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Full Length Research Paper

A derivation of the Kerr metric by ellipsoid coordinate transformation

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Einstein's general relativistic field equation is a nonlinear partial differential equation that lacks an easy way to obtain exact solutions. The most famous of which are Schwarzschild and Kerr's black hole solutions. Kerr metric has astrophysical meaning because most cosmic celestial bodies rotate. Kerr metric is even harder than Schwarzschild metric to be derived directly due to off-diagonal term of metric tensor. In this paper, a derivation of Kerr metric was obtained by ellipsoid coordinate transformation which causes elimination of large amount of tedious derivation. This derivation is not only physically enlightening, but also further deducing some characteristics of the rotating black hole.

Key words: General relativity, Schwarzschild metric, Kerr metric, ellipsoid coordinate transformation, exact solutions.

INTRODUCTION

The theory of general relativity proposed by Albert Einstein in 1915 was one of the greatest advances in modern physics. It describes the distribution of matter to determine the space-time curvature, and the curvature determines how the matter moves. Einstein's field equation is very simple and elegant, but based on the fact that the Einstein' field equation is a set of nonlinear differential equations, it has proved difficult to find the exact analytic solution. The exact solution has physical meanings only in some simplified assumptions, the most famous of which is Schwarzschild and Kerr's black hole solution, and Friedman's solution to cosmology. One year after Einstein published his equation, the spherical symmetry, static vacuum solution with center singularity was found by Schwarzschild (Schwarzschild, 1916). Nearly 50 years later, the fixed axis symmetric rotating black hole was solved in 1963 by Kerr (Kerr, 1963). Some of these exact solutions have been used to explain topics related to the gravity in cosmology, such as Mercury's precession of the perihelion, gravitational lens, black hole, expansion of the universe and gravitational waves.

Today, many solving methods of Einstein field equations are proposed. For example: Pensose-Newman's method (Penrose and Rindler, 1984) or Bäcklund transformations (Kramer et al., 1981). Despite their great success in dealing with the Einstein equation, these methods are technically complex and expert-oriented.

The Kerr solution is important in astrophysics because most cosmic celestial bodies are rotating and are rarely completely at rest. Traditionally, the general method of

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the Kerr solution can be found here, *The Mathematical Theory of Black Holes*, by the classical works of Chandrasekhar (1983). However, the calculation is so lengthy and complicated that the College or Institute students also find it difficult to understand. Recent literature review showed that it is possible to obtain Kerr metric through the oblate spheroidal coordinate's transformation (Enderlein, 1997). This encourages me to look for a more concise way to solve the vacuum solution of Einstein's field equation.

The motivation of this derivation simply came from the use of relatively simple way of solving Schwarzschild metric to derive the Kerr metric, which can make more students to be interested in physics for the general relativity of the exact solution to self-deduction.

In this paper, a more enlightening way to find this solution was introduced, not only simple and elegant, but also further deriving some of the physical characteristics of the rotating black hole.

SCHWARZSCHILD AND KERR SOLUTIONS

The exact solution of Einstein field equation is usually expressed in metric. For example, Minkowski space-time is a four-dimension coordinates combining three-dimensional Euclidean space and one-dimension time can be expressed in Cartesian form as shown in Equation 1. In all physical quality, we adopt *c* = *G* = 1.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

and in polar coordinate form in Equation 2:

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2}$$

Schwarzschild employed a non-rotational sphere-symmetric object with polar coordinate in Equation 2 with two variables from functions v(r), λ(r), which is shown in Equation 3:

$$ds^2 = e^{2v(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{3}$$

In order to solve the Einstein field equation, Schwarzschild used a vacuum condition, let $R_{\mu\nu} = 0$, to calculate Ricci tensor from Equation 3, and got the first exact solution of the Einstein field equation, Schwarzschild metric, which is shown in Equation 4 (Schwarzschild, 1916):

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - d\Omega^2$$
$$d\Omega^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4}$$

