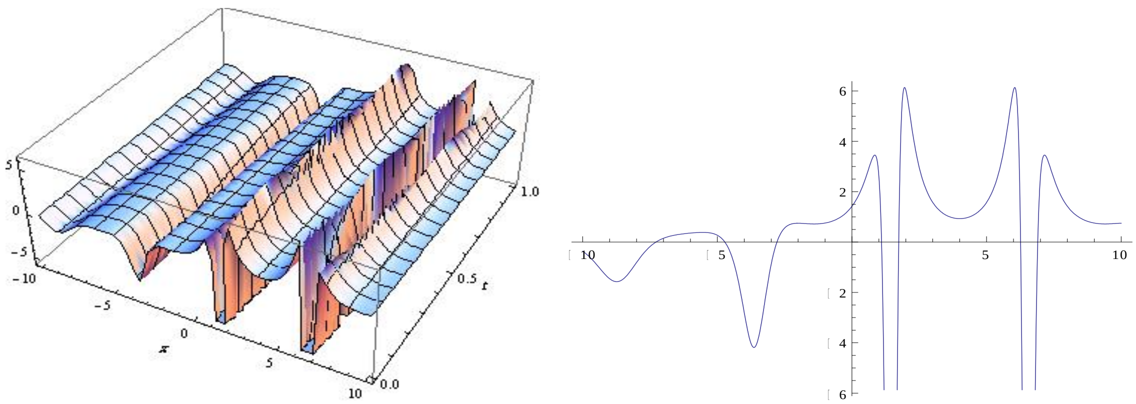


Figure 2. The real part of the traveling wave solution (Equation 51) and its projection at $t=0$ when the parameters take special values $\alpha_1=2,\ N_1=0.5,\ N_2=0.25,\ \lambda_1=-2.5,\ \lambda_3=-0.5,\ \lambda_4=1.05,\ V=1,\ \beta=-1$ **and** $\eta_0=4.$

$$+\frac{\xi_0}{\xi_4}=0. \tag{55}$$

To integrate Equation 54, we discuss the roots of Equation 55 as the following families:

Family 5. If Equation 55 has four equal repeated roots α_1, consequently we can write the Equation 55 in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3-\frac{3[2\tau_2(V^2-1)-\beta\lambda_1(4\tau_0\tau_2+\tau_1^2)]}{4\tau_2^2\beta\lambda_1}Y^2-\frac{\tau_1[6\tau_2(V^2-1)+\beta\lambda_1(-12\tau_0\tau_2+\tau_1^2)]}{4\tau_2^3\beta\lambda_1}Y+\frac{\xi_0}{\xi_4}-(Y-\alpha_1)^4=0. \tag{56}$$

From equating the coefficients of Y to both sides of Equation 56, we get a system of algebraic equations in $\xi_0,\xi_4,\tau_0,\tau_1$ and τ_2, which can be solved by using the Maple software package to get the following results:

$$\xi_0=\alpha_1^4\xi_4,\tau_0=-\frac{12\alpha_1^2\beta(\lambda_4V^2-\lambda_3)-V^2+1}{2\beta\lambda_1},$$
$$\tau_1=\frac{12\alpha_1(\lambda_4V^2-\lambda_3)}{\lambda_1},\ \tau_2=-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}. \tag{57}$$

Equations 57, 53 and 54 lead to:

$$\xi_1=-4\alpha_1^3\xi_4,\ \xi_2=6\alpha_1^2\xi_4,\ \xi_3=-4\alpha_1\xi_4,\ \zeta_0=\xi_4,\ k=-\frac{V^4-2V^2+1}{4\beta\lambda_1}. \tag{58}$$

where ξ_4 is an arbitrary constant and

$$\pm(\xi-\eta_0)=\int\frac{dY}{(Y-\alpha_1)^2}=\frac{-1}{Y-\alpha_1}, \tag{59}$$

Then

$$Y=\alpha_1\mp\frac{1}{(x-Vt-\eta_0)}. \tag{60}$$

Substituting Equations 60, 58 and 57 into Equation 52, we get the traveling wave solution of nonlinear strain wave Equation 16 taking the following form:

$$u_6(\xi)=-\frac{12\alpha_1^2\beta(\lambda_4V^2-\lambda_3)-V^2+1}{2\beta\lambda_1}+\frac{12\alpha_1(\lambda_4V^2-\lambda_3)}{\lambda_1}\left[\alpha_1\mp\frac{1}{(x-Vt-\eta_0)}\right]-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}\left[\alpha_1\mp\frac{1}{(x-Vt-\eta_0)}\right]^2. \tag{61}$$