However, Schwarzschild metric cannot be used to describe rotation, axial-symmetry and charged heavenly bodies. From the examination of the metric tensor $g_{\mu\nu}$ in Schwarzschild metric, one can obtain the components:

$$g_{00} = 1 - \frac{2M}{r}, g_{11} = -\left(1 - \frac{2M}{r}\right)^{-1},$$
$$g_{22} = -r^2, g_{33} = -r^2 \sin^2\theta$$

Which can also be presented as:

$$g_{tt} = 1 - \frac{2M}{r}, g_{rr} = -\left(1 - \frac{2M}{r}\right)^{-1},$$
$$g_{\theta\theta} = -r^2, g_{\phi\phi} = -r^2 \sin^2\theta \tag{5}$$

Differences of metric tensor $g_{\mu\nu}$ between the Schwarzschild metric in Equation 4 and Minkowski space-time in Equation 2 are only in time-time terms (g_{tt}) and radial-radial terms (g_{rr}).

Kerr metric is the second exact solution of Einstein field equation, which can be used to describe space-time geometry in the vacuum area near a rotational, axial-symmetric heavenly body (Kerr, 1963). It is a generalized form of Schwarzschild metric. Kerr metric in Boyer-Lindquist coordinate system can be expressed in Equation 6:

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4Mra \sin^2\theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2$$
$$-\rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2 \tag{6}$$

Where $\rho^2 \equiv r^2 + a^2 \cos^2\theta$ and $\Delta \equiv r^2 - 2Mr + a^2$

M is the mass of the rotational material, a is the spin parameter or specific angular momentum and is related to the angular momentum J by $a = J/M$.

By examining the components of metric tensor $g_{\mu\nu}$ in Equation 6, one can obtain:

$$g_{00} = 1 - \frac{2Mr}{\rho^2}, g_{11} = -\frac{\rho^2}{\Delta}, g_{22} = -\rho^2,$$
$$g_{03} = g_{30} = \frac{2Mra \sin^2\theta}{\rho^2}$$
$$g_{33} = -\left(r^2 + a^2 + \frac{2Mra^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta \tag{7}$$

Comparison of the components of Schwarzschild metric Equation 4 with Kerr metric Equation 6:

Both $g_{03}(g_{t\phi})$ and $g_{30}(g_{\phi t})$ off-diagonal terms in Kerr metric are not present in Schwarzschild metric, apparently due to rotation. If the rotation parameter $a = 0$, these two terms vanish.

$g_{00}g_{11} = g_{tt}g_{rr} = -1$ in Schwarzschild metric, but not in Kerr metric when spin parameter $a = 0$, Kerr metric turns into Schwarzschild metric and therefore is a generalized form of Schwarzschild metric.

TRANSFORMATION OF ELLIPSOID SYMMETRIC ORTHOGONAL COORDINATE

To derive Kerr metric, if we start from the initial assumptions, we must introduce g_{00}, g_{11}, g_{22}, g_{03}, g_{33} (five variables) all are function of (r, θ), and finally we will get monster-like complex equations. Apparently, due to the off-diagonal term, Kerr metric cannot be solved by the spherical symmetry method used in Schwarzschild metric.

Different from the derivation methods used in classical works of Chandrasekhar (1983), the author used the changes in coordinate of Kerr metric into ellipsoid symmetry firstly to get a simplified form, and then used Schwarzschild's method to solve Kerr metric. First of all, the following ellipsoid coordinate changes were apply to Equation 1 (Landau and Lifshitz, 1987):

$$\begin{aligned} &x \to (r^2+a^2)^{\frac{1}{2}} \sin\theta \cos\phi, \\ &y \to (r^2+a^2)^{\frac{1}{2}} \sin\theta \sin\phi, \\ &z \to r\cos\theta, \\ &t \to t \end{aligned} \tag{8}$$

Where a is the coordinate transformation parameter. The metric under the new coordinate system becomes Equation 9:

$$ds^2 = dt^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\phi^2 \tag{9}$$