The behavior of the exact Solution 61 has been illustrated in Figure 3

Family 6. If the Equation 55 has two equal repeated roots α_1 and α_2, $\alpha_1\neq\alpha_2$ consequently we can write Equation 55 in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3-\frac{3[2\tau_2(V^2-1)-\beta\lambda_1(4\tau_0\tau_2+\tau_1^2)]}{4\tau_2^2\beta\lambda_1}Y^2-\frac{\tau_1[6\tau_2(V^2-1)+\beta\lambda_1(-12\tau_0\tau_2+\tau_1^2)]}{4\tau_2^3\beta\lambda_1}Y+\frac{\xi_0}{\xi_4}-(Y-\alpha_1)^2(Y-\alpha_2)^2=0. \tag{62}$$

From equating the coefficients of Y to both sides of Equation (62), we get a system of algebraic equations in $\xi_0,\xi_4,\tau_0,\tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$\xi_0=\alpha_1^2\alpha_2^2\xi_4,\qquad \tau_0=-\frac{\beta(\lambda_4V^2-\lambda_3)(\alpha_1^2+\alpha_2^2+10\alpha_1\alpha_2)-V^2+1}{2\beta\lambda_1},$$
$$\tau_1=\frac{6(\lambda_4V^2-\lambda_3)(\alpha_1+\alpha_2)}{\lambda_1},\quad \tau_2=-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}. \tag{63}$$

Equations 63, 53 and 54 lead to:

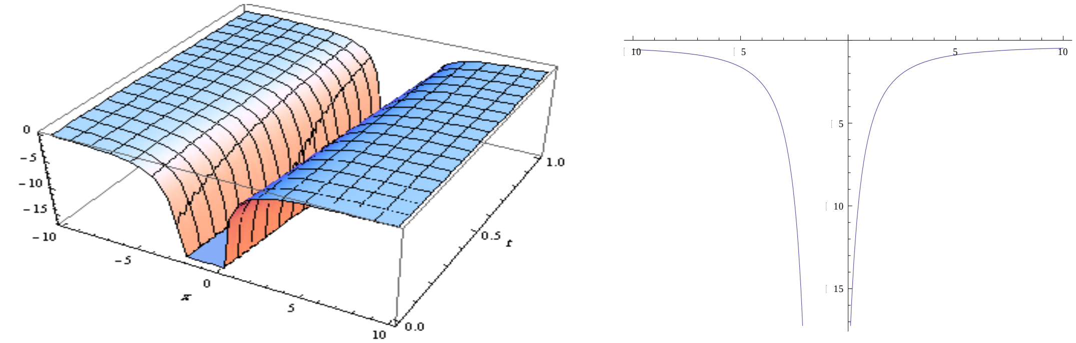


Figure 3. The traveling wave solution (Equation 61) for nonlinear strain wave Equation (Equation 16) when $\alpha_1 = 1.5,\ \lambda_1 = 2,\ \lambda_3 = -3,\ \lambda_4 = 1,\ V = 2,\ \beta = -2.5$ and $\eta_0 = -1.$

$$\xi_1 = -2(\alpha_1 + \alpha_2)\alpha_1\alpha_2\xi_4, \qquad \xi_2 = \xi_4(\alpha_1^2 + 4\alpha_1\alpha_2 + \alpha_2^2), \qquad \xi_3 = -2(\alpha_1 + \alpha_2)\xi_4,$$
$$k = \frac{\beta^2(\alpha_1^4 + \alpha_2^4 - 4\alpha_1\alpha_2^3 + 6\alpha_1^2\alpha_2^2 - 4\alpha_2\alpha_1^3)(\lambda_4 V^2 - \lambda_3)^2 + 2V^2 - V^4 - 1}{4\beta\lambda_1}, \qquad \zeta_0 = \xi_4, \tag{64}$$

where ξ_4 is an arbitrary constant and

$$\pm(\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)(Y - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2} \ln\left|\frac{Y - \alpha_1}{Y - \alpha_2}\right|. \tag{65}$$

or

$$Y = \frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}. \tag{66}$$

Substituting Equations 66, 64 and 63 into Equation 52, we get the traveling wave solution of the strain wave Equation 16 takes the form:

$$u_7(\xi) = -\frac{\beta(\lambda_4 V^2 - \lambda_3)(\alpha_1^2 + \alpha_2^2 + 10\alpha_1\alpha_2) - V^2 + 1}{2\beta\lambda_1} + \frac{6(\lambda_4 V^2 - \lambda_3)(\alpha_1 + \alpha_2)}{\lambda_1}\left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}\right] - \frac{6(\lambda_4 V^2 - \lambda_3)}{\lambda_1}\left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(x - Vt - \eta_0)}}\right]^2. \tag{67}$$

The behavior of the exact Solution 67 has been illustrated in Figure 4.

Family 7. If Equation 55 has four different roots α_1, α_2, α_3 and α_4, consequently we can write Equation 55 in the following form:

$$Y^4 + \frac{2\tau_1}{\tau_2}Y^3 - \frac{3[2\tau_2(V^2 - 1) - \beta\lambda_1(4\tau_0\tau_2 + \tau_1^2)]}{4\tau_2^2\beta\lambda_1}Y^2 - \frac{\tau_1[6\tau_2(V^2 - 1) + \beta\lambda_1(-12\tau_0\tau_2 + \tau_1^2)]}{4\tau_2^3\beta\lambda_1}Y + \frac{\xi_0}{\xi_4} - (Y - \alpha_1)(Y - \alpha_2)(Y - \alpha_3)(Y - \alpha_4) = 0. \tag{68}$$

From equating the coefficients of Y to both sides of Equation 68, we get a system of algebraic equations in $\xi_0, \xi_4, \tau_0, \tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$\xi_0 = -\xi_4\alpha_2\alpha_3\alpha_4(\alpha_2 - \alpha_3 - \alpha_4), \qquad \alpha_1 = -\alpha_2 + \alpha_3 + \alpha_4,$$
$$\tau_0 = \frac{\beta(\lambda_4 V^2 - \lambda_3)(4\alpha_2^2 - \alpha_3^2 - \alpha_4^2 - 4\alpha_2\alpha_3 - 4\alpha_2\alpha_4 - 6\alpha_3\alpha_4) + V^2 - 1}{2\beta\lambda_1},$$
$$\tau_1 = \frac{6(\lambda_4 V^2 - \lambda_3)(\alpha_3 + \alpha_4)}{\lambda_1}, \quad \tau_2 = -\frac{6(\lambda_4 V^2 - \lambda_3)}{\lambda_1}. \tag{69}$$

Equations 69, 53 and 54 lead to:

$$\xi_1 = (\alpha_3 + \alpha_4)(-\alpha_3\alpha_4 + \alpha_2^2 - \alpha_2\alpha_4 - \alpha_2\alpha_3)\zeta_0,$$
$$\xi_2 = \zeta_0(3\alpha_4\alpha_3 + \alpha_2\alpha_3 + \alpha_3^2 + \alpha_2\alpha_4 + \alpha_4^2 - \alpha_2^2), \quad \xi_3 = -2(\alpha_3 + \alpha_4)\zeta_0, \quad \xi_4 = \zeta_0,$$
$$k = \frac{1}{4\beta\lambda_1}\Big[\beta^2(\alpha_3^4 + \alpha_4^4 + 16\alpha_2^4 + 56\alpha_2^2\alpha_3\alpha_4 - 28\alpha_2\alpha_3\alpha_4^2 - 28\alpha_2\alpha_3^2\alpha_4 - 32\alpha_2^3\alpha_3 - 32\alpha_2^3\alpha_4 + 20\alpha_2^2\alpha_3^2 + 20\alpha_2^2\alpha_4^2 - 4\alpha_2\alpha_3^3 - 4\alpha_2\alpha_4^3 + 14\alpha_4^2\alpha_3^2)(\lambda_4 V^2 - \lambda_3)^2 + 2V^2 - V^4 - 1\Big], \tag{70}$$

where ξ_4 is an arbitrary constant and

$$\pm(\xi - \eta_0) = \int \frac{dY}{\sqrt{(Y - (-\alpha_2 + \alpha_3 + \alpha_4))(Y - \alpha_2)(Y - \alpha_3)(Y - \alpha_4)}} = \frac{2i}{(\alpha_2 - \alpha_4)} EllipticF\left[\sqrt{\frac{(\alpha_4 - \alpha_2)(Y - \alpha_4)}{(\alpha_3 - \alpha_2)(Y - \alpha_3)}}, \frac{(\alpha_2 - \alpha_3)}{(\alpha_2 - \alpha_4)}\right]. \tag{71}$$

or

$$Y = \frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 - \alpha_3^2)sn^2\left(\frac{i}{2}(\alpha_2 - \alpha_4)(x - Vt - \eta_0), \frac{(\alpha_2 - \alpha_3)}{(\alpha_2 - \alpha_4)}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 - \alpha_3)sn^2\left(\frac{i}{2}(\alpha_2 - \alpha_4)(x - Vt - \eta_0), \frac{(\alpha_2 - \alpha_3)}{(\alpha_2 - \alpha_4)}\right)}. \tag{72}$$