Equation 9 has physics significance, which represents the coordinate with ellipsoid symmetry in vacuum, it can also be obtained by assigning mass $M = 0$ to the Kerr metric in Equation 6. Due to the fact that most of the celestial bodies, stars and galaxy for instance, are ellipsoid symmetric, Bijan started from this vacuum ellipsoid coordinate and derived Schwarzschild-like solution for ellipsoidal celestial objects following Equation 10 (Bijan, 2011):

$$\begin{aligned} ds^2 = &\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}\frac{\rho^2}{r^2+a^2}dr^2 \\ &-\rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\varphi^2 \end{aligned} \tag{10}$$

Equation 10 morphs into the Schwarzschild's solution in Equation 4 when the coordinate transformation parameter $a = 0$ and therefore Equation 10 is also a generalization of Schwarzschild's solution.

In order to eliminate the difference between Kerr metric and Schwarzschild metric described earlier, we can assume to rewrite the Kerr metric in the following coordinates:

$$ds^2 = G'_{00}dT^2 + G'_{11}dr^2 + G'_{22}d\theta^2 + G'_{33}d\emptyset^2 \tag{11}$$

To eliminate the off-diagonal term:

$$dT \equiv dt - p d\varphi,\ d\emptyset \equiv d\varphi - q dt \tag{12}$$

to obtain

$$G'_{00}G'_{11} = -1 \tag{13}$$

By comparing the coefficient, Equations 14 to 18 were obtained.

$$G'_{00}p + G'_{33}q = \frac{-2Mra\sin^2\theta}{\rho^2} \tag{14}$$

$$G'_{00} + G'_{33}q^2 = 1 - \frac{2Mr}{\rho^2} \tag{15}$$

$$G'_{00}p^2 + G'_{33} = -\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta \tag{16}$$

$$G'_{22} = -\rho^2 \tag{17}$$

$$G'_{11} = -\frac{\rho^2}{\Delta} \tag{18}$$

By solving six variables $G'_{00}, G'_{11}, G'_{22}, G'_{33}, p, q$ in the six dependent Equations 13 to 18, the results shown in Equation (19) were obtained:

$$p = \pm a\sin^2\theta\text{, take the positve result}$$

$$q = \pm\frac{a}{r^2+a^2}\text{, take the positive result}$$

$$\begin{aligned} &G'_{00} = \frac{\Delta}{\rho^2}, G'_{11} = -\frac{\rho^2}{\Delta}, G'_{22} = -\rho^2 \\ &G'_{33} = -\frac{(r^2+a^2)^2\sin^2\theta}{\rho^2} \end{aligned} \tag{19}$$

Put them into Equation 8 and obtain Equation 20:

$$\begin{aligned} ds^2 = &\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \\ &-\frac{(r^2+a^2)^2\sin^2\theta}{\rho^2}\left(d\varphi - \frac{a}{r^2+a^2}dt\right)^2 \end{aligned} \tag{20}$$

Equation 20 can be found in the literature and also textbook by O'Neil. It is also called Kerr metric with Boyer-Lindquist in orthonormal frame (O'Neil, 1995). There is no off-diagonal terms, and $g_{00}g_{11} = -1$ after the coordinate transformation.

CALCULATING THE RICCI TENSOR

From pervious discussion, Equation 9 can be recognized as the coordinate under the ellipsoid symmetry in vacuum. Therefore, when the mass M approached 0, Kerr metric Equation 20 will also be transformed into Equation 21, which equals Equation 9. The differences of metric tensor components are in time-time and radial-radial terms, just the same as between Schwarzschild metric (Equation 4) and Minkowski space-time (Equation 2). dT and $d\emptyset$ defined in Equation 22 are ellipsoid coordinate transformation functions.

$$ds^2 = \frac{r^2+a^2}{\rho^2}dT^2 - \frac{\rho^2}{r^2+a^2}dr^2 - \rho^2 d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\rho^2}d\emptyset^2 \quad (21)$$

$$dT \equiv dt - a\sin^2\theta\, d\varphi$$
$$d\emptyset \equiv d\varphi - \frac{a}{r^2+a^2}dt \quad (22)$$

In this paper, Schwarzschild method was used to solve Kerr metric starting from Equations 21 to 22 by introducing two new functions $e^{2v(r,\theta)}, e^{2\lambda(r,\theta)}$:

$$ds^2 =$$
$$e^{2v(r,\theta)}dT^2 - e^{2\lambda(r,\theta)}dr^2 - \rho^2 d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\rho^2}d\emptyset^2 \quad (23)$$

Define the parameters ρ^2 and h in Equation (24):

$$\rho^2 \equiv r^2 + a^2 cos^2\theta$$
$$h \equiv r^2 + a^2 \quad (24)$$

Metric tensor in the matrix form shown in Equations 25 to 26:

$$g_{\mu\nu} = \begin{pmatrix} e^{2v(r,\theta)} & 0 & 0 & 0 \\ 0 & -e^{2\lambda(r,\theta)} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -\frac{h^2 sin^2\theta}{\rho^2} \end{pmatrix} \quad (25)$$

$$g^{\mu\nu} = \begin{pmatrix} e^{-2v(r,\theta)} & 0 & 0 & 0 \\ 0 & -e^{-2\lambda(r,\theta)} & 0 & 0 \\ 0 & 0 & -\rho^{-2} & 0 \\ 0 & 0 & 0 & -\frac{\rho^2}{h^2 sin^2\theta} \end{pmatrix} \quad (26)$$

Chrostoffel symbols can be obtained by the following steps in Equation 27:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \quad (27)$$

Non-zero Chrostoffel symbols are listed in Equation 28 to 37:

$$\Gamma^1_{00} = e^{2(v-\lambda)}\partial_1 \nu \quad (28)$$

$$\Gamma^1_{11} = \partial_1\lambda \quad (29)$$

$$\Gamma^0_{10} = \Gamma^0_{01} = \partial_1\nu \quad (30)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{r}{\rho^2} \quad (31)$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{2r}{h} - \frac{r}{\rho^2} \quad (32)$$

$$\Gamma^1_{22} = -re^{-2\lambda} \quad (33)$$

$$\Gamma^3_{32} = \Gamma^3_{23} = cot\,\theta\left(\frac{h}{\rho^2}\right) \quad (34)$$

$$\Gamma^1_{33} = -re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right) \quad (35)$$

$$\Gamma^2_{22} = -\frac{h^2 \sin\theta\cos\theta}{\rho^2} \quad (36)$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta\left(\frac{h^3}{\rho^6}\right) \quad (37)$$

The calculation of Ricci curvature tensor can be derived by the following (Equation 38), and the results are listed in Equations 39 to 50:

$$R_{\alpha\beta} = R^{\rho}_{\alpha\rho\beta} = \partial_\rho\Gamma^\rho_{\beta\alpha} - \partial_\beta\Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\rho\alpha} \quad (38)$$

$$R^0_{101} = \partial_0\Gamma^0_{11} - \partial_1\Gamma^0_{01} + \Gamma^0_{0\lambda}\Gamma^\lambda_{11} - \Gamma^0_{1\lambda}\Gamma^\lambda_{01}$$
$$= \partial_1\nu\,\partial_1\lambda - (\partial_1\nu)^2 - \partial_1^2\nu \quad (39)$$

$$R^0_{202} = \partial_0\Gamma^0_{22} - \partial_2\Gamma^0_{02} + \Gamma^0_{0\lambda}\Gamma^\lambda_{22} - \Gamma^0_{2\lambda}\Gamma^\lambda_{02} = -re^{-2\lambda}\partial_1\nu \quad (40)$$