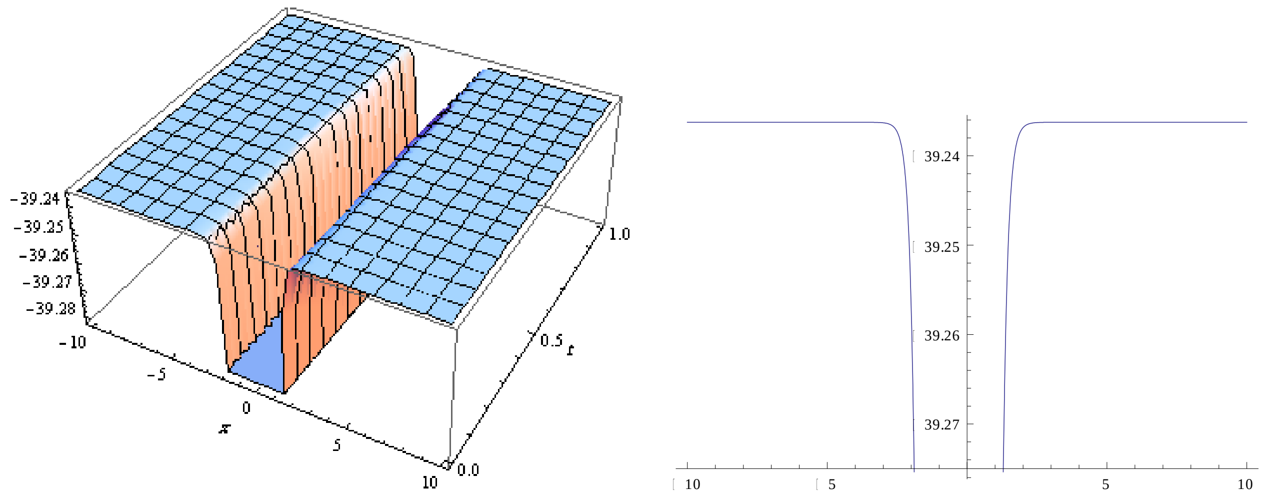


Figure 4. The traveling wave solution (Equation 67) for nonlinear strain wave Equation (Equation 16) at $\alpha_1=-0.9,\ \alpha_2=5,\ \lambda_1=2,\ \lambda_3=-3,\ \lambda_4=6,\ V=0.5,\ \beta=2.5$ and $\eta_0=-0.3$.

Substituting Equations 72, 70 and 69 into Equation 52, we get the traveling wave solution of nonlinear strain wave Equation 16 takes the form:

$$u_8(\xi)=\frac{\beta(\lambda_4V^2-\lambda_3)(4\alpha_2^2-\alpha_3^2-\alpha_4^2-4\alpha_2\alpha_3-4\alpha_2\alpha_4-6\alpha_3\alpha_4)+V^2-1}{2\beta\lambda_1}$$
$$+\frac{6(\lambda_4V^2-\lambda_3)(\alpha_3+\alpha_4)}{\lambda_1}\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)sn^2\left(\frac{i}{2}(\alpha_2-\alpha_4)(x-Vt-\eta_0),\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)sn^2\left(\frac{i}{2}(\alpha_2-\alpha_4)(x-Vt-\eta_0),\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right)}\right]$$
$$-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}\left[\frac{\alpha_4^2-\alpha_2\alpha_4+(\alpha_2\alpha_3-\alpha_3^2)sn^2\left(\frac{i}{2}(\alpha_2-\alpha_4)(x-Vt-\eta_0),\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right)}{\alpha_4-\alpha_2+(\alpha_2-\alpha_3)sn^2\left(\frac{i}{2}(\alpha_2-\alpha_4)(x-Vt-\eta_0),\frac{(\alpha_2-\alpha_3)}{(\alpha_2-\alpha_4)}\right)}\right]^2. \quad (73)$$