$$R^0_{303} = \partial_0\Gamma^0_{33} - \partial_3\Gamma^0_{03} + \Gamma^0_{0\lambda}\Gamma^\lambda_{33} - \Gamma^0_{3\lambda}\Gamma^\lambda_{03}$$
$$= -re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)\partial_1\nu \quad (41)$$

$$R^1_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1\lambda}\Gamma^\lambda_{22} - \Gamma^1_{2\lambda}\Gamma^\lambda_{12}$$
$$= e^{-2\lambda}\left(r\partial_1\lambda - 1 + \frac{r^2}{\rho^2}\right) \quad (42)$$

$$R^1_{313} = \partial_1\Gamma^1_{33} - \partial_3\Gamma^1_{13} + \Gamma^1_{1\lambda}\Gamma^\lambda_{13} - \Gamma^1_{3\lambda}\Gamma^\lambda_{13}$$
$$= re^{-2\lambda}sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)(\partial_1\lambda) \quad (43)$$

$$R^2_{323} = \partial_2\Gamma^2_{33} - \partial_3\Gamma^2_{23} + \Gamma^2_{2\lambda}\Gamma^\lambda_{33} - \Gamma^2_{3\lambda}\Gamma^\lambda_{23}$$
$$= sin^2\theta\left[\frac{h^4}{\rho^8}\left(\frac{5r^2-4\rho^2}{h}\right) - \frac{r^2h}{\rho^4}\left(2 - \frac{h}{\rho^2}\right)e^{-2\lambda}\right] \quad (44)$$

$$R^1_{010} = g^{11}g_{00}R^0_{101} = e^{2(v-\lambda)}\left[-\partial_1\nu\,\partial_1\lambda + (\partial_1\nu)^2 + \partial_1^2\nu\right] \quad (45)$$

$$R^2_{020} = g^{22}g_{00}R^0_{202} = e^{2(v-\lambda)}\frac{r}{\rho^2}\partial_1\nu \quad (46)$$

$$R^3_{030} = g^{33}g_{00}R^0_{303} = e^{2(v-\lambda)}\left(\frac{2r}{h} - \frac{r}{\rho^2}\right)\partial_1\nu \quad (47)$$

$$R^2_{121} = g^{22}g_{11}R^1_{212} = \frac{1}{\rho^2}\left(r\partial_1\lambda + \frac{r^2}{\rho^2} - 1\right) \quad (48)$$

$$R^3_{131} = g^{33}g_{11}R^1_{313} = \left(\frac{2}{h} - \frac{1}{\rho^2}\right)\left(r\partial_1\lambda + \frac{2r^2}{\rho^2} - \frac{2r^2}{h}\right) \quad (49)$$

$$R^3_{232} = g^{33}g_{22}R^2_{323} = \left[\frac{h^2}{\rho^4}\left(\frac{5r^2-4\rho^2}{h}\right) - \frac{r^2}{h}\left(2 - \frac{h}{\rho^2}\right)e^{-2\lambda}\right] \quad (50)$$

$R_{\mu\nu}$ can be calculated by the Equations 51 to 54:

$$R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = e^{2(\nu\text{-}\lambda)}\left[-\partial_1\nu\,\partial_1\lambda + (\partial_1\nu)^2 + \partial_1^2\nu + \frac{2r}{h}\partial_1\nu\right] \quad (51)$$

$$R_{11} = R^0_{101} + R^2_{121} + R^3_{131} = \partial_1\nu\,\partial_1\lambda - (\partial_1\nu)^2 - \partial_1^2\nu + \frac{2r}{h}\partial_1\lambda \quad (52)$$

$$R_{22} = R^0_{202} + R^1_{212} + R^3_{232} = e^{\text{-}2\lambda}\left(r(\partial_1\lambda - \partial_1\nu) - 1 + \frac{2r^2}{\rho^2} - \frac{2r^2}{h}\right) + \frac{h^2}{\rho^4}\left(\frac{5r^2-4\rho^2}{h}\right) \quad (53)$$

$$R_{33} = R^0_{303} + R^1_{313} + R^2_{323} = sin^2\theta\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)\left[e^{\text{-}2\lambda}\left(r(\partial_1\lambda - \partial_1\nu) - \frac{r^2}{\rho^2}\right) + \frac{h^2}{\rho^4}\left(\frac{5r^2-4\rho^2}{h}\right)\left(\frac{2h}{\rho^2} - \frac{h^2}{\rho^4}\right)^{-1}\right] \quad (54)$$

FINDING A SOLUTION OF THE VACCUM EINSTEIN FIELD EQUATIONS

To solve vacuum Einstein's field equations, first, the Ricci tensor was set to zero, which means: $R_{\mu\nu} = 0, R = 0,$ in the empty space, θ is approximately constant. Then combine with R_{00} and R_{11} to get Equation 55, and solve the equation, Equations 56 to 58 were obtained:

$$e^{\text{-}2(\nu\text{-}\lambda)}\mathrm{R}_{00} + \mathrm{R}_{11} = \frac{2r}{h}(\partial_1\nu + \partial_1\lambda) = 0 \quad (55)$$

$$\partial_1\nu + \partial_1\lambda = \partial_1(\nu + \lambda) = 0 \quad (56)$$

$$\nu = -\lambda + c,\ \nu(r,) = -\lambda(r,) + c \quad (57)$$

$$e^{\nu} = e^{\text{-}\lambda} \quad (58)$$

To solve this partial differential equation, one has to remember that when the angular momentum approaches zero ($a \to 0$), the Kerr metric Equation 6 turns into Schwarzschild metric (Equation 4). Then Equation 59 to 61 were obtained:

$$\lim_{a\to 0} R_{33} = sin^2\theta\left[e^{\text{-}2\lambda}(r(\partial_1\lambda - \partial_1\nu) - 1) + 1\right] = sin^2\theta R_{22} \quad (59)$$

$$\lim_{a\to 0} R_{22} = e^{\text{-}2\lambda}(r(\partial_1\lambda - \partial_1\nu) - 1) + 1 = e^{2\nu}\left(-2r\,\partial_1\nu - 1\right) + 1 = 0 \quad (60)$$

$$e^{2\nu} = 1 + \frac{C}{r}, let\ C = -2M \quad (61)$$

So under the limit condition of angular momentum approaching zero ($a \to 0$), the equations could be solved as shown in Equation 62:

$$\lim_{a\to 0} e^{2\nu} = 1 - \frac{2M}{r} = \frac{r^2 - 2Mr}{r^2} \quad (62)$$

One could also demand the other limit condition of flat space-time, where the mass approaches zero ($\mathrm{M} \to 0$) in the Equation 18, which could be represented as in Equation 63:

$$\lim_{M\to 0} e^{2\nu} = \frac{r^2+a^2}{\rho^2} = \frac{r^2+a^2}{r^2+a^2\cos^2\theta} \quad (63)$$

Deduced from the above conditions in Equations 62 to 63, the equations of Ricci tensor could be solved as in Equation 64:

$$e^{2\nu} = \frac{r^2-2Mr+a^2}{r^2+a^2\cos^2\theta} \qquad e^{2\lambda} = \frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2} \quad (64)$$

Finally, the Kerr metric was gotten as shown in Equation 65:

$$ds^2 = \frac{r^2-2Mr+a^2}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \frac{\rho^2}{r^2-2Mr+a^2}dr^2 - \rho^2 d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\rho^2}\left(d\varphi - \frac{a}{r^2+a^2}dt\right)^2 \quad (65)$$

DISCUSSION

It is proved that Kerr metric equation (Equation 65) can be obtained by combining the ellipsoid coordinate transformation and the assumptions listed in Equations 21 to 23 following these steps: transforming the Euclidian four-dimension space-time in Equation 1 to vacuum Minkowski space-time with ellipsoid symmetry in Equation 9; transforming from (t, r, θ, ϕ) to $(T, r, \theta, \emptyset)$ under the new coordinate system to eliminate the major difference in metric tensor components between Kerr metric and Schwarzschild metric, and the product of $dtd\phi$ and $g_{00}g_{11}$ becomes -1; solving vacuum Einstein's equation by using Schwarzschild method from Equation 23; applying limit method to calculate Ricci curvature tensor; and finally deducting Kerr metric.