Family 8. If Equation 55 has four complex roots, $\alpha_1=N_1+iN_2$, $\alpha_2=N_1-iN_2$, $\alpha_3=N_3+iN_4$ and $\alpha_4=N_3-iN_4$, $N_j,\ j=1\ldots,4$ are real numbers, consequently we can write Equation 55 in the following form:

$$Y^4+\frac{2\tau_1}{\tau_2}Y^3-\frac{3[2\tau_2(V^2-1)-\beta\lambda_1(4\tau_0\tau_2+\tau_1^2)]}{4\tau_2^2\beta\lambda_1}Y^2-\frac{\tau_1[6\tau_2(V^2-1)+\beta\lambda_1(-12\tau_0\tau_2+\tau_1^2)]}{4\tau_2^3\beta\lambda_1}Y$$
$$+\frac{\xi_0}{\xi_4}-(Y-(N_1+iN_2))(Y-(N_1-iN_2))(Y-(N_3+iN_4))(Y-(N_3-iN_4))=0. \quad (74)$$

From equating the coefficients of Y to both sides of Equation 74, we get a system of algebraic equations in $\xi_0,\xi_4,\tau_0,\tau_1$ and τ_2 which can be solved by using the Maple software package to get the following results:

$$N_1=N_3,\quad \xi_0=\xi_4(N_3^2N_4^2+N_2^2N_3^2+N_2^2N_4^2+N_3^4),\ \tau_1=\frac{12N_3(\lambda_4V^2-\lambda_3)}{\lambda_1}, \quad (75)$$
$$\tau_0=-\frac{\beta(\lambda_4V^2-\lambda_3)(12N_3^2+4N_4^2+4N_2^2)-V^2+1}{2\beta\lambda_1},\ \tau_2=-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}.$$

Equations 75, 66 and 67 lead to get:

$$\xi_1=-2N_3(2N_3^2+N_2^2+N_4^2)\xi_4,\ \xi_2=(6N_3^2+N_4^2+N_2^2)\xi_4,\quad \xi_3=-4N_3\xi_4,\ \zeta_0=\xi_4,$$
$$k=\frac{1}{4\beta\lambda_1}\left[\beta^2(16N_2^4-16N_2^2N_4^2+16N_4^4)(\lambda_4V^2-\lambda_3)^2+2V^2-V^4-1\right], \quad (76)$$

where ξ_4 is an arbitrary constant and

$$\pm(\xi-\eta_0)=\int\frac{dY}{\sqrt{(Y^2-2N_3Y+N_3^2+N_2^2)(Y^2-2N_3Y+N_3^2+N_4^2)}} \quad (77)$$
$$=\frac{2}{(N_2-N_4)}\mathit{EllipIticF}\left[\sqrt{\frac{(N_2-N_4)(Y-N_3-iN_4)}{(N_2+N_4)(Y-N_3+iN_4)}},\frac{(N_2+N_4)}{(N_2-N_4)}\right],$$

then

$$Y=\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}{(N_4-N_2)+(N_4+N_2)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}. \quad (78)$$

Substituting Equations 78, 76 and 75 into Equations 52, we get the traveling wave solution of the strain wave Equation 16 taking the form:

$$u_9(\xi)=-\frac{\beta(\lambda_4V^2-\lambda_3)(12N_3^2+4N_4^2+4N_2^2)-V^2+1}{2\beta\lambda_1}$$
$$+\frac{12N_3(\lambda_4V^2-\lambda_3)}{\lambda_1}\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}{(N_4-N_2)+(N_4+N_2)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}\right]$$
$$-\frac{6(\lambda_4V^2-\lambda_3)}{\lambda_1}\left[\frac{(N_4-N_2)(N_3+iN_4)+(N_4+N_2)(N_3-iN_4)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}{(N_4-N_2)+(N_4+N_2)sn^2\left(\frac{1}{2}(N_2-N_4)(\xi-\eta_0),\frac{(N_2+N_4)}{(N_2-N_4)}\right)}\right]^2. \quad (79)$$

RESULTS AND DISCUSSION

This method allowed the construction of many types of the traveling wave solutions in the hyperbolic functions, trigonometric functions, and Jacobian elliptic functions. The balance number of this method is not constant as in other methods but changes when the trial equation changes. This method has generalized the tanh-function method, Jacobian elliptic functions methods, and Exp function method.

Conclusion

In this paper, the modified extended trial equation method was used to construct series of some new analytic solutions for some nonlinear partial differential equations in mathematical physics when the balance numbers is positive integer. The exact solutions were constructed in many different functions such as hyperbolic function solutions, trigonometric function solutions and Jacobi elliptic functions solutions and rational solutions for nonlinear strain wave equation. The performance of this method is reliable, effective and powerful for solving more complicated nonlinear partial differential equations in mathematical physics. This method is more powerful than other method for solving the nonlinear partial differential equations. This method can be used to solve many nonlinear partial differential equations in mathematical physics.

Conflict of Interests

The authors have not declared any conflict of interests.

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