Table 1 shows the list of the metric tensor components discussed in previous sections, including the Minkowski space-time, the Schwarzschild solution, empty ellipsoid, a Schwarzschild-like ellipsoid solution and the Kerr solution. The Minkowski space-time and the Schwarzschild solution have spherical symmetry, and the others have ellipsoid symmetry.

Further, some of characteristics with deeper physics meaning of ellipsoid symmetry, Kerr metric and rotating black hole can be obtained from this new coordinate function $\mathrm{d}T, d\emptyset$. Remember when a approaches to zero $(a \to 0)$, $\mathrm{d}T, d\emptyset$ degenerate to $\mathrm{d}t, d\phi$.

Table 1. Metric tensor components and symmetry.

Metric tensor	$dt^2(dT^2)$	dr^2	$d\theta^2$	$d\phi^2(d\Phi^2)$	Symmetry and state
Minkowski	*1*	*-1*	$-r^2$	$-r^2 sin^2\theta$	Spherical, empty
Schwarzschild	$\frac{r^2-2M}{r^2}$	$-\frac{r^2}{r^2-2M}$	$-r^2$	$-r^2 sin^2\theta$	Spherical, static, mass
Ellipsoid	$\frac{r^2+a^2}{\rho^2}$	$-\frac{\rho^2}{r^2+a^2}$	$-\rho^2$	$-\frac{(r^2+a^2)^2 sin^2\theta}{\rho^2}$	Ellipsoid, empty
Schwarzschild-like	$\frac{r^2-2M}{r^2}$	$-\frac{r^2}{r^2-2M}\frac{\rho^2}{r^2+a^2}$	$-\rho^2$	$-(r^2+a^2)sin^2\theta$	Ellipsoid, static, mass
Kerr	$\frac{r^2-2M+a^2}{\rho^2}$	$-\frac{\rho^2}{r^2-2M+a^2}$	$-\rho^2$	$-\frac{(r^2+a^2)^2 sin^2\theta}{\rho^2}$	Ellipsoid, axisymmetric, mass

Ellipsoid symmetry and Kerr metric

While metric with spherical symmetry in vacuum has the following expression:

$$-r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{66}$$

And metric of ellipsoid symmetric in vacuum has the following expression in Equation 67, where $\frac{a}{r^2+a^2}$ and $a\,sin^2\theta$ term can be seen in multiply and divide combination:

$$\begin{aligned}&-\rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\phi^2\\ &-\rho^2 d\theta^2 - \left(\frac{\boldsymbol{r^2+a^2}}{\boldsymbol{a}}\right)(\boldsymbol{a\sin^2\theta})\,d\phi^2\end{aligned} \tag{67}$$

Terms of $d\theta^2, d\phi^2$ in Kerr metric is shown in Equation 68, where $\frac{a}{r^2+a^2}$ and $a\,sin^2\theta$ term can also be seen in linear combination:

$$\begin{aligned}&-\rho^2 d\theta^2 - (r^2+a^2+\frac{2Mra^2\,sin^2\theta}{\rho^2})\,sin^2\theta d\phi^2\\ &-\rho^2 d\theta^2 - (\frac{\boldsymbol{r^2+a^2}}{\boldsymbol{a}}+\frac{2Mr\boldsymbol{a}\,\boldsymbol{sin^2\theta}}{\rho^2})(\boldsymbol{asin^2\theta})\,d\phi^2\end{aligned} \tag{68}$$

The a in Equation 67 represents a parameter in the ellipsoid symmetric coordinate transformation, however, *a* in Kerr metric Equation (Equation 68) represents a spin parameter, which is proportional to angular momentum. Both $a's$ are equivalent in mathematic perspective and used to transform the space-time into ellipsoid symmetry with a rotational symmetric z-axis. As $a \to 0$, both Equation (67) and Equation (68) degenerate into spherical symmetry equation (Equation 66).

Frame-dragging angular momentum

In physics, a spinning heavenly body with a non-zero mass will generate a frame-dragging phenomenon along the equator's direction, which has been proven by Gravity Probe B experiment (Everitt et al., 2011). Therefore, an extra term $\frac{2Mra^2\sin^2\theta}{\rho^2}$ in Kerr metric is found in Equation 68 as compared to the vacuum ellipsoid symmetry in Equation 67. As the mass approaches zero $M \to 0$, Equation 68 degenerates into Equation 67.

In order to describe frame-dragging, Kerr metric can be re-written as Equation 69:

$$\begin{aligned}ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\varphi^2\\ &= \left(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\right)dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}\left(d\varphi + \frac{g_{t\phi}}{g_{\phi\phi}}\right)^2\end{aligned} \tag{69}$$

The definition of angular momentum (Ω) in frame-dragging:

$$\begin{aligned}\Omega &= -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{\frac{2Mra\sin^2\theta}{\rho^2}}{\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta}\\ &= \frac{2Mra}{\rho^2(r^2+a^2)+2Mra^2\sin^2\theta} = \frac{2Mr}{\rho^2\left(\frac{\boldsymbol{r^2+a^2}}{\boldsymbol{a}}\right)+2Mr(\boldsymbol{a\sin^2\theta})}\end{aligned} \tag{70}$$

So, we see both the $a\sin^2\theta$ and $\frac{\boldsymbol{a}}{\boldsymbol{r^2+a^2}}$ term in Ω, which means $\mathrm{d}T, d\emptyset$ would have some relation with frame-dragging angular momentum.

Black hole angular velocity

Its close relationship with the black hole angular velocity (Ω_H) can be easily identified by examining $d\emptyset$ term in Equation 71.

$$\begin{aligned}&d\emptyset = d\phi - \frac{a}{r^2+a^2}dt\\ &\Omega_H = \frac{a}{r_+^2+a^2}\\ &from\ \Delta = 0,\ solve\ r_\pm = M \pm \sqrt{M^2-a^2}\end{aligned} \tag{71}$$

Base on this derivation, in the future, we will further study whether the method mentioned in this paper can be extended to other more general cases. For example, suppose we start with three functions $e^{2\mathrm{v}(r,\theta)}, e^{-2\mathrm{v}(r,\theta)}, e^{2\lambda(r,\theta)}, e^{2\mu(r,\theta)}$ as shown in Equation (72):

$$ds^2 = e^{2\mathrm{v}(r,\theta)}dT^2 - e^{-2\mathrm{v}(r,\theta)}dr^2 - e^{2\lambda(r,\theta)}d\theta^2 - e^{2\mu(r,\theta)}d\emptyset^2 \quad (72)$$

Besides, as $dT, d\emptyset$ is shown to be related to ellipsoid symmetry, frame-dragging angular momentum, and black hole angular velocity, which are all rotation parameters, it deserves further study if this method could be extended to solve the other axial-symmetry exact solutions of vacuum Einstein's field equation.

CONCLUSION

In this paper, the Kerr metric was derived from the coordinate transformation method. Firstly, the Kerr Metric was obtained with Boyer-Lindquist in orthonormal frame, and then it was proven that it is possible to derive the Kerr metric from the vacuum ellipsoid symmetry, and this derivation allows us to better understand the physical properties of the rotating black hole, such as the frame-dragging effect, the angular velocity. This deduction method is different from classical papers written by Kerr and Chandrasekhar, and is expected to encourage future study in this subject.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